Physics 1 Mechanics – Universal Gravitation

When solving problems related to the force of gravity, we have been using the following equation for the gravitational force:

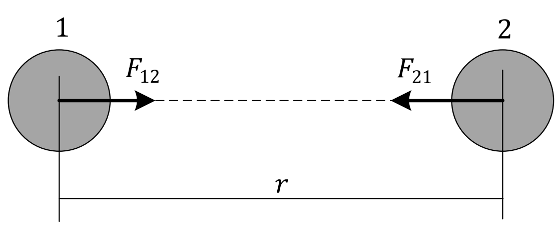
$$\boldsymbol{F}_g = m\boldsymbol{g}$$

Where, we said that g was the acceleration that all falling objects experience *near the surface of the earth*. Why did we say, "near the surface of the earth"? What about objects *far* from the surface of the earth? What about objects near the surface of a different planet? These questions, and more, are answered by *Newton's Universal Law of Gravitation*.

Newton's Universal Law of Gravitation
Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. The force is a vector that acts along the line connecting the center of mass of the two objects. $$\boldsymbol{F}_g = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}$$ Where, G is the universal gravitational constant: $G = 6.67\ E^{-11}$

Let's do an example to illustrate.

Example 1: Find the mutual gravitational force the two objects below exert on each other where $r = 10\ m$, $m_1 = 5\ kg$, and $m_1 = 15\ kg$.



Solution 1: The magnitude of the force *on* object 1 *by* object 2 is shown as F_{12}, and the force *on* object 2 *by* object 1 is shown as F_{21}. Each object attracts the other along the dashed line shown.

$$\mathrm{F}_{12} = (6.67\ \mathrm{E}^{-11})\frac{5 \cdot 15}{10^2} = 5\ \mathrm{E}^{-11}\ \mathrm{N}$$

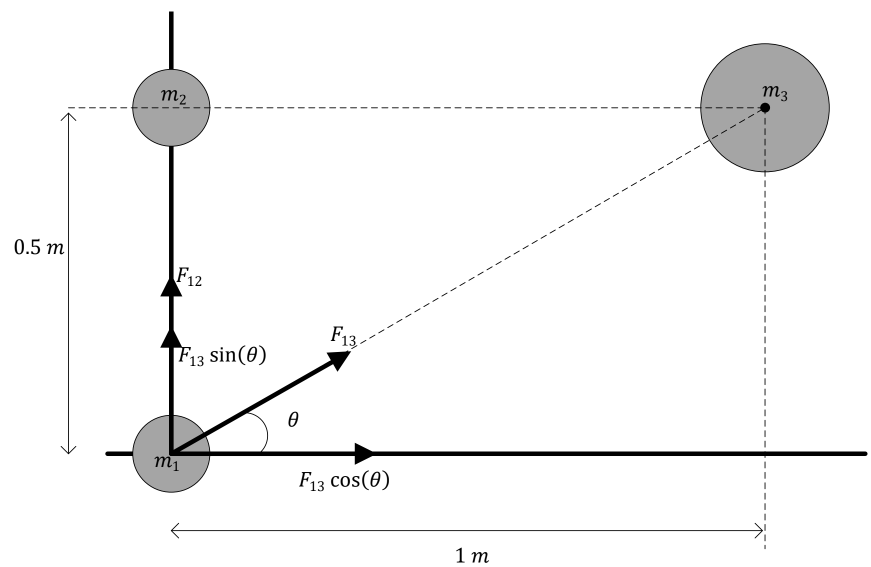
$$\mathrm{F}_{21} = (6.67\ \mathrm{E}^{-11})\frac{15 \cdot 5}{10^2} = 5\ \mathrm{E}^{-11}\ \mathrm{N}$$

As expected, the magnitude of the forces are equal, as they form a force pair from Newton's 3rd law.

Principle of Superposition: Since *all* objects attract *all other* objects, how do we compute the total gravitational force from many objects on any *one* object? We use what is called the *Principle of Superposition*, which is just a fancy way of saying that we can independently compute the force that each of the other objects exerts and add them up (vectorially).

Principle of Superposition for the Force of Gravity
For a collection of N objects, the force on the m^{th} object is computed as follows: $$\boldsymbol{F}_{m,net} = \sum_{\substack{n=1 \\ n\neq m}}^{N} \boldsymbol{F}_{m,n}$$

Example 2: Given the configuration below with $m_1 = m_2 = 3\ kg$, and $m_3 = 6\ kg$, find the net force on m_1 from the other two mases.



