Physics 2 E&M - Electric Potential

In physics 1 the concept of energy provided additional insight into various mechanical phenomena while simultaneously proving very useful for solving mechanics based problems - mainly because energy is a conserved quantity. In this section, energy will again prove to be very useful in dealing with various electrical phenomena. Recall the principle of the conservation of mechanical energy as stated below.

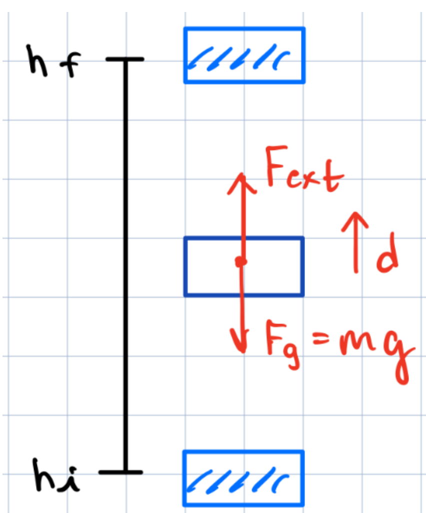
The Principle of the Conservation of Mechanical Energy
In a closed system where only conservative forces are acting; the kinetic and potential energy may change but the total mechanical energy remains constant. $$E_{mec} = K + U$$ $$U_i + K_i = U_f + K_f$$

Rearranging we can also write:

$$\Delta K = -\Delta U$$

Which can be interpreted as: *A loss/gain of kinetic energy is associated with a gain/loss in potential energy.*

An example of a conservative force is the force of gravity, while the force of friction is non-conservative. It should come as no surprise, (since the electric force and the gravitational force are both governed by the inverse square law), that the electric force is also a conservative force, and therefore electrical potential energy can be defined. We'll start, however, by recalling how gravitational potential energy is defined. The figure below shows an object being lifted at a constant rate by an external force, F_{ext}, from y_1 to y_2.



The work done by the external force is given as:

$$W_{ext} = \boldsymbol{F}_{ext} \cdot \boldsymbol{d} = mgd\,cos(0) = mg(h_f - h_i)$$

$$\boxed{W_{ext} = mg\Delta h}$$

The work done by the gravitational force if given as:

$$W_g = \boldsymbol{F}_g \cdot \boldsymbol{d} = mgd\,cos(180) = -mg(h_f - h_i)$$

$$\boxed{W_g = -mg\Delta h}$$

Next, we can define the change in gravitational potential energy of the object, ΔU, as equal to the work done by the external force.

$$\Delta U = U_f - U_i = W_{ext} = mg\Delta h$$

Similarly, it can be defined as the negative of the work done by the gravitational force.

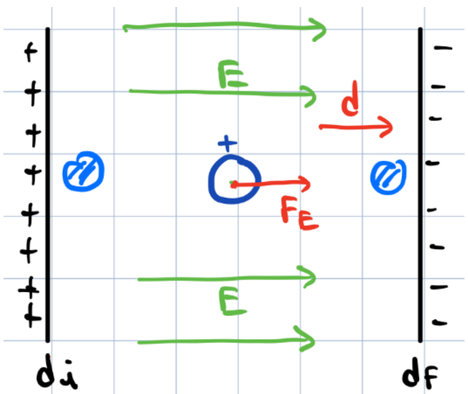
$$\Delta U = U_f - U_i = -W_g = mg\Delta h$$

Therefore, **the change in the gravitational potential energy** is given by

$$\boxed{\Delta U_g = mg\Delta h}$$

Note that when the object moves to a higher elevation, i.e., Δh *is positive*, the potential energy *increases*. However, when the object falls, i.e., Δh *is negative*, the potential energy *decreases*.

Using this as a baseline, let's now turn to the electrical potential energy of a *charged* object that is placed in an electric field. In this case, we place a positive charge in a uniform electric field and compute the work done by the electrical force as shown.



$$W_E = \boldsymbol{F}_E \cdot \boldsymbol{d} = qEd\,cos(0) = qE(d_f - d_i)$$
$$W_E = qE\Delta d$$

Next, as we did with the gravitational potential energy, we define the *electrical* potential energy as the negative of the work done by the electrical force.

$$\boxed{\Delta U = -W_E}$$

Therefore, **the change in the electrical potential energy** is given by

$$\boxed{\Delta U_E = -qE\Delta d}$$

The major difference in this case is the fact that the mass is replaced by the charge, which can be positive or negative. Referring to the figure above when a positive charge moves to the right, i.e., Δd *is positive*, the potential energy de*creases*. However, if a negative charge moves in the same direction, the potential energy *increases.* Note that moving a negative charge to the right would require an external force. Note that if a charge, (*whether positive or negative*), is under the influence of the electric force only it will always move from a state of higher potential energy to lower potential energy.

Electric Potential from Electric Potential Energy

The electric force is defined between two charged objects. Recall however, that in order to describe the effects of a single charged object the electric field was defined as shown.

$$\boldsymbol{E} \stackrel{\text{def}}{=} \frac{\boldsymbol{F}}{q}$$

And once the electric field is known in some vicinity of space we can find the force on a charged particle, q, brought into that space as:

$$\boldsymbol{F} = q\boldsymbol{E}$$

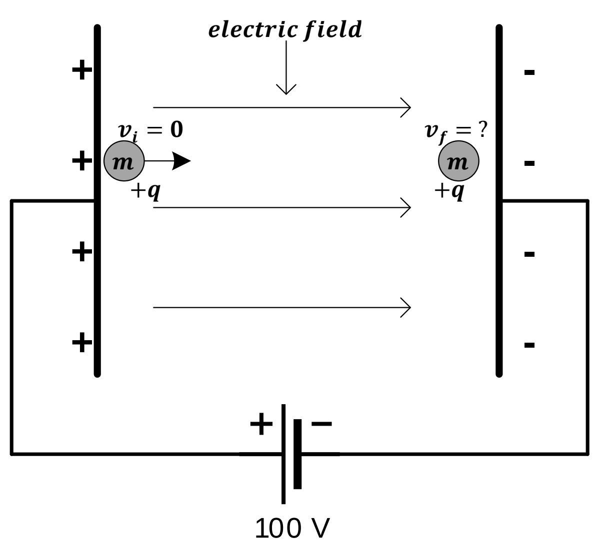
The relationship between the electric potential, symbolized as V, and the electrical potential energy, U, is exactly analogous to the relationship between the electric field and the electric force. Therefore, we have.

$$\Delta V \stackrel{\text{def}}{=} \frac{\Delta U}{q} \qquad\qquad \Delta U = q\Delta V$$

The units of electric potential are J/C. However, as this quantity is so frequently used, it was given its own unit; (volt, V). It is named after Alessandro Volta, who is credited with creating the first electric battery.

