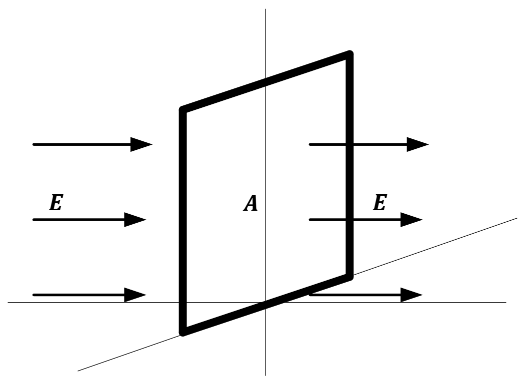
Physics 2 E&M - Gauss's Law

Gauss's law is one of the four, so called, Maxwell's Equations. Together these four equations form the basis of classical electrodynamics, and their importance to our understanding of the natural world cannot be understated. In previous lessons we learned how Coulomb's law can be used to relate electric charges to electric fields. In this section we learn that Gauss's law can provide us with a much more general and elegant mathematical relationship between electric charges and electric fields. However, before introducing Gauss's law, we will need to learn a new quantity called *electric flux,* as it plays a central role in the law.

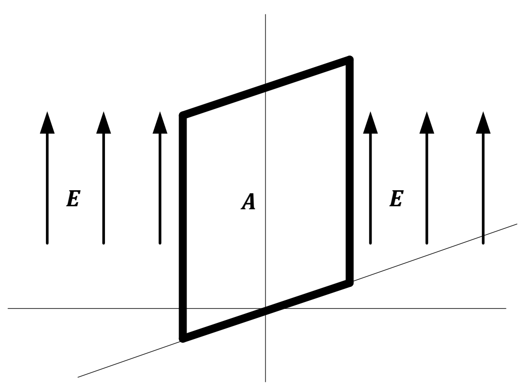
Electric Flux: The term flux is generally used to describe the flow of a certain substance through a surface, e.g., water flow through a net. As the electric field does not *flow*, we define the *electric flux,* Φ, as the *distribution* of the electric field through a surface. To illustrate we take a flat surface with area, A, and place it in uniform electric field, E, as shown below.



In this case, the electric flux is given by the magnitude of the electric field multiplied by the area of the surface.

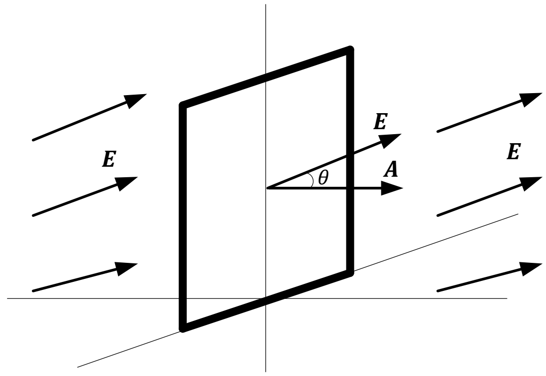
$$\Phi = EA$$

The computation may seem quite trivial for the example above, however, one could imagine that as we rotate the surface, *or equivalently the direction of the field*, less and less of the field goes *directly through* the surface. In the extreme case, i.e., the field is rotated by 90°, the electric flux should be zero since none of the field lines go *through* the surface.

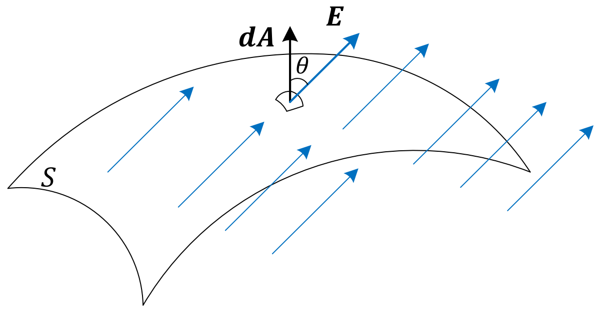


This behavior resembles that of work, for which we used the vector dot product to capture, i.e., $W = \boldsymbol{F} \cdot \boldsymbol{d}$. We can do the same here by defining an area vector, $\boldsymbol{A}$, that is normal to the surface and has a magnitude equal to the surface area. The electric flux is then defined as the vector dot product of the electric field and this area vector.

$$\Phi = \boldsymbol{E} \cdot \boldsymbol{A} = EA\,cos(\theta)$$



Lastly, and similar to the when we defined work, if the surface area is not perfectly flat and/or if the electric field is not uniform, we must use the integral to define the flux as shown below.



