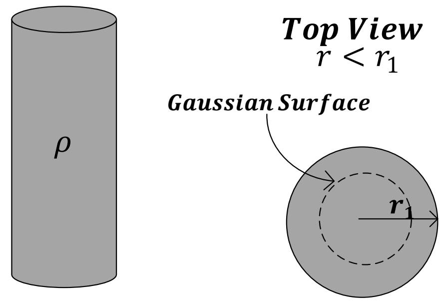
Physics 2 E&M - Gauss's Law and Conductors

In example 5 from the previous lesson, we found that the electric field inside a **uniformly charged insulator**, in this case a cylinder, increases linearly as shown below.



$$E = \left(\frac{\rho}{2\epsilon_o}\right) r, \quad r < r_1$$

ρ is the charge density C/m^3

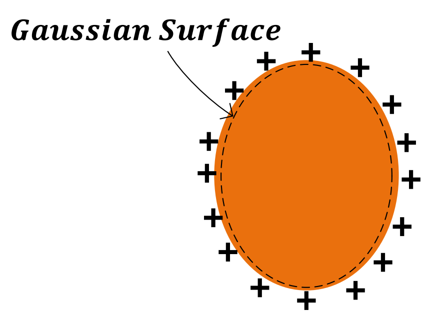
In this lesson we investigate the electric field inside **charged conductors**.

Recall that conductors are objects that allow electrons to move freely from atom to atom within the object, while insulators do not easily allow this movement of electrons. Therefore, if an insulator is somehow given an excess of electrons, e.g., via friction, the electrons will tend to stay at the location where they were initially placed. On the other hand, when excess electrons are imparted on a conductor, they are free to move throughout the material. The repulsive force between electrons will tend to make the excess electrons move as far apart from each other as possible, thereby migrating to the surface of the material. This phenomenon is succinctly stated as a theorem given below.

Theorem for Static Charges and Conductors
If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

Next, we look at the implication this theorem has on the electric field inside conductors.

The figure below shows a charged conductor, where all of the charge is located on the surface. To find the electric field inside the conductor we draw a Gaussian surface just inside the walls of the conductor as shown below.



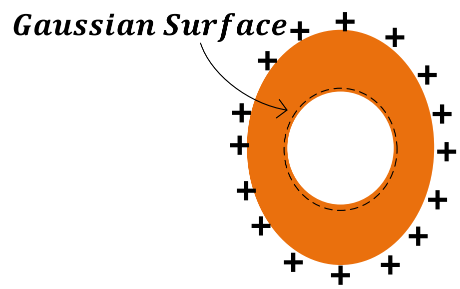
Then we write Gauss's law:

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$

Since all of the excess charge is located on the surface of the object, we have $Q_{enc} = 0$, which implies that the electric field is zero inside the object.

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = 0 \rightarrow \boldsymbol{E} = \boldsymbol{0}$$

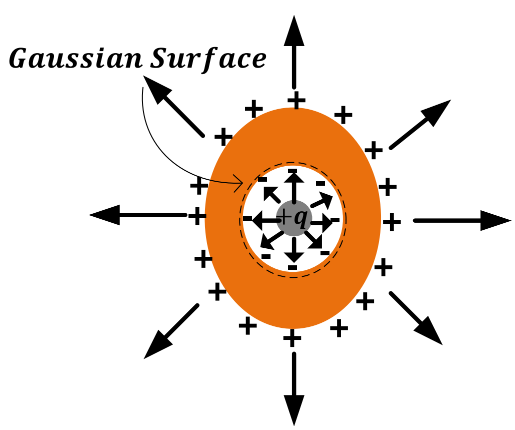
Taking this example one step further, assume we "scooped out" some material in the center of the object, creating a sort of spherical shell as shown below.



Using the Gaussian surface shown we see that Q_{enc} is still zero, implying again that the electric field inside a charged conducting shell is also zero.

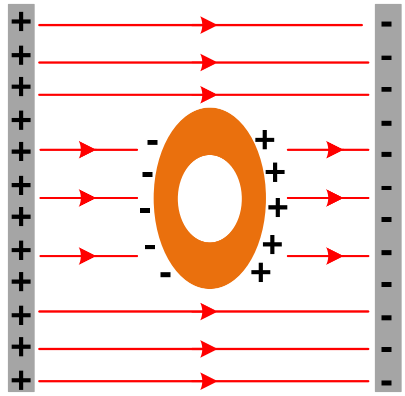
$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = 0 \rightarrow \boldsymbol{E} = \boldsymbol{0}$$

Next, we place a charge, e.g., $+q$, inside a conducting shell.