Let's look at a few examples to illustrate the above concepts.

Example 1: Suppose we connect a $100\ V$ battery between two conducting plates. We then place a proton near the positive plate. What speed will the proton attain when it reaches the negative plate?



Solution 1: The proton has a positive charge and will therefore move towards the negative plate, i.e., from a higher electric potential to lower, once released. Assuming the electrical force is the only acting force, we can use the conservation of mechanical energy as shown.

$$U_f + K_f = U_i + K_i$$

With the initial kinetic energy being zero we have.

$$K_f = U_i - U_f$$

$$\frac{1}{2}mv^2 = -\Delta U$$

Finally, using $\Delta U = q\Delta V$, we can solve for the speed of the proton when it reaches the negative plate as follows.

$$\frac{1}{2}mv^2 = -q\Delta V$$

$$v = \sqrt{\frac{-2q(V_f - V_i)}{m}}$$

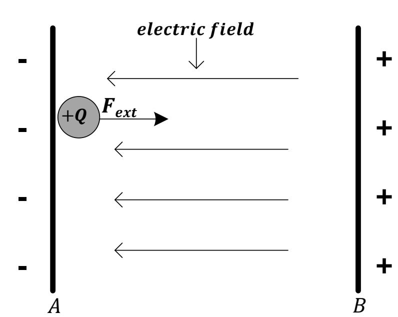
The proton ends at the negative terminal, which is at a lower potential. We can arbitrarily set this to voltage to $V_f = 0$, which gives us $V_i = 100$. Substituting we have the following:

$$v = \sqrt{\frac{-2(1.6\ X\ 10^{-19})(0 - 100)}{1.67\ X\ 10^{-27}}}$$

$$v = 1.38\ X\ 10^5\ {}^{m}/_{s}$$

Example 2: If the work required to move $10\ C$ of charge without accelerating from point A to point B is $20\ J$, then what is the electrical potential difference between these points.

Solution 2: Since an external force is required to move the positive charge, we can assume point A is at a lower potential than point B, which can be illustrated in the figure below.



In this case, the conservation of energy can be used with an added term to describe the work performed by the external non-conservative force.

$$U_i + K_i + W_{ext} = U_f + K_f$$
$$\Delta U = -\Delta K + W_{ext}$$
$$\boxed{\Delta U = W_{ext}}$$

Where we let $\Delta K = 0$, since the acceleration is zero. Note the final equation is the same result we found with the gravitational example above. Finally, with $\Delta U = Q\Delta V$, we have

$$Q\Delta V = W_{ext}$$
$$\Delta V = \frac{W_{ext}}{Q}$$
$$\Delta V = \frac{20}{10}$$
$$\Delta V = 2\ V$$

Note since $\Delta V = V_B - V_A$ is a positive number, point B is at a higher potential than point A, as we originally suggested.

Example 3: To recharge a $12\ V$ battery, a battery charger must move $2.2E5\ C$ of charge from the negative terminal to the positive terminal. How much work is done by the battery charger?

Solution 3: The external work can be found using the equation derived from example 2 above.

$$W_{ext} = Q\Delta V$$
$$W_{ext} = 2.2E^5 \cdot 12$$
$$W_{ext} = 2.64E^6\ J$$

Relationship Between the Electric Field and the Electric Potential

In the first few lessons we solved various problems with a force based approach using the relationship between the electric force and the electric field, i.e., $\boldsymbol{F} = q\boldsymbol{E}$. The examples above make it clear that problems can also be solved with an energy based approach using the relationship between the electric potential energy and the electric potential, i.e., $\Delta U = q\Delta V$. One of the major advantages of using an energy based approach is that the electric potential is a scalar quantity, whereas the electric field is a vector quantity. For this reason, we now explore the relationship between these two quantities.

At any point in space where there is a vector-based value for the electric field there is a corresponding scalar value for the electric potential. To find a relationship between these two quantities we start with the general expression for the work done by the electrical force in moving a charged particle from an initial position, d_i, to a final position, d_f.

$$W_E = \int_{d_i}^{d_f} \boldsymbol{F_e} \cdot \boldsymbol{dl}$$

From our introductory example, we know that the work done by the electric force is equal to the negative change in the electrical potential energy of the particle, i.e., $W_e = -\Delta U = -q\Delta V$. Using this along with the relationship between the electric force and the electric field, i.e., $\boldsymbol{F} = q\boldsymbol{E}$, we can find an integral expression for the change in the electric potential as shown.

$$-q\Delta V = \int_{d_i}^{d_f} q\boldsymbol{E} \cdot \boldsymbol{dl}$$

$$\Delta V = -\int_{d_i}^{d_f} \boldsymbol{E} \cdot \boldsymbol{dl}$$

The potential difference between any two points, i and f, in an electric field is equal to negative line integral, (integral along a given path), of $\boldsymbol{E} \cdot \boldsymbol{dl}$ from i to f.

We can derive the inverse relationship by considering an infinitesimal potential difference, dV, between two points that are dl apart. In this case we can remove the integral as shown.

$$dV = -\boldsymbol{E} \cdot \boldsymbol{dl} = -E\,cos(\theta)\,dl$$

Finally, if we let $E_l = E\,cos(\theta)$, be the component of the electric field in the direction of dl, we can create a differential relationship shown below.

$$E_l = -\frac{dV}{dl}$$

In other words, the value of the electric field in a certain direction, e.g., $\boldsymbol{dl}$, is equal to the rate of change of the scalar electric potential along that same direction.

To further generalize, we can express the electric field vector, $\boldsymbol{E}$, using three partial derivatives as follows. Therefore, given the electric potential function, $V(x, y, z)$, the electric field vector is given as.

$$\boldsymbol{E} = -\langle\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\rangle$$

A shorthand notation that is widely used in mathematics to express the derivative of multivariable functions uses the *nabla* symbol, ∇, referred to as the del operator.

$$\nabla \stackrel{\text{def}}{=} \frac{\partial}{\partial x}\boldsymbol{\hat{\imath}} + \frac{\partial}{\partial x}\boldsymbol{\hat{\jmath}} + \frac{\partial}{\partial x}\boldsymbol{\hat{k}}$$

When this operator is applied to a scalar function, e.g., $V(x, y, z)$, the results is referred to as the gradient vector of the function operated on. Note, the gradient vector, ∇V, is a vector that points in the direction of maximum increase of V. In this case, the electric field vector is equal to the negative gradient of the electric potential function. In other words, given a scalar electric potential field, the electric field vector will consistently point in the direction where the scalar potential decreases the most, i.e., (direction of maximum descent).