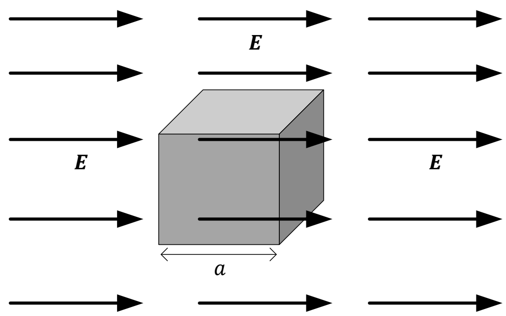
The electric flux, Φ, through a surface, S.

$$\Phi = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$$

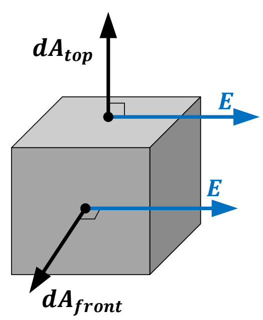
Where, $\boldsymbol{dA}$ is the area vector that represents an infinitesimal surface area element. The S at the bottom of the integral represents the fact that the integral is taken over the entire surface.

In the most general cases this integral can be difficult to evaluate, however by using surfaces that display certain symmetries we can mostly avoid fully computing the integral. We will illustrate this with the example below.

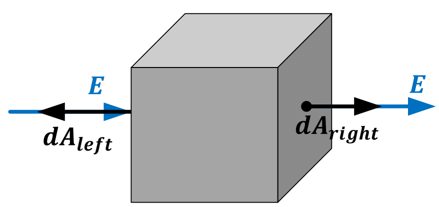
Example 1: Suppose we have a cube with side length of a that is placed in a uniform electric field with a magnitude E and oriented as shown in the figure below. Find the electric flux through the surface of the cube.



Solution 1: The problem may initially seem difficult because the overall surface area of the cube is not all contained to a single plane. However, we can compute the total flux by taking one side of the cube at a time. The cube has six sides, four of which are aligned with the direction of the electric field, i.e., front, back, top, and bottom. As shown below, using the top and front surfaces, the area vector is always perpendicular to the field, resulting in zero flux.



The remaining two surfaces, i.e., the right and left sides, have surface area vectors that are parallel with the electric field, resulting in a nonzero flux, as illustrated below.



$$\Phi_{left} = \int_S \boldsymbol{E} \cdot \boldsymbol{dA} \qquad \Phi_{right} = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$$

$$\Phi_{left} = \int_S EdA\,cos(180^\circ) \qquad \Phi_{right} = \int_S EdA\,cos(0^\circ)$$

$$\Phi_{left} = -E\int_S dA \qquad \Phi_{right} = E\int_S dA$$

$$\Phi_{left} = -Ea^2 \qquad \Phi_{right} = Ea^2$$

The flux through the entire cube is found by adding the flux through each of its surfaces.

$$\Phi_{net} = \Phi_{top} + \Phi_{bottom} + \Phi_{front} + \Phi_{back} + \Phi_{right} + \Phi_{left}$$

$$\Phi_{net} = 0 + 0 + 0 + 0 + Ea^2 + -Ea^2$$

$$\Phi_{net} = 0$$

Therefore, the flux through the surface of a cube placed in a uniform electric field is zero. We can think of this as all the electric field lines that enter the cube through the left side exit the cube through the right side. With this we are now ready to introduce Gauss's Law.

Gauss's Law: In the previous example we computed electric flux through a closed surface, i.e., a cube. As a matter of notation, the electric flux through a closed surface is written as follows:

$$\oint \boldsymbol{E} \cdot \boldsymbol{dA}$$

It is this integral, *the electric flux through a closed surface*, that is central in Gauss's law. In fact, Gauss's law provides us with knowledge about what is inside an enclosed surface, (in terms of the amount of electric charge), based on computing the electric flux through it. Specifically, Gauss's law is stated as follows.