Interestingly, the electric field within the conductor is still zero. This is due to the fact that the positive charge inside the shell will now induce an equal magnitude negative charge on the inside surface of the shell. A Gaussian surface just inside the conductor shows that $Q_{enc} = +q - q = 0$, implying again that $\boldsymbol{E} = \boldsymbol{0}$ within the conductor. Also note that the negative charge induced on the inside surface of the shell will induce a positive charge on the outside surface of the shell. Therefore, an electric field exists both inside and outside the shell, but not within the conductor itself.

Lastly, we place a conducting shell in an electric field as shown below.



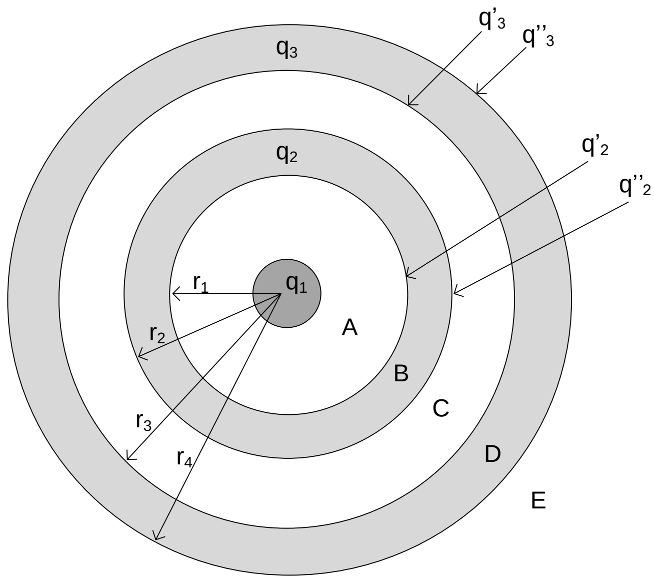
Since the conductor contains electrons that are free to move from atom to atom, they will redistribute themselves at the surface. As shown, the electric field lines emanating from the positively charged plate terminate on the left surface of the conductor. Furthermore, the electric field lines that terminate at the negatively charged plate emanate from the right surface of the conductor. From this we see that the electric field inside the hollow conducting box is zero, i.e., the hollow portion of the shell is “shielded” from an eternal electric field.

The fact that a hollow metal structure will have an electric field of zero everywhere inside has various practical applications. One example is for research laboratories where sensitive electronic measurements need to be made. These structures are usually built enclosed by a conducting material which allows the delicate measurements to be unaffected by outside sources. Another example is in various electronic equipment, where some circuits are built within metal structures to “shield” them from the neighboring circuits. Finally, even in everyday life we can see that your car, (assuming it’s made of metal material), can be safe place in a lightning storm.

Example 1: A point charge of q_1 is concentric with two spherical conducting thick shells. The smaller shell has a net charge of q_2 and the larger shell has a net charge of q_3. The charge on the inner and outer surfaces of the shells are represented as q'_x and q''_x respectively. Charge and radii values are given below.

$$q_1 = -6\ pC \qquad q_2 = -23\ pC \qquad q_3 = 12\ pC$$

$$r_1 = 1.2\ cm \qquad r_2 = 2.5\ cm \qquad r_3 = 4.4\ cm \qquad r_4 = 5.9\ cm$$



Find the charge on the outer and inner surface of both shells. Then find the magnitude of the electric field at the points, A, B, C, D, and E. These points correspond to the following distances from the center, $r_A = 0.6\ cm,\ r_B = 1.85\ cm,\ r_C = 3.45\ cm,\ r_D = 5.15\ cm,\ r_E = 6.65\ cm.$

Solution 1:

Charge on Smaller Shell:

The center charge of q_1 induces an equal in magnitude but opposite sign charge on the inner surface of the smaller shell.