$$\boldsymbol{E} = -\nabla V$$

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}\right)$$

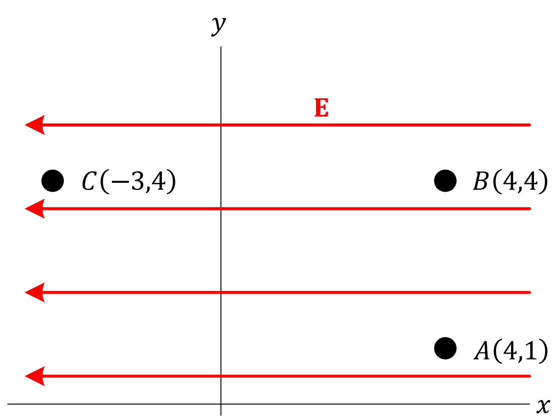
To summarize, we have developed two formulas to describe the relationship between the electric field, a vector quantity, and the electric potential, a scalar quantity.

Electric potential from the electric field	*Electric field from the electric potential*
$\Delta V = -\int_{d_i}^{d_f} \boldsymbol{E} \cdot \boldsymbol{dl}$	$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}\right)$

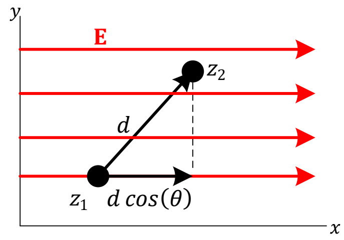
Let's explore these relationships with some examples.

Example 4: A uniform electric field with magnitude $300\ N/C$ points in the negative x direction as shown below. Determine the following electric potential differences.

a. $V_{AB} = V_B - V_A$
b. $V_{CB} = V_B - V_C$
c. $V_{AC} = V_C - V_A$



Solution 4: Let's start by investigating a general case for a change in electric potential when the electric field is uniform, i.e., constant magnitude and direction.



We can use the figure above, which shows a uniform electric field of magnitude E pointing in the positive x direction. The potential difference between two arbitrary points, Z_1 and Z_2, is

$$\Delta V = -\int_{Z_1}^{Z_2} \boldsymbol{E} \cdot \boldsymbol{dl}$$

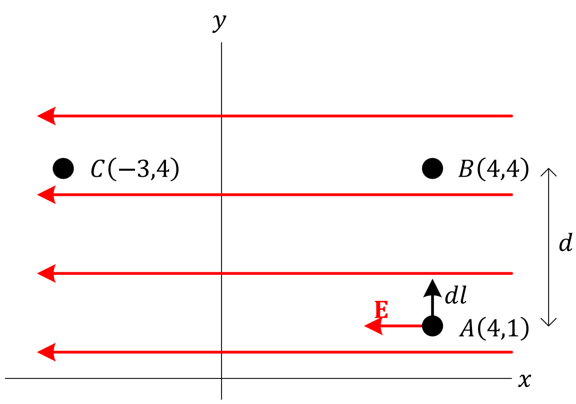
Since the electric field is constant, the dot product is given as: $\boldsymbol{E} \cdot \boldsymbol{dl} = E\,cos(\theta)\,|\boldsymbol{dl}|$**.** Leaving us with the following expression

$$\Delta V = -E\,cos(\theta)\int_{Z_1}^{Z_2} |\boldsymbol{dl}|$$

Where, $\int_{Z_1}^{Z_2}|\boldsymbol{dl}|$, represents the straight line distance from Z_1 to Z_2, i.e., $\int_{Z_1}^{Z_2}|\boldsymbol{dl}| = d$.
Therefore, we can write the general formula.

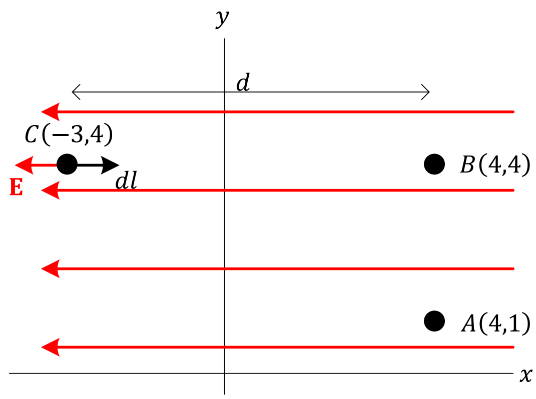
> The potential difference between Z_1 and Z_2 in a ***uniform electric field*** is given as:
>
> $$\Delta V = -Ed\,cos(\theta)$$
>
> Where, E, is the magnitude of the electric field and $d\,cos(\theta)$ represents the distance between $Z_1$and Z_2 that is parallel to the field.

Solution 4a: In this case, since the electric field vector is perpendicular to the displacement vector, the difference in the electric potential is zero.



$$V_{AB} = -Ed\,cos(\theta)$$
$$V_{AB} = -(300)3\,cos(90^{\circ})$$
$$V_{AB} = 0$$

4b: In this case the electric field vector is anti-parallel to the displacement vector.



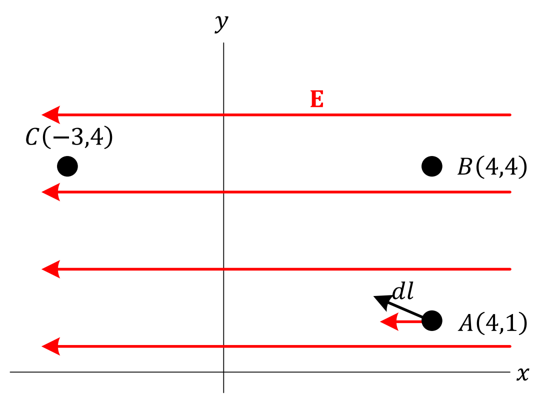
$$V_{CB} = -Ed\cos(180^{\circ})$$
$$V_{CB} = -(300)7(-1)$$
$$V_{CB} = 2100\ V$$
$$V_B - V_C = 2100\ V$$

The last equation is meant the emphasize that the point B is at a higher electric potential than point C, as should be evident from the direction of the electric field.