Using vector addition according to the principle above, we can write the following

$$\boldsymbol{F}_{1,net} = \sum_{n=2}^{3} \boldsymbol{F}_{1,n}$$

$$\boldsymbol{F}_{1,net} = \boldsymbol{F}_{1,2} + \boldsymbol{F}_{1,3}$$

$$\boldsymbol{F}_{1,net} = \langle F_{1,2,x}, F_{1,2,y}\rangle + \langle F_{1,3,x}, F_{1,3,y}\rangle$$

$$\boldsymbol{F}_{1,net} = \langle (F_{1,2,x} + F_{1,3,x}), (F_{1,2,y} + F_{1,3,y})\rangle$$

Next, we compute the x and y components separately as follows

$$F_{1,net,x} = F_{1,2,x} + F_{1,3,x}$$
$$F_{1,net,x} = 0 + F_{1,3}\,cos(\theta)$$
$$F_{1,net,x} = G\frac{m_1 m_2}{{r_{12}}^2}cos(\theta)$$

$$F_{1,net,y} = F_{1,2,y} + F_{1,3,y}$$
$$F_{1,net,y} = F_{1,2} + F_{1,3}\,sin(\theta)$$
$$F_{1,net,y} = G\frac{m_1 m_2}{{r_{12}}^2} + G\frac{m_1 m_3}{{r_{13}}^2}sin(\theta)$$

The angle, θ, and squared distances, ${r_{12}}^2$and ${r_{13}}^2$, are found below.

$$\theta = tan^{-1}\left(\frac{0.5}{1}\right) = 26.6^{\circ} \qquad {r_{12}}^2 = 0.5^2 = 0.25 \qquad {r_{13}}^2 = 0.5^2 + 1^2 = 1.25$$

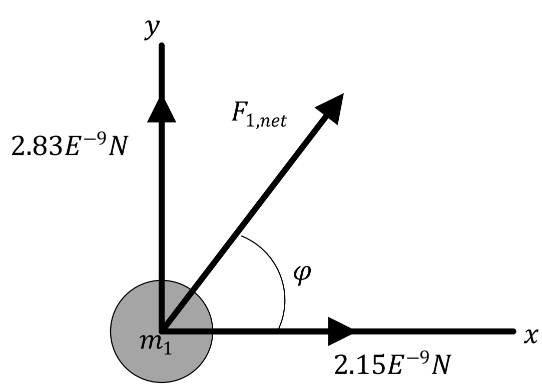
Substituting, we can solve for $F_{1,net,x}$ and $F_{1,net,y}$.

$$F_{1,net,x} = G\frac{m_1 m_2}{{r_{12}}^2}cos(\theta)$$
$$F_{1,net,x} = 6.67E^{-11}\cdot\frac{3\cdot 3}{0.25}cos(26.6^{\circ})$$
$$F_{1,net,x} = 2.15E^{-9}$$

$$F_{1,net,y} = Gm_1\left(\frac{m_2}{{r_{12}}^2} + \frac{m_3}{{r_{13}}^2}sin(\theta)\right)$$
$$F_{1,net,y} = 6.67E^{-11}\cdot 3\left(\frac{3}{0.25} + \frac{6}{1.25}sin(26.6^{\circ})\right)$$
$$F_{1,net,y} = 2.83E^{-9}$$

Finally, the net vector, as well as its magnitude and angle is shown below.

$$\boldsymbol{F}_{1,net} = \langle 2.15E^{-9}, 2.83E^{-9}\rangle\ N$$
$$F_{1,net} = \sqrt{(2.15E^{-9})^2 + (2.83E^{-9})^2} = 3.55E^{-9}\ N$$
$$\varphi = tan^{-1}\left(\frac{2.83}{2.15}\right) = 52.8^{\circ}$$

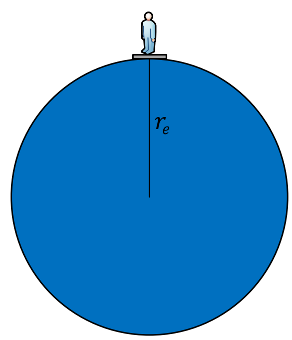


Planetary Gravity: When we step on a scale to measure our "weight", it is actually the force of gravity that we are measuring, which we previously computed as $F_g = mg$, where $g = 9.8$. Since g is a constant it looks as though our weight is a function of our mass *only*. However, we now know the force of gravity is a function of not only your mass, but the mass of the earth and your distance from the center of the earth. Let's investigate this idea using the following example.

Example 3: Using Newton's Universal Law of Gravitation, compute the weight of an $80\ kg$ person from 4 different locations:

a. Standing on the surface of the earth
b. From $11\ km$ above the surface of the earth (typical height of a commercial flight)
c. $385{,}000\ km$ from the center of the earth (average earth-moon distance)
d. Standing on the surface of the moon.