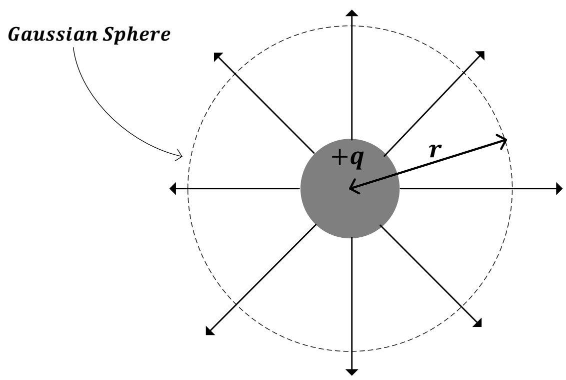
Gauss's Law
The electric flux through a ***closed*** surface is equal to the charge enclosed by that surface divided by a constant. $$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$ Where, Q_{enc} is the total charge enclosed in the surface, and ϵ_o is the permittivity of free space, $\epsilon_o = 8.85\ X\ 10^{-12}\ \ C^2/N \cdot m^2$. Note: The enclosed surface, (sometimes referred to as a Gaussian surface), can be, and usually is, imaginary.

Example 2: Find the electric field due to a point charge, q, using Gauss's Law.

Solution 2: Coulomb's Law tells us that the electric field at a distance r from a point charge is given as

$$E(r) = k\frac{q}{r^2}$$

Using Gauss's Law, we start by creating a Gaussian surface in the shape of a sphere with radius r around the point charge. We then write Gauss's las as shown below.



$$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$

The spherical Gaussian surface helps to simplify this problem in two specific ways:

1. At every point on the surface of the sphere the electric field is parallel to the surface area vector so we can remove the dot product.
2. The magnitude of the electric field is constant with respect to the surface integral and can be factored out.

With these observations we can write the following

$$E\oint dA = \frac{Q_{enc}}{\epsilon_o}$$

Furthermore, $\oint dA$ represents the surface area of the sphere, i.e., $4\pi r^2$, therefore we have

$$E4\pi r^2 = \frac{Q_{enc}}{\epsilon_o}$$

And since we know that $Q_{enc} = +q$, we can write the magnitude of the electric field as follows.

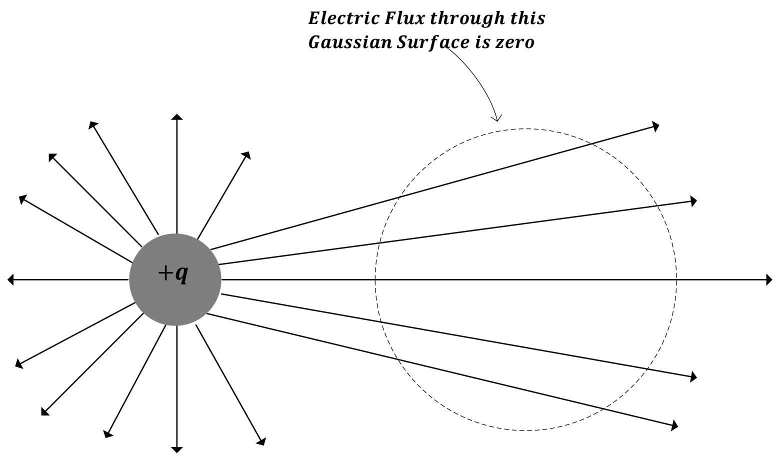
$$E(r) = \frac{q}{4\pi r^2 \epsilon_o}$$

Finally, since $k = \frac{1}{4\pi\epsilon_o}$, we have:

$$E(r) = k\frac{q}{r^2}$$

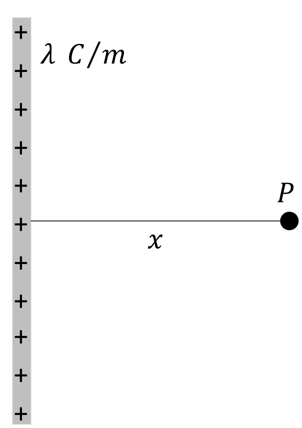
Which matches what is given by Coulomb's Law!

What if, in the previous example, the point charge was located outside of our Gaussian sphere? Since $Q_{enc} = 0$, the net flux through the surface is zero. Conversely, if we were to start by computing the electric flux, we could conclude that there is no net charge within our constructed gaussian surface, i.e., $Q_{enc} = 0$. The concept is illustrated in the figure below, where it is shown that each electric field line that enters the Gaussian sphere also exits so that the electric flux, i.e., distribution of the electric field through the entire surface, is zero.