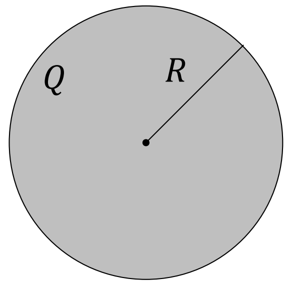
4c: In this case the displacement vector is neither parallel nor perpendicular to the electric field vector. The problem can be solved by finding the angle between the two vectors and evaluating the dot product. However, the problem can be made much simpler if we recall from 4a that point A and point B are at the same potential. Therefore, $V_{AC} = V_{BC}$. Furthermore, it should be clear that $V_{BC} = -V_{CB}$. Therefore,

$$V_{AC} = -2100\ V$$
$$V_C - V_A = -2100$$

In this case the last equation shows that point C is at a lower electric potential than point A, which again should be evident from the direction of the electric field



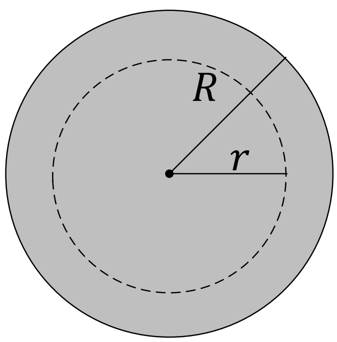
Example 5: Determine the potential at a distance r from the center of a non-conducting sphere that is uniformly charged with a total charge Q. Assume $V = 0$ at $r = \infty$.



Solution 5: In this case, we'll first find the electric field using Gauss's law.

- **Inside the sphere, $r \le R$**

Since the charge is uniformly distributed the charge enclosed in the Gaussian surface is given as shown below.



$$\frac{Q_{enc}}{Q} = \left(\frac{Volume\ with\ radius\ r}{Volume\ with\ radius\ R}\right)$$

$$Q_{enc} = \left(\frac{{}^4/_3\,\pi r^3}{{}^4/_3\,\pi R^3}\right) Q$$

$$Q_{enc} = \left(\frac{r^3}{R^3}\right) Q$$

The electric field inside the sphere can then be found using Gauss's law.

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$

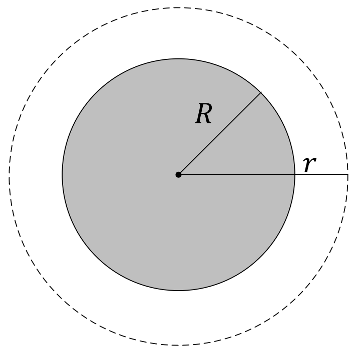
$$E \oint dA = \frac{Q}{\epsilon_o}\left(\frac{r^3}{R^3}\right)$$

$$E 4\pi r^2 = \frac{Q}{\epsilon_o}\left(\frac{r^3}{R^3}\right)$$

$$\boxed{E(r) = \frac{kQ}{R^3} r, \quad r \le R}$$

- **Outside the sphere, $r \geq R$**

In this case all the charge is enclosed in the Gaussian surface so that the electric field is that of a point charge.



$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$

$$E \oint dA = \frac{Q}{\epsilon_o}$$

$$E 4\pi r^2 = \frac{Q}{\epsilon_o}$$

$$\boxed{E(r) = \frac{kQ}{r^2}, \quad r \geq R}$$

With these results, along with our new relationships from above, we can now find the electric potential, starting with $r \geq R$.

$$\Delta V = -\int_r^\infty \boldsymbol{E}(r) \cdot \boldsymbol{dr}$$

Since $\boldsymbol{E}(r)$ and $\boldsymbol{dr}$ point in the same positive direction, $\boldsymbol{E}(r) \cdot \boldsymbol{dr} = E(r)dr$.

$$\Delta V = -\int_r^\infty \frac{kQ}{r^2} dr$$

$$\Delta V = kQ\left(\frac{1}{\infty} - \frac{1}{r}\right)$$

$$V(\infty) - V(r) = -\frac{kQ}{r}$$

Finally, letting $V(\infty) = 0$, we have

$$\boxed{V(r) = \frac{kQ}{r}, r \geq R}$$

Which, we should note, is also the electric potential of any point charge.

To find the potential inside the sphere, we integrate from a point on the surface to a point inside the sphere, using $E(r) = \frac{kQ}{R^3}r, \ \ r \le R$.

$$V(R) - V(r) = -\int_r^R \boldsymbol{E}(r) \cdot \boldsymbol{dr}$$

$$V(R) - V(r) = -\frac{kQ}{R^3}\int_r^R r dr$$

$$V(R) - V(r) = -\frac{kQ}{2R^3}(R^2 - r^2)$$

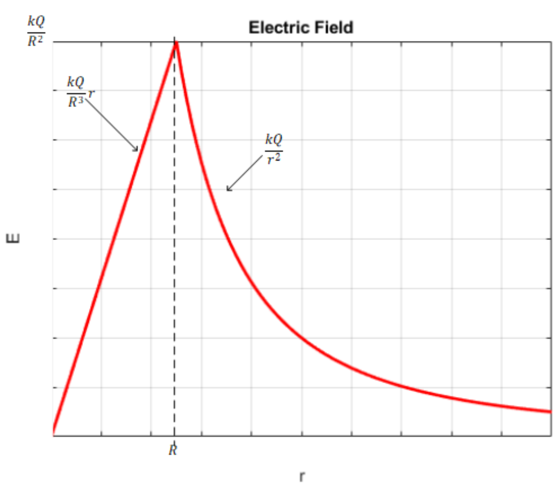
$$V(r) = \frac{kQ}{2R} - \frac{kQr^2}{2R^3} + V(R)$$

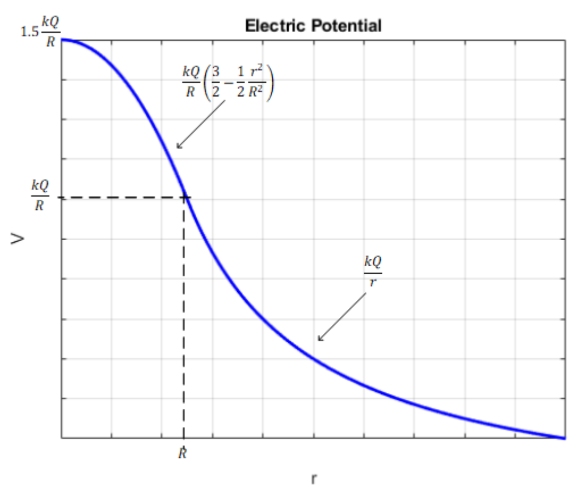
$$V(r) = \frac{kQ}{2R} - \frac{kQr^2}{2R^3} + \frac{kQ}{R}$$

$$V(r) = \frac{3kQ}{2R} - \frac{kQr^2}{2R^3}$$

$$\boxed{V(r) = \frac{kQ}{R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), r \le \ R}$$

Where we used $V(R)$ from the previous expression. We plot the both the electric field and the electric potential below for illustrative purposes.