$$q'_2 = -q_1$$

And since the net charge on the smaller shell is given as q_2, there must be a balancing charge on the outside surface such that $q_2 = q'_2 + q''_2$. Solving for q''_2, we have

$$q''_2 = q_2 - q'_2$$

Finally, using the given values we find the charges as shown below.

$$q'_2 = -(-6) = 6\ pC \qquad q''_2 = -23 - 6 = -29\ pC$$

Charge on Larger Shell:

The charge of q''_2 on the outer surface of the smaller shell induces an equal in magnitude but opposite sign charge on the inner surface of the larger shell.

$$q'_3 = -q''_2$$

To find the charge on the outer surface we again use the fact that the net charge is known.

$$q''_3 = q_3 - q'_3$$

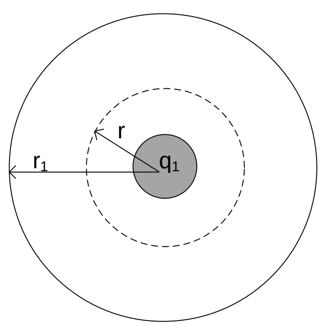
Using the given values, we find the charges as shown below.

$$q'_3 = -(-29) = 29\ pC \qquad\qquad q''_3 = 12 - 29 = -17\ pC$$

Electric Field:

We can use Gauss's Law to find the electric field at the various locations given.

$\boldsymbol{A}$: $\boldsymbol{r < r_1,\ \ r = 0.6\ cm}$



$$\oint \boldsymbol{E \cdot dA} = \frac{Q_{enc}}{\epsilon_o}$$

$$E4\pi r^2 = \frac{q_1}{\epsilon_o}$$

$$E = \frac{1}{\epsilon_o 4\pi} \cdot \frac{q_1}{r^2}$$

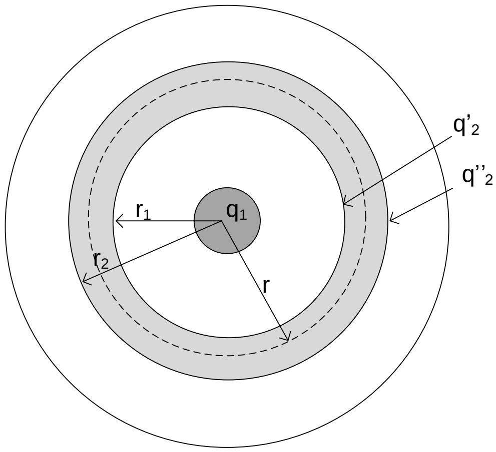
$$E = k \cdot \frac{q_1}{r^2}$$

Which is the expression for Coulomb's Law for a point charge as expected. Using the given values, we find the magnitude of the electric field as follows.

$$|E| = 9E^9 \cdot \frac{6E^{-12}}{(0.006)^2} = 1500\ N/C$$

$\boldsymbol{B}$: $\boldsymbol{r_1 < r < r_2,\ \ r = 1.85\ cm}$

In this case, since the shell in conducting we know the electric field is zero. We show this more explicitly using Gauss's Law below.

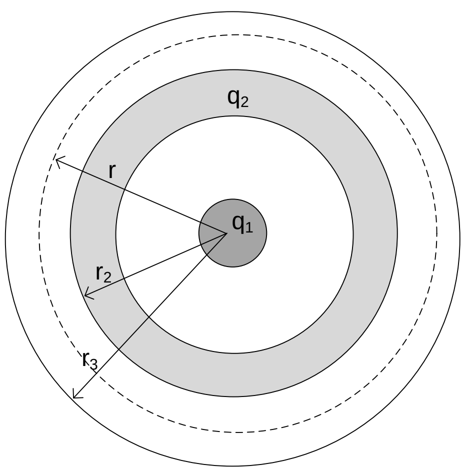


$$\oint \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{Q_{enc}}{\epsilon_o}$$
$$E4\pi r^2 = \frac{q_1 + q'_2}{\epsilon_o}$$
$$E = \frac{q_1 + q'_2}{4\pi\epsilon_o r^2}$$