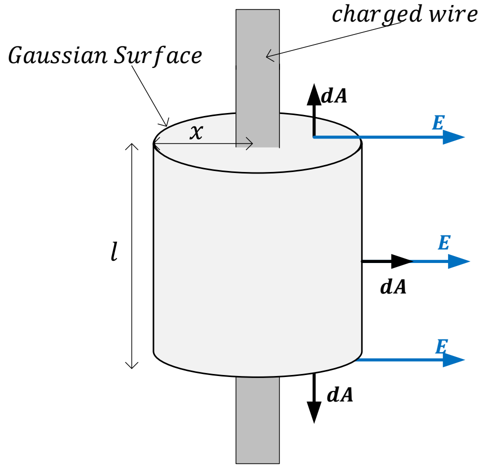


As you can see Gauss's Law can be much more convenient to use compared to Coulomb's Law when trying to find the electric field in a given region of space. However, two key elements should be present; 1. The charge distribution should display some sort of symmetry, and 2. The Gaussian surface should be chosen appropriately based on the geometry of the problem. The electric field due to an infinite line of charge and an infinite plane of charge was computed in the previous lesson using Coulomb's Law. We repeat these examples below to illustrate how Gauss's law can be applied to make the computation of the electric field much easier.

Example 3: Infinite Line of Charge - Determine the magnitude of the electric field at a point P that is a distance x from a very long wire with a uniformly distributed charge. Let λ be the charge per unit length of the wire in C/m.



Solution 3: The charge distribution is symmetrical in the sense that the electric field is constant for all points that are a horizontal distance of x from the wire. For this case it is most convenient to use a cylindrical Gaussian surface centered on the wire as shown below.



$$\oint \boldsymbol{E}d\boldsymbol{A} = \frac{Q_{enc}}{\epsilon_o}$$

Gauss's Law integral from above can be simplified using the following observations.

- The electric field is perpendicular to the top and bottom surface area vectors of the cylinder and so the flux through these surfaces is zero.
- The electric field is parallel to the surface area vector of the body of the cylinder at all points and so the dot product can be removed.
- The electric field is constant for all points on the body of the cylinder and can be pulled out of the integral.

With this we can rewrite the integral as shown.

$$E\oint dA = \frac{Q_{enc}}{\epsilon_o}$$

Where, $\oint dA$ represents the surface area for the body of the cylinder, i.e., $2\pi xl$, and the charge enclosed by the cylinder can be expressed in terms of the given charge density as $Q_{enc} = \lambda l$.

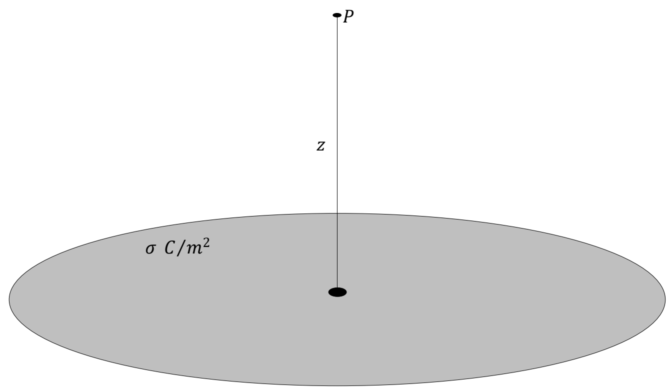
We can now solve for the electric field as follows.

$$E2\pi xl = \frac{\lambda l}{\epsilon_o} \rightarrow E = \frac{\lambda}{\epsilon_o 2\pi x}$$

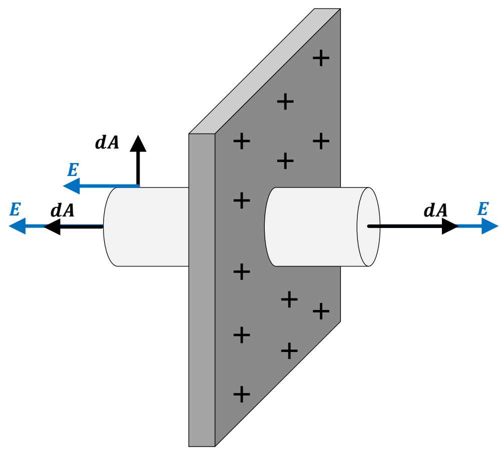
Finally, substituting $\epsilon_o = \frac{1}{4\pi k}$, we find the same expression we computed in the previous lesson, this time with much less effort.