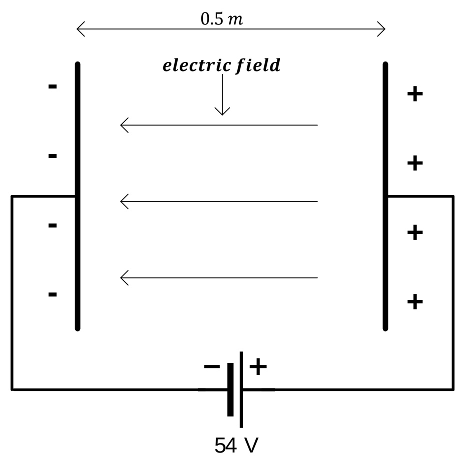




Plot highlights:

- From $r = 0$ to $r = R$
  - The electric field starts at zero and *increases* linearly to $\frac{KQ}{R^2}$ at the surface,
  - The electric potential starts at 1.5 times the value at the surface decreasing according to an inverted parabola.
- From $r = R$ to $r = \infty$
  - The electric field decreases as $1/r^2$, as expected for a point charge.
  - The electric potential decreases as $1/r$ to a value of 0 at $r = \infty$.

Example 6: The potential difference between two plates is $54\ V$. If the plates are $0.5\ m$ apart, what is the value of the electric field between them?



Solution 6: The electric field is equal to the negative of gradient of the potential as shown below.

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} + \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}\right)$$

In this case, the potential varies linearly in the x direction only, therefore the electric field is constant and can be written as follows.

$$E = -\frac{dV}{dx}$$

Assuming the negative plate is at $0\ V$, the potential function can be written as

$$V(x) = \frac{54}{0.5}x$$

Therefore, the value of the electric field is given as

$$E = -\frac{d}{dx}\left(\frac{54}{0.5}x\right) = -108\ V/m$$

Example 7: The electric potential in a certain region is

$$V(x) = 10x^2 - 18x + 63\ V$$

Determine the magnitude electric field at $x = 2\ m$.

Solution 7: Again, since the potential varies in the x direction only we have

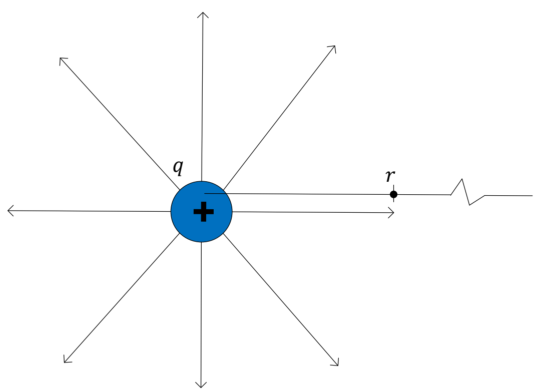
$$E(x) = -\frac{d}{dx}(V(x)) = 20x - 18$$
$$E(x) = -\frac{d}{dx}(10x^2 - 18x + 63)$$
$$E(x) = 20x - 18$$

Therefore,

$$E(2) = 20 \cdot 2 - 18 = 22\ V/m$$

Electric Potential due to a Group of Charged Particles

As we saw in example 5 the electric potential due to a single point charge is given as follows.



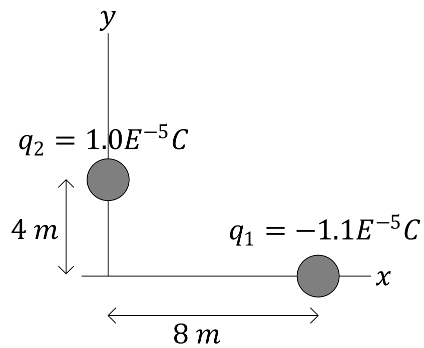
$$V(r) = k\frac{q}{r}$$

Just as we did with the electric field due to multiple sources, we can use the superposition principle to find the potential from a group of N charged particles.

$$V_P = k\sum_{i=1}^{N}\frac{q_i}{r_{Pi}}$$

Where, P is the point where we are computing the potentials and r_{Pi} is the distance from that point to the i^{th} charged particle.

Example 8: Two charges are placed in a $2D$ plane as shown below. a.) Find the electric potential at the origin resulting from this arrangement. b.) Find the electric field at the origin resulting from this arrangement.



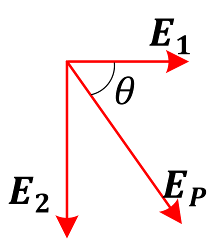
Solution 8a: Since the electric potential is a scalar quantity it is much easier to compute compared to the electric field.

$$V_P = k\sum_{i=1}^{N}\frac{q_i}{r_{Pi}} = k\left(\frac{q_1}{r_{P1}}+\frac{q_2}{r_{P2}}\right) = 9E^9\left(\frac{-1.1E^{-5}}{8}+\frac{1.0E^{-5}}{4}\right) = 10125\ V$$

8b: The electric field, being a vector quantity, is much more difficult to compute compared to the electric potential. The complexity in this case is somewhat reduced since each vector is in one dimension only.

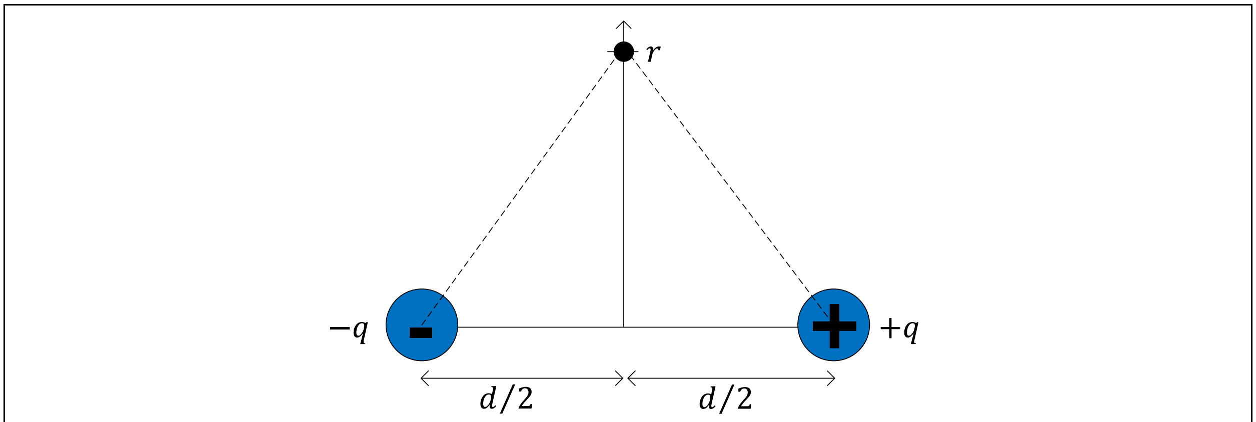
$$\boldsymbol{E}_P = \sum_{i=1}^{N}\boldsymbol{E}_i = k\left\langle\frac{|q_1|}{{r_{P1}}^2},0\right\rangle + k\left\langle 0,-\frac{|q_2|}{{r_{P2}}^2}\right\rangle = k\left\langle\frac{1.1E^{-5}}{8^2},-\frac{1.0E^{-5}}{4^2}\right\rangle = k\langle 1.72E^{-7},-6.25E^{-7}\rangle$$

The magnitude and angle are computed below.