Solution 3a: Standing on the surface of the earth



Newton's universal law of gravity is written as:

$$F_g = G\frac{m_e m_p}{{r_e}^2} = \left(G\frac{m_e}{{r_e}^2}\right)m_p$$

The final expression can be matched to $F_g = m_p g$, if we let $g = \left(G\frac{m_e}{{r_e}^2}\right)$. Let's first solve for this value using the mass, m_e, and radius, r_e, of the earth, which are equal to $5.98E^{24}\ kg$ and $6.37E^6\ m$ respectively.

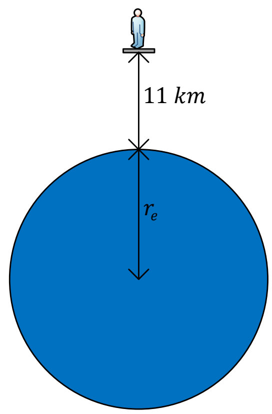
$$g = G\frac{m_e}{{r_e}^2}$$

$$g = 6.67E^{-11}\frac{5.98E^{24}}{(6.37E^6)^2} = 9.829878576$$

Which is approximately equal to 9.8, i.e., the value we have been using throughout our lessons! However, we now realize that this value is only applicable for objects that are *located on the surface of the earth*. In this, case we measure out weight to be

$$F_g = (g)m_p = 9.8 \cdot 80 = 784\ N$$

Solution 3b: Typical height of a commercial flight: $11\ km$ above the surface of the earth.



In this case, the force of gravity is given as

$$F_g = G\frac{m_e m_p}{(r_e + h)^2} = \left(G\frac{m_e}{(r_e + h)^2}\right)m_p$$

Where, $h = 11{,}000\ m$.

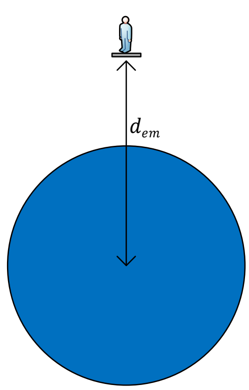
$$F_g = \left(6.67E^{-11}\frac{5.98E^{24}}{(6.37E^6 + 11E^3)^2}\right)\cdot 80$$

$$F_g = (9.79601696967)\cdot 80$$

$$F_g = 783.7\ N$$

As you can see, g is very close to what it was on the surface of the earth, as is our weight.

Solution 3c: Average earth-moon distance: $385{,}000\ km$ from the center of the earth.



$$F_g = G\frac{m_e m_p}{(d_{em})^2}$$

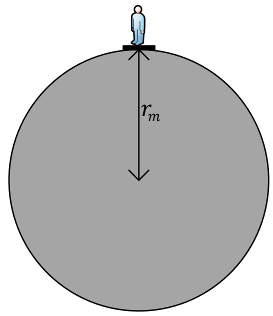
$$F_g = \left(6.67E^{-11}\frac{5.98E^{24}}{(3.85E^8)^2}\right)80$$

$$F_g = (0.0026909496)\cdot 80$$

$$F_g = 0.215\ N$$

In this case, $g = 0.0026909496$, and therefore the earth exerts very little gravitational force on you, making you feel “weightless”.

Solution 3d: Standing on the surface of the moon



The equation for the gravitational force, of course, remains the same.

$$F_g = G\frac{m_m m_p}{(r_m)^2} = \left(G\frac{m_m}{(r_m)^2}\right)m_p$$

However, the mass and radius are now with respect to the moon and not the earth. The mass, m_m, and radius, r_m, of the moon are equal to $7.35E^{22}\ kg$ and $1.737E^{6}\ m$ respectively.

$$F_g = \left(6.67E^{-11}\frac{7.35E^{22}}{(1.737E^{6})^2}\right)\cdot 80$$
$$F_g = (1.624850987)\cdot 80$$
$$F_g = 130\ N$$

Note, the value of g on the surface of the moon is quite different from earth!

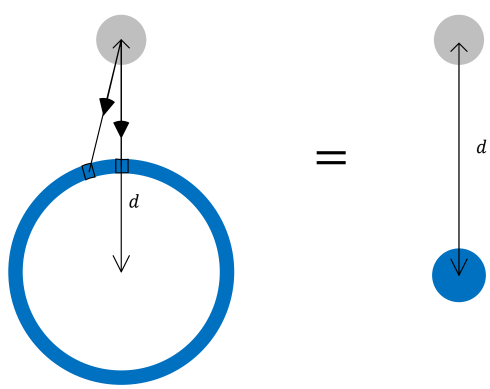
In summary, we see that the value of g we've been using, i.e., $g = 9.8$, is applicable on for objects that are on the surface of the earth. The table below shows our weight, converted to pounds, (1 pound is approximately 4.45 Newtons), for the four cases above.