Electric Field at a distance x, due to an infinite length wire with charge density, λ.	$E = 2k\frac{\lambda}{x}$

Example 4: Infinite Plane of Charge - Determine the magnitude of the electric field due to an infinite plane with a uniformly distributed charge. Let σ be the charge per unit area in C/m^2.



Solution 4: A cylinder can again be used as our Gaussian surface. However, in this case, the flux will be zero through the body of the surface, but non-zero through each lid. We illustrate this below using a vertically orientated plane.



With A representing the surface area of each of the cylinder lids, the charge enclosed by the cylinder can be expressed in terms of the charge density as follows.

$$Q_{enc} = \sigma A$$

With this, we can easily evaluate Gauss's Law as shown.

$$\oint \boldsymbol{EdA} = \frac{Q_{enc}}{\epsilon_o}$$

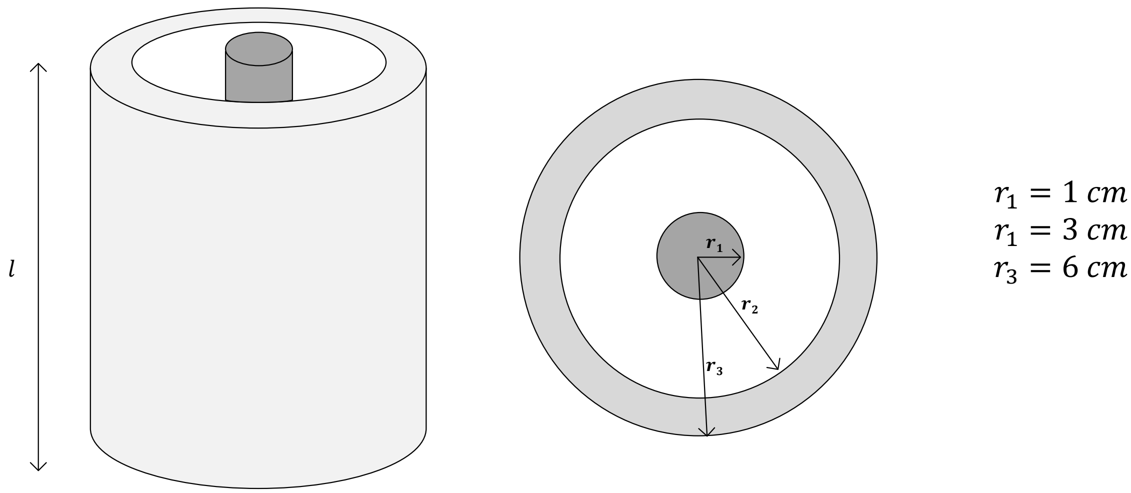
$$EA + EA = \frac{\sigma A}{\epsilon_o}$$

$$E = \frac{\sigma}{2\epsilon_o}$$

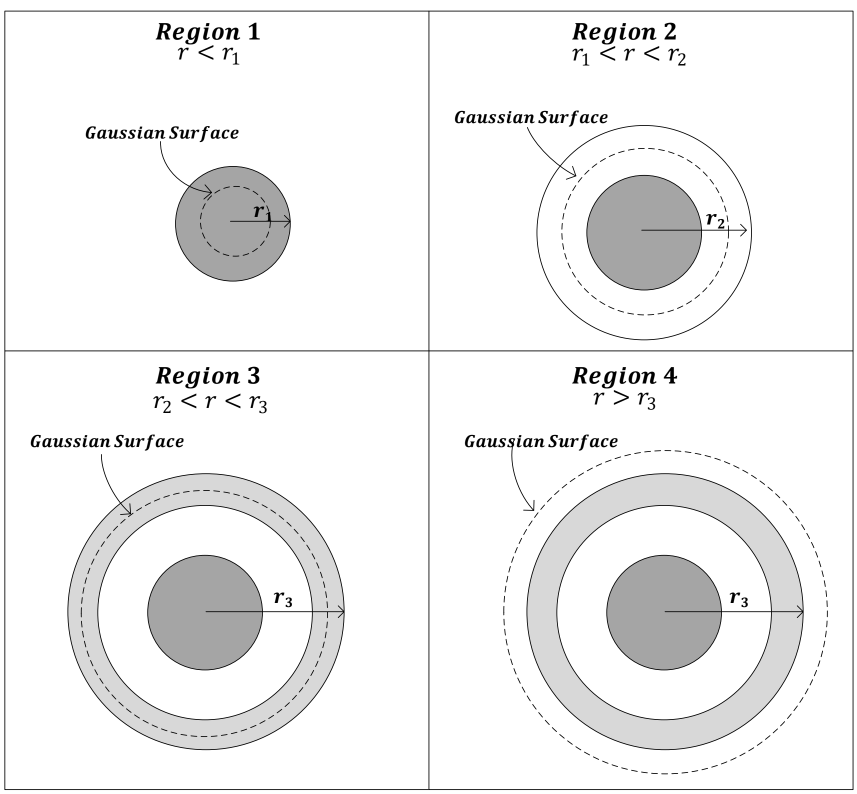
Similar to previous example, we find the same expression from the previous lesson, but with very little computation!