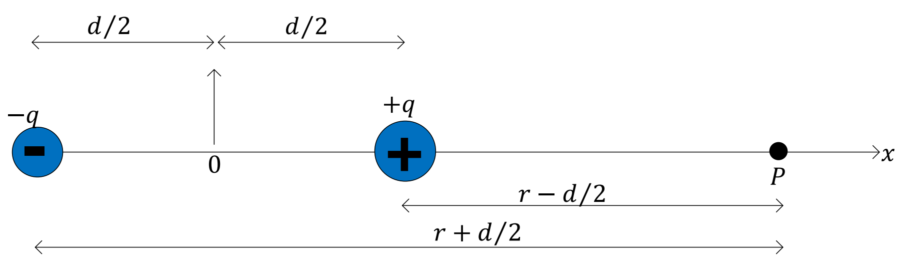
$$|\boldsymbol{E}_P| = 9E^9\sqrt{(1.72E^{-7})^2+(6.25E^{-7})^2}$$
$$= 5834\ N/C$$
$$\theta = tan^{-1}\left(\frac{6.25}{1.72}\right) = 74.6^{\circ}$$

Example 9: In a previous lesson on the electric field, we looked at the electric dipole. Specifically, we found that the electric field along the perpendicular bisector as shown below. In this example we will find the electric potential along this same vertical line, as well as the potential along x-axis.



Electric Field at a distance r, due to an Electric Dipole with separation distance d.	$E = k\dfrac{qd}{r^3}$	$r >> d$

Solution 9: We'll start by finding the electric potential for a point along the x-axis. We'll use a point on the right side of the dipole, as shown below.



Using the principle of superposition, the electric potential at the point P along the x-axis is

$$V_P = k\sum_{i=1}^{N}\frac{q_i}{r_{Pi}}$$

$$V_P = k\left(\frac{q}{r - d/2} - \frac{q}{r + d/2}\right)$$

Which can be simplified as follows

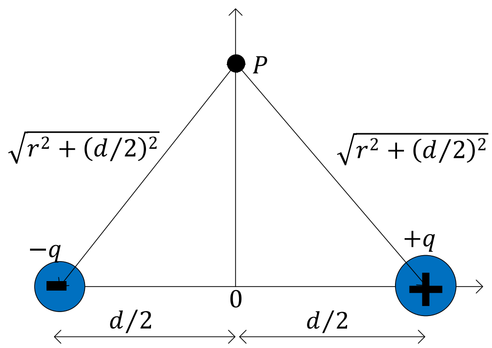
$$V_P = kq\left(\frac{(r + d/2) - (r - d/2)}{(r + d/2)(r - d/2)}\right)$$

$$V_P = kq\left(\frac{d}{(r^2 - d^2/4)}\right)$$

Next, as before, we let $r >> d$, allowing us to remove the $d^2/4$ term in the denominator.

Electric Potential at a distance r **along the parallel axis**, due to an Electric Dipole with separation distance d.	$V_P = kqd\left(\frac{1}{r^2}\right)$	$r >> d$

Next, we will find the electric potential for a point along the y-axis.

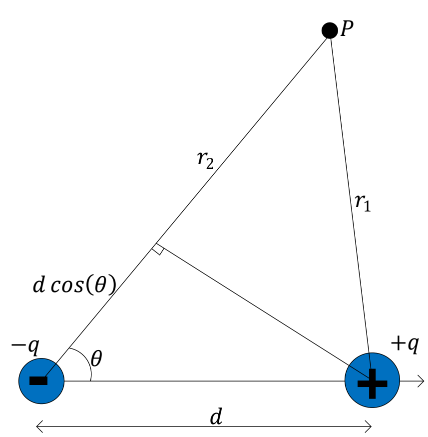


In this case, since the distance to the point P is equal from both charges, the electric potential is zero everywhere on the y-axis.

$$V_P = k\sum_{i=1}^{N}\frac{q_i}{r_{Pi}} = k\left(\frac{q}{\sqrt{r^2+(d/2)^2}} - \frac{q}{\sqrt{r^2+(d/2)^2}}\right) = 0$$

Electric Potential at a distance r **along the transverse axis**, due to an Electric Dipole with separation distance d.	$V_P = 0$

Finally, to find a more general formula we will place the point P at an arbitrary location in the x-y plane as shown below.



The configuration above assumes that the point P is much farther away from the dipole compared to the distance d, which allows us to approximate $r_1 = r_2 = r$. Therefore, the negative charge is $d\, cos(\theta)$ further from point P than the positive charge, and we can again use the superposition principle to find the electric potential at P.

$$V_P = k \sum_{i=1}^{N} \frac{q_i}{r_{Pi}}$$

$$V_P = k \left(\frac{q}{r} - \frac{q}{r + d\, cos(\theta)} \right)$$

$$V_P = kq \left(\frac{(r + d\, cos(\theta)) - r}{r(r + d\, cos(\theta))} \right)$$

$$V_P = kq \left(\frac{d\, cos(\theta)}{r(r + d\, cos(\theta))} \right)$$

And since $r >> d$, we can remove the $d\, cos(\theta)$ term in the denominator. With this, we have derived a much more general formula for the electric potential due to a dipole at an arbitrary point P, which is summarized below.

Electric Potential at a distance r from an Electric Dipole with separation distance d.	$V_P = kqd\, cos(\theta) \left(\frac{1}{r^2} \right)$	$r >> d$

Note when $\theta = 0^\circ$, i.e., P is on the x-axis, the above reduces to the equation found above.

$$V_P = kqd\, cos(0^\circ) \left(\frac{1}{r^2} \right) = kqd \left(\frac{1}{r^2} \right)$$

And when $\theta = 90^\circ$, i.e., P is on the y-axis, the electric potential reduces to 0, as expected.

$$V_P = kqd\, cos(90^\circ) \left(\frac{1}{r^2} \right) = 0$$

Equipotential Surfaces

From our lesson on electric fields, we found the following.