Recall, we found above that $q'_2 = -q_1$, therefore we have

$$E = \frac{q_1 + (-q_1)}{4\pi\epsilon_o r^2} = 0$$

$\boldsymbol{C}$: $\boldsymbol{r_2 < r < r_3,\ \ r = 3.45\ cm}$



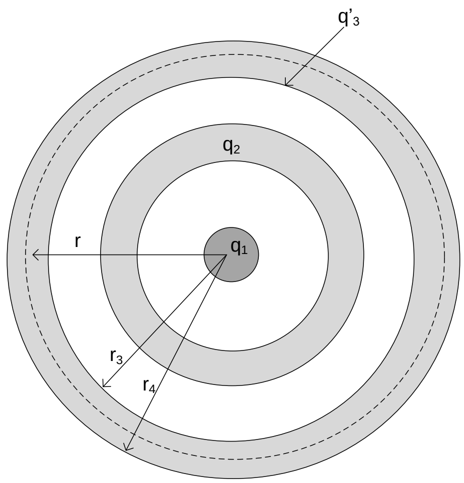
$$\oint \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{Q_{enc}}{\epsilon_o}$$
$$E4\pi r^2 = \frac{(q_1 + q_2)}{\epsilon_o}$$
$$E = \frac{1}{\epsilon_o 4\pi}\cdot\frac{(q_1 + q_2)}{r^2}$$
$$E = k\cdot\frac{(q_1 + q_2)}{r^2}$$

We again find that the electric field is given by Coulomb's Law. Using the given values, we find the magnitude of the electric field as follows.

$$|E| = 9E^9\cdot\frac{|-6E^{-12} + (-23E^{-12})|}{(0.0345)^2} = 219.3\ N/C$$

D: $\boldsymbol{r_3 < r < r_4,\ \ r = 5.15\ cm}$

As was the case in region B, the shell is conducting and therefore the electric field is zero. For completeness we again show the computation using Gauss's Law.

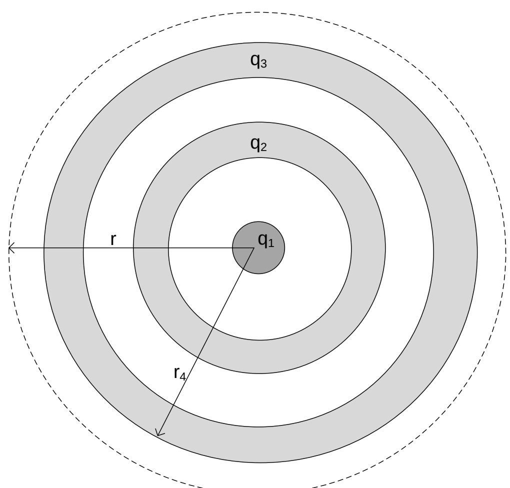


$$\oint \boldsymbol{E \cdot dA} = \frac{Q_{enc}}{\epsilon_o}$$
$$E 4\pi r^2 = \frac{(q_1 + q_2 + q'_3)}{\epsilon_o}$$
$$E = \frac{(q_1 + q_2 + q'_3)}{4\pi\epsilon_o r^2}$$

Recall, we found above that $q'_3 = -q''_2$, and $q''_2 = (q_1 + q_2)$. Therefore, we have

$$E = \frac{((q_1 + q_2) - (q_1 + q_2))}{4\pi\epsilon_o r^2} = 0$$

E: $\boldsymbol{r > r_4,\ \ r = 6.65\ cm}$

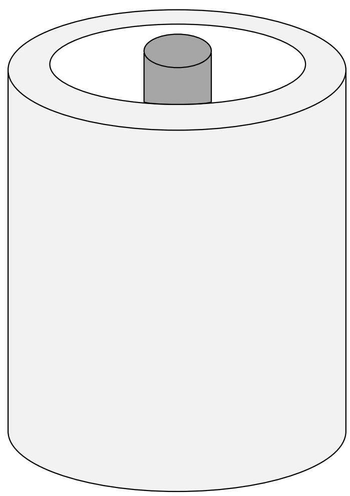


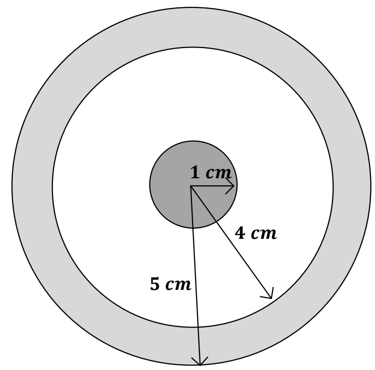
$$\oint \boldsymbol{E \cdot dA} = \frac{Q_{enc}}{\epsilon_o}$$
$$E 4\pi r^2 = \frac{(q_1 + q_2 + q_3)}{\epsilon_o}$$
$$E = \frac{1}{\epsilon_o 4\pi} \cdot \frac{(q_1 + q_2 + q_3)}{r^2}$$
$$E = k \cdot \frac{(q_1 + q_2 + q_3)}{r^2}$$