Electric Field due to an infinite plane of charge with a charge density, σ.	$E = \dfrac{\sigma}{2\epsilon_o}$

Example 5: Consider two very long ***nonconducting*** concentric cylinders arranged as shown in the figure below. Both cylinders are uniformly charged with a charge density, $\rho = 2\ uC/m^3$. Find an expression for the electric field for all r.



Solution 5: We can use a cylindrical Gaussian surface as we did in example 3. However, in this case we need to analyze 4 different regions separately, as specified below.



Similar to example 3, in all four cases, the flux in non-zero only for the body of the cylinder, which has a surface area of $2\pi rl$.

Region 1: The electric field is constant for any distance, r. Therefore, Gauss's Law is as follows.

$$E \oint dA = \frac{Q_{enc}}{\epsilon_o}$$
$$E 2\pi r l = \frac{Q_{enc}}{\epsilon_o}$$

Furthermore, since the charge is uniformly distributed, the enclosed charge can be expressed using the volume charge density as $Q_{enc} = \rho V_{enc} = \rho \pi r^2 l$. The electric field is then

$$\boxed{E = \left(\frac{\rho}{2\epsilon_o}\right) r, \ \ r < r_1}$$

Region 2: In this case, the Gaussian surface encloses the entire inner cylinder, and therefore $Q_{enc} = \rho \pi {r_1}^2 l$. The electric field in this region is then found as shown.

$$E \oint dA = \frac{Q_{enc}}{\epsilon_o}$$
$$E 2\pi r l = \frac{\rho \pi {r_1}^2 l}{\epsilon_o}$$
$$\boxed{E = \left(\frac{\rho {r_1}^2}{2\epsilon_o}\right)\frac{1}{r}, \ \ r_1 < r < r_2}$$

Region 3: In this case, the Gaussian surface encloses the entire inner cylinder but only a portion of the outer cylinder. The volume of the outer cylinder that is enclosed by the Gaussian surface is given as follows.

$$V_{enc} = \pi r^2 l - \pi {r_2}^2 l$$

And the total charge enclosed is then given by the sum of the charge from the inner cylinder and the charge in the portion of the outer cylinder.

$$\begin{aligned} Q_{enc} &= \rho \pi {r_1}^2 l + \rho(\pi r^2 l - \pi {r_2}^2 l) \\ &= \rho \pi l ({r_1}^2 + r^2 - {r_2}^2) \\ &= \rho \pi l ({r_1}^2 - {r_2}^2 + r^2) \end{aligned}$$

The electric field in this region is then

$$E \oint dA = \frac{Q_{enc}}{\epsilon_o}$$
$$E 2\pi r l = \frac{\rho \pi l ({r_1}^2 - {r_2}^2 + r^2)}{\epsilon_o}$$
$$\boxed{E = \left(\frac{\rho}{2\epsilon_o}\right)\frac{({r_1}^2 - {r_2}^2 + r^2)}{r}, \ \ r_2 < r < r_3}$$