Electric Field at a distance r **along the transverse axis**, due to an Electric Dipole with separation distance d.	$E = k\dfrac{qd}{r^3}$	$r >> d$

How is it then, as we have seen above, that the electric potential along the transverse line is zero? The answer is related to *Equipotential Surfaces*, which are locations in the space where $\Delta V = 0$. Recall the equation we developed for the change in electric potential as a function of the electric field.

$$\Delta V = -\int_i^f \boldsymbol{E} \cdot \boldsymbol{dl}$$

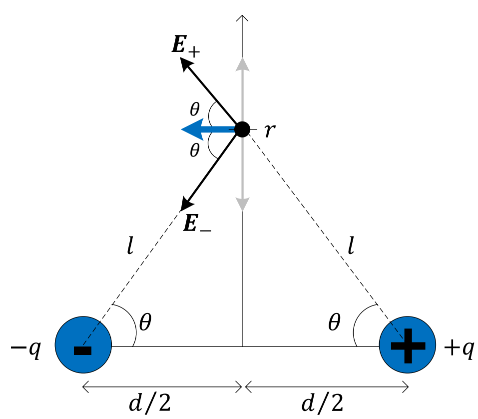
For equipotential surfaces we have:

$$0 = -\int_i^f \boldsymbol{E} \cdot \boldsymbol{dl}$$

$$0 = \int_i^f E \, \cos(\theta) \, dl$$

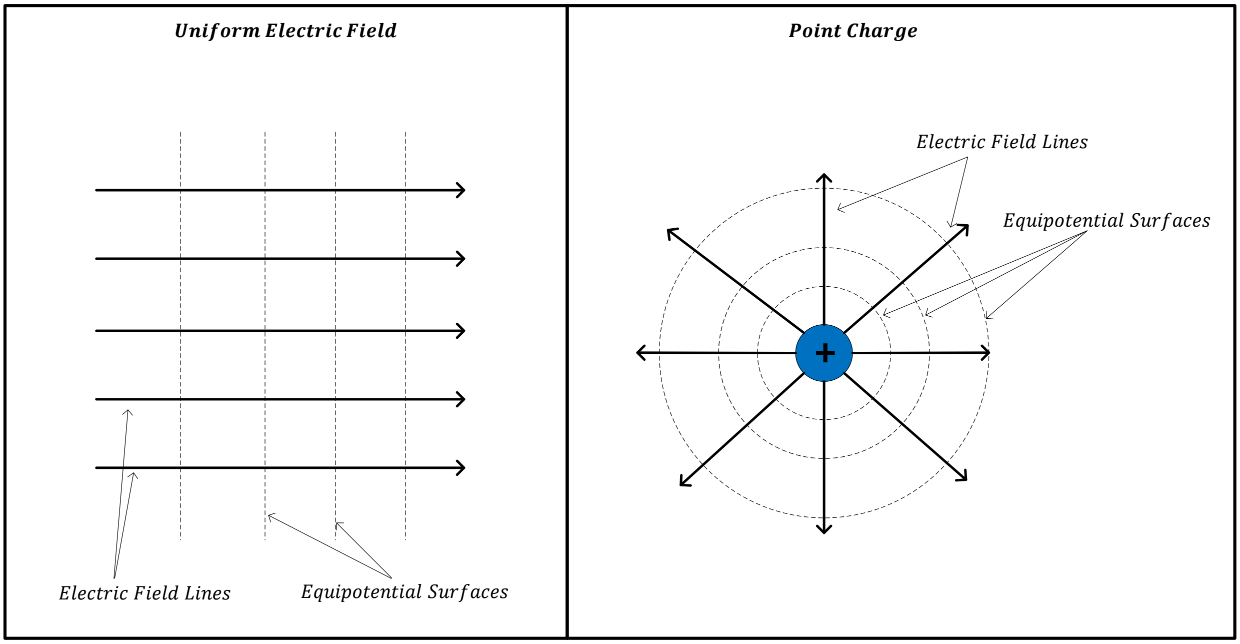
And since $cos(90) = 0$ we can define equipotential surfaces as those where $\boldsymbol{E}$ and $\boldsymbol{dl}$ are perpendicular.

This is exactly the case we found with the electric field of a dipole. Note the figure below copied from an earlier lesson.



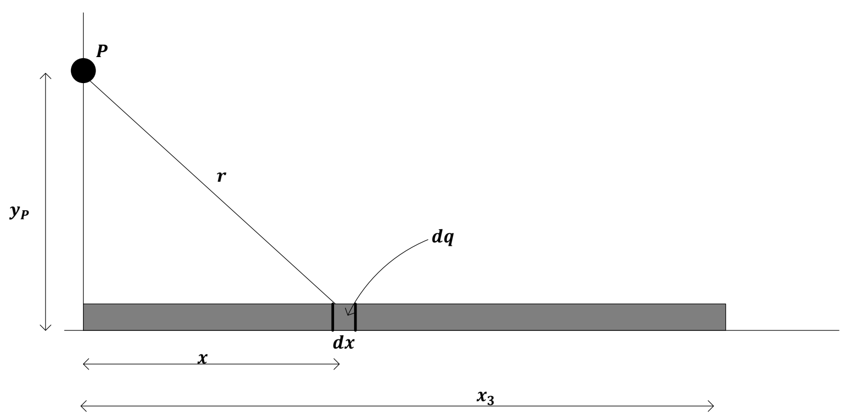
Since the vertical components of the electric field cancel, the resultant field vector points along the x-axis all along the transverse line. Therefore, as we travel along this line, downward from infinity, the electric potential remains constant (note we assume zero potential at infinity). The transverse line of a dipole is then an equipotential line. Also note that since the work done by an external force to move a test charge is proportional to the change in electric potential, there is no work done to move along equipotential surfaces.

Two additional examples showing equipotential surfaces are shown in the figures below. The left figure is for a uniform electric field and the right is for a single point charge. .



Electric Potential Due to Continuous Charge Distributions

As we have seen, to find the potential from a group of charged particles we sum the potentials due to each particle. With this in mind let's look at how we can find the electric potential for a continuous line of charge with a charge density of $\lambda \ C/m$ placed along the along the x-axis as shown below.