As usual we find that the electric field is given by Coulomb's Law. Using the given values, we find the magnitude of the electric field as follows.

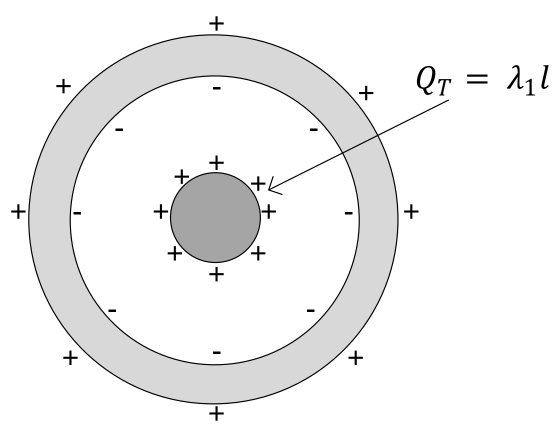
$$|E| = 9E^9 \cdot \frac{|-6E^{-12} + (-23E^{-12}) + (12E^{-12})|}{(0.0665)^2} = 34.6\ N/C$$

Example 2: A coaxial cable is an electrical cable that has an inner conductor surrounded by an insulating layer, and then surrounded again by a conducting shield. Consider an infinitely long cable illustrated below that carries a charge of $\lambda_1 = 5\ nC/m$ on its inner conductor, while the outer conducting shield in uncharged. a.) Find the charge density on the inside and outside surface of the conducting shield. b.) Find an expression for the electric field for all r.





Solution 2a: The charge on the inner conductor is located on its outside surface. As we know, this will induce an equal magnitude but opposite sign charge on the inner surface of the outer conducting shell. Finally, since the outer shell is uncharged there must exist the same amount of positive charge on its outside surface. This is shown in the figure below.



The total charge on the inner conductor is given as follows.

$$Q_T = \lambda_1 l$$

Where, l is the length of the cable.

This total charge is then spread over the inside, (and outside), surface of the outer conducting shell. The charge density on the inside surface, λ_2, is then given as

$$\lambda_2 = \frac{Q_T}{2\pi r_2 l} = \frac{\lambda_1 l}{2\pi r_2 l} = \frac{\lambda_1}{2\pi r_2}$$

The density on the outside surface of the outer shell, λ_3, is similarly given by

$$\lambda_3 = \frac{\lambda_1}{2\pi r_3}$$

Substituting the given values, we compute the charge densities below.

Inside Surface	***Outside Surface***
$\lambda_2 = \dfrac{5}{2\pi 0.04}$ $\lambda_2 = 19.9\ nC/m^2$	$\lambda_3 = \dfrac{5\ X\ 10^{-9}}{2\pi 0.05}$ $\lambda_3 = 15.9\ nC/m^2$

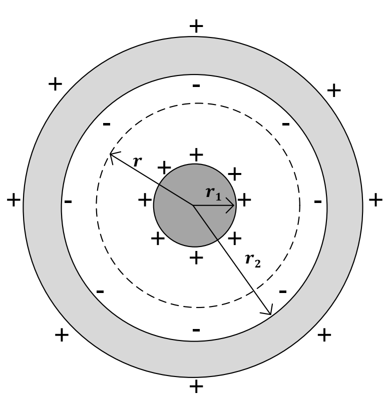
2b. To find the electric field we use Gauss's Law for the different regions.