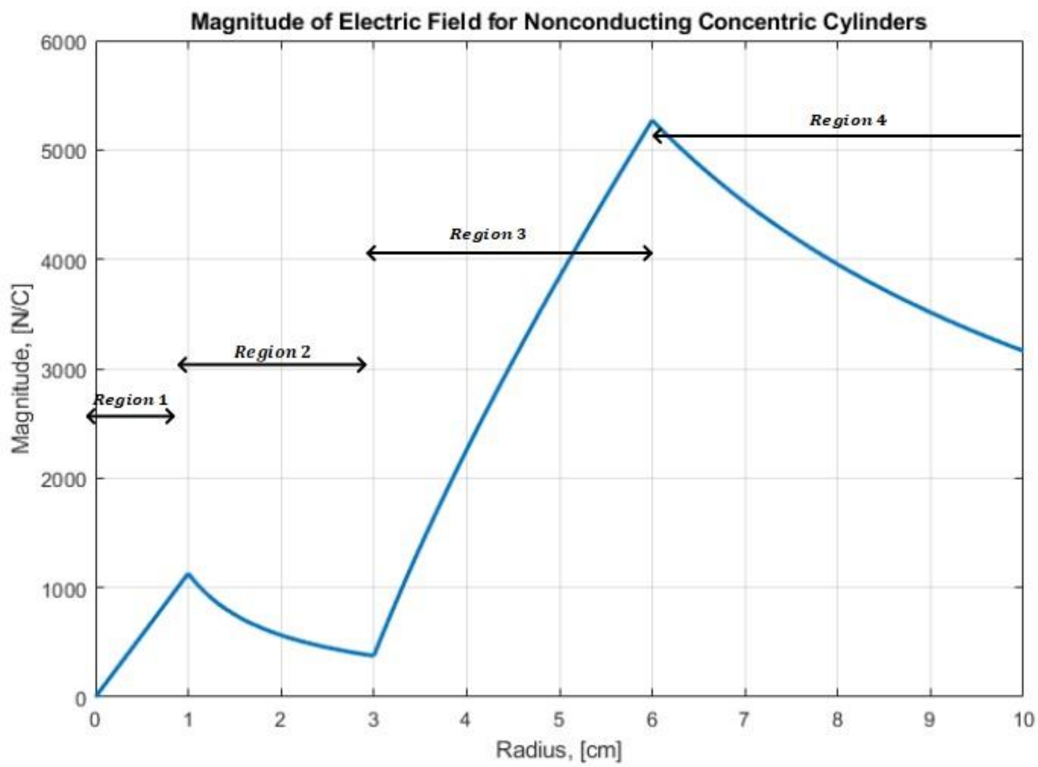
Region 4: Finally, in the last region the charge from both cylinders is enclosed by the Gaussian surface. Therefore, the electric field is given as follows.

$$E\oint dA = \frac{Q_{enc}}{\epsilon_o}$$

$$E2\pi rl = \frac{\rho\pi {r_1}^2 l + \rho\pi({r_3}^2 - {r_2}^2)l}{\epsilon_o}$$

$$\boxed{E = \left(\frac{\rho}{2\epsilon_o}\right)\frac{({r_1}^2 + {r_3}^2 - {r_2}^2)}{r}, \quad r > r_3}$$

For illustration purposes we substitute the given values and plot the magnitude of the electric field below.



The behavior of the electric field strength in the four regions is summarized below.

Region 1: The electric field strength grows linearly as the radius increases since we are enclosing more total charge.

Region 2: The electric field strength decreases as $1/r$ since the enclosed charge remains constant as we increase the radius.

Region 3: The electric field strength again grows linearly as the radius increases since we are enclosing more total charge.

Region 4: The electric field strength again begins to decrease as $1/r$ since the enclosed charge remains constant as we increase the radius.

Final Summary for Gauss's Law

Electric Flux
The electric flux through a surface is a measure of the distribution of the electric field through that surface. Mathematically the electric flux through a small area on a surface is computed with a vector dot product as follows: $$d\Phi = \boldsymbol{E} \cdot \boldsymbol{dA}$$ Where $\boldsymbol{dA}$ is a vector that is oriented perpendicular to the surface and has a magnitude equal to the area of the small patch. The compute the total electric flux through a surface we integrate over the surface of interest. $$\Phi = \int_S \boldsymbol{E} \cdot \boldsymbol{dA}$$
Gauss's Law
The electric flux through a ***closed*** surface, Φ_{net}, is equal to the charged enclosed in that surface divided by a constant. $$\oint \boldsymbol{E} \cdot \boldsymbol{dA} = \frac{Q_{enc}}{\epsilon_o}$$ Where, $\oint$ represents the integral over a ***closed*** surface, Q_{enc} is the total charge enclosed in the sufcae, and ϵ_o is the permittivity of free space, $\epsilon_o = 8.85\, X\, 10^{-12}\ C^2/N \cdot m^2$.

By: ferrantetutoring