We start as we have done in the past by considering a small piece of the charged line with length dx, which will contain a charge of dq. Treating this infinitesimal element as a charged point particle the electric potential is given as:

$$dV = k\frac{dq}{r}$$

And since the charge is uniformly distributed along the rod, the charge contained in dx is:

$$dq = \lambda dx$$

Substituting this, along with solving for r in terms of x and y_p, we have:

$$dV = k\lambda \frac{1}{r} dx$$

$$dV = k\lambda \frac{1}{\sqrt{{y_p}^2 + x^2}} dx$$

As we are dealing with infinitesimal sections of the rod, we replace our superposition summation with an integral as follows:

$$V = k\lambda \int_0^{x_3} \frac{1}{\sqrt{{y_p}^2 + x^2}} dx$$

This resulting integral is not straightforward to evaluate but can be easily looked up in a table of integrals. The solution is given below:

$$V = k\lambda \left[\ln \left(\frac{x_3 + \sqrt{{y_p}^2 + {x_3}^2}}{y_p} \right) \right]$$

As you can see the idea is exactly like finding the electric field for continuous charges except, we are working with scalar values instead of vectors, which generally makes the calculations easier.

Electric <u>Potential Energy</u> from Multiple Charges:

We are now familiar with the equation for the electric potential associated with a point charge.

$$V(r) = k\frac{q_1}{r}$$

If we now place another charge at a distance r from the first charge, we can compute the electrical *potential energy* of this arrangement as.

$$U(r) = q_2 V(r)$$

$$U(r) = k\frac{q_1 q_2}{r}$$

Where, r is the distance between the two charges.

What happens when we bring another charge near these charges? We can explain the scenario as follows:

To bring a single charge (without accelerating) to an otherwise empty location in space requires no work since there is no force to repel this action. Hence there is no associated potential energy with a single charge. However, bringing a second charge to a distance of r_{12} from the first charge will require work since an electric field now exists due to the first charge. In this case there will be an associated potential energy associated with this arrangement of charges given by:

$$U = k\frac{q_1 q_2}{r_{12}}$$

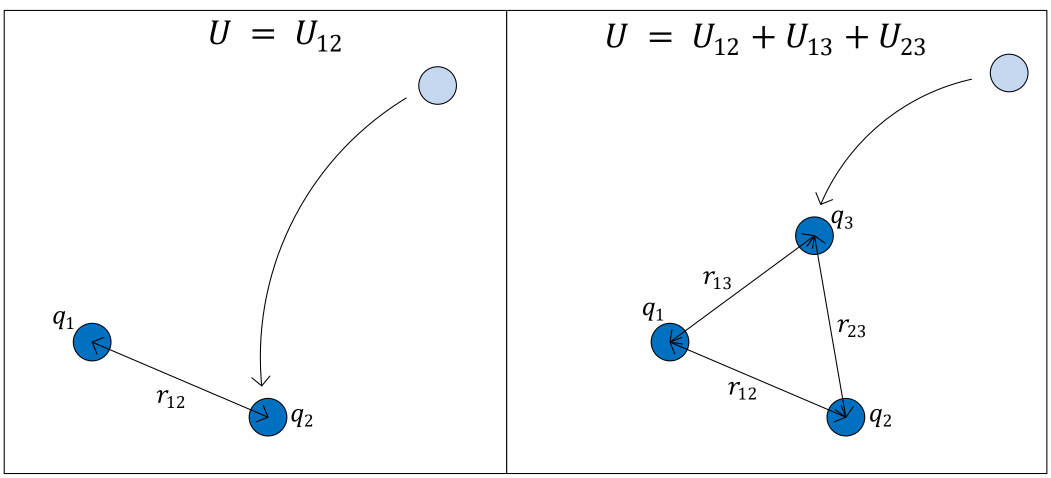
We now attempt to bring a third charge near the first two. The work now required results from the electric field from *both* charges. Using the superposition principle, the potential energy associated with this arrangement of charges can be computed as follows:

$$U = U_{12} + U_{13} + U_{23}$$

$$U = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}}$$

$$U = k\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

Where, U_{12} is the potential energy associated with bringing charge 2 near charge 1, U_{13} is the potential energy associated with bringing charge 3 near charge 1, and U_{23} is the potential energy associated with bringing charge 3 near charge 2.



Of course, this procedure will continue for any number of charges.

Final Summary for Electric Potential

Work and Electric Potential Energy
The work done by the conservative electric force by a charged object on a test charge, q, is equal to the negative change in potential energy of the test charge – object system. $$W_C = -\Delta U = -(U_f - U_i)$$ If the test charge starts at an infinite distance from the charged object where the potential energy is zero, i.e., ($U_i = 0$), we can write: $$W_C = -U(r)$$ Where, r is the distance between the charged object and the test charge.
Electric Potential
The change in electric potential, ΔV, is defined as the change in electric potential energy per unit charge. $$\Delta V \stackrel{\text{def}}{=} \frac{\Delta U}{q} \qquad \Delta U = q\Delta V$$
Relationship Between the Electric Field and the Electric Potential
The electric potential difference between two points is found by computing the following integral taken over ant path between two points, e.g., d_i and d_f. $$\Delta V = -\int_{d_i}^{d_f} \boldsymbol{E} \cdot \boldsymbol{dl}$$ In the special case of a uniform electric field, the electric potential difference is given as $$\Delta V = -E\, cos(\theta)\, \Delta d$$ Where, E is the magnitude of the field, d is the straight line distance between the two points, and theta is the angle between $\boldsymbol{E}$ and a vector in the direction of Δd. The electric field is equal to the negative gradient of the electric potential. $$\boldsymbol{E} = -\nabla \mathrm{V}$$ $$\boldsymbol{E} = -\frac{\partial V}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{\jmath}} - \frac{\partial V}{\partial x}\hat{\boldsymbol{k}}$$ Where each component of the electric field is equal to the partial derivative of the electric potential.

The Electric Potential from a Point Charge
The electric potential from a point charge, q, at a distance r from the charge is: $$V(r) = k\frac{q}{r}$$
Electric Potential due to a Collection of Charged Particles
The electric potential at a point P in space due to a N charged particles is found using the superposition principle. $$V_P = k\sum_{i=1}^{N}\frac{q_i}{r_i}$$ Where, r_i is the distance from the i^{th} charge to the point P.
Electric Potential due to a Continuous Charge Distribution
The electric potential from a continuous charge distribution is found by integrating over the entire distribution. $$V = k\int\frac{1}{r}dq$$
Electric Potential Energy of a System of Charged Particles
The electric potential energy of a system of charged particles can be found by imagining the assembly of the system one charged particle at a time, where each particle is brought to its location from infinitely far away. For example, the potential energy of a system of three particles can be written as follows: $$U = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$ Where, r_{12} is the distance between particle 1 and 2, r_{13} is the distance between particle 1 and 3, and r_{23} is the distance between particle 2 and 3.
Equipotential Surfaces
Equipotential surfaces are ones where the potential remains constant. Moving along an equipotential surface requires zero work. The electric field is always perpendicular to equipotential surfaces.

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