Region 1: $\boldsymbol{r \leq r_1}$

As we know by now the electric field inside a conductor is zero.

$$E(r) = 0, \qquad r \leq r_1$$

Region 2: $\boldsymbol{r_1 < r < r_2}$

	$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \dfrac{Q_{enc}}{\epsilon_o}$ $E 2\pi r l = \dfrac{\lambda_1 l}{\epsilon_o}$ $E = \dfrac{1}{\epsilon_o 2\pi} \cdot \dfrac{\lambda_1}{r}$ $E = 2k\lambda_1 \cdot \dfrac{1}{r}$

Which shows that the electric drops off as $1/r$. Using the given values, we find the magnitude of the electric field can be expresses as follows.

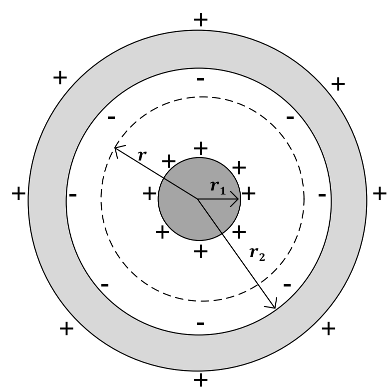
$$|E| = 2 \cdot 9E^9 \cdot 5E^{-9} \frac{1}{r} = 90 \frac{1}{r}, \quad r_1 < r < r_2$$

Region 3: $r_2 \leq r \leq r_3$

Here again we are inside a conductor and therefore we have zero electric field.

$$E(r) = 0, \quad r_2 \leq r \leq r_3$$

Region 4: $r > r_3$



$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$

$$E 2\pi r l = \frac{\lambda_1 l - \lambda_1 l + \lambda_1 l}{\epsilon_o}$$

$$E = \frac{1}{\epsilon_o 2\pi} \cdot \frac{\lambda_1}{r}$$

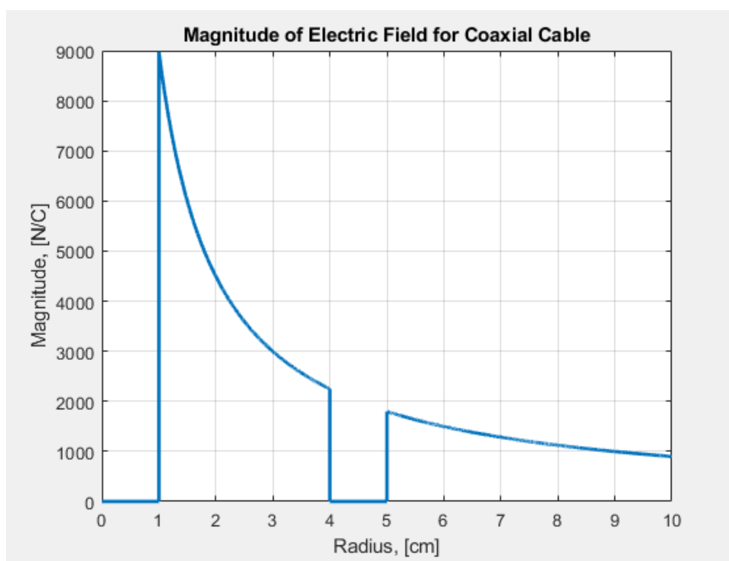
$$E = 2k\lambda_1 \cdot \frac{1}{r}$$

Since the outer shell is neutral, the expression for the electric field is identical to region 3.

$$|E| = 90\frac{1}{r}, \quad r > r_3$$

The full expression, along with the plot for additional insight, is shown below.

$$E(r) = \begin{cases} 0, & r \leq r_1 \\ 2k \cdot \dfrac{\lambda_1}{r}, & r_1 < r < r_2 \\ 0, & r_2 \leq r \leq r_3 \\ 2k \cdot \dfrac{\lambda_1}{r}, & r > r_3 \end{cases}$$



Final Summary for Gauss's Law and Conductors

Theorem for Static Charges and Conductors
If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor. This theorem implies, via Gauss's Law, that the electric field inside a statically charged conductor is zero.
Conducting Shells
An electrically charged hollow conducting structure still contains all charge on the **outside** surface, resulting in zero electric field within the hollow cavity. A hollow conducting structure placed in an electric field will redistribute its electrons on the outside surface, resulting in zero electric field within the hollow cavity. A hollow conducting structure can be used as an electric "shield", protecting objects inside from outside electric field influences. Some practical applications: • Research laboratories where sensitive electronic measurements need to be made. • Individual circuits in a multi-circuit device.

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