Location	**Weight, [lbs.]**
On surface of the earth	176.2
11 km above the surface of the earth	176.1
385,000 km from the center of the earth	0.048 (weightless)
On the surface of the moon	29.2

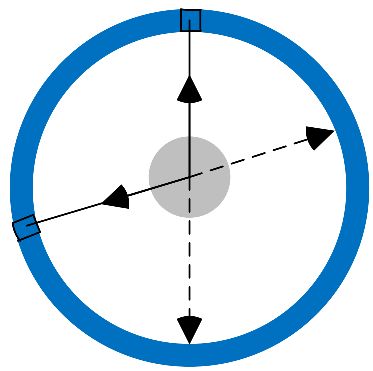
Newton's Shell Theorem: The shell theorem applies to spherical shells (hollow balls) of uniform density and can be used to simplify various gravitational problems. We'll state the theorem without proof, although we will provide some intuition. More importantly we will show how this theorem can be used to solve for the gravitational force for a particle located inside a *solid* sphere. There are two statements associated with the theorem.

Newton's Shell Theorem
1. A uniform shell of matter attracts *particles located outside the shell* as if all the matter were concentrated at the center of the shell. 2. A uniform shell of matter exerts no net force on a *particle located inside the shell*, regardless of its location.

The first statement of the theorem can help us convert a seemingly difficult problem, shown below on the left to the much simpler problem, shown on the right. With the configuration on the left one way to compute the force on the small particle from the shell would be to use the methods of calculus and divide the shell into infinitesimal parts and sum the contributions from each part using integration. On the right we consider the total mass of the shell to be located at its geometrical center and use the universal law of gravitation directly. We can attempt to justify this simplification intuitively as follows. When using the figure on the right-hand side we could imagine that we would *overestimate* the force for the portions of the shell located further form the center and *underestimate* the force for the portions of the shell located closer to the center, thereby averaging to the correct answer. As it turns out the combined "errors" cancel completely, and we get the correct force using this much simpler approach!

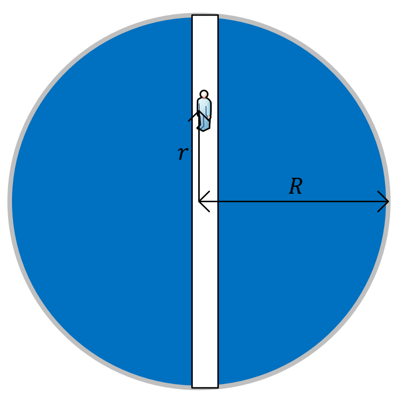


The second statement of the theorem tells us that the force on a particle located inside a shell will be exactly zero. Using this theorem allows us to not have to perform any computation at all! To get an intuitive feel for why this is true we can make a similar argument as above. In this case, for each force vector from one part of the shell there seems to be another that points in the completely opposite direction, thereby cancelling the effects of the first. As it turns out this is precisely the case, and if we add all force vectors on a particle located anywhere inside the shell, the sum will always be zero.



Let's apply this theorem to determine the gravitational force for an object inside a **solid** sphere.

Example 4: Imagine a person traveling through a hypothetical tunnel made through the center of the earth as shown below. If we approximate the earth as a solid sphere, find the gravitational force on a person as a function of their distance from the center, r.



Solution 4: To use the shell theorem, we can imagine the earth is built up from many spherical shells within each other. With this we can state the following:

The force due to the mass of the shells located inside of your current location can be computed by assuming their mass is located at the center, whereas the mass from the shells located outside of your current location will provide a zero net force. Therefore, we can write the gravitational force as:

$$F_g = F_{g,inside} + F_{g,outside}$$

$$F_g = G\frac{M_{inside}m}{r^2} + 0$$

Since the mass is uniformly distributed, we can construct a proportion to solve for M_{inside}.

$$\frac{M_{inside}}{M_{total}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$M_{inside} = M_{total}\frac{r^3}{R^3}$$

Substituting this expression into the one for F_g from above we have

$$F_g = G\frac{\left(\frac{M_{total}r^3}{R^3}\right)m}{r^2}$$

$$F_g = \left(\frac{GM_{total}m}{R^3}\right)r$$

Which shows that the gravitational force varies linearly with respect to your current distance from the center of the earth, r. To obtain some intuition for the expression we can use two extreme locations, $r = R$, and $r = 0$. In the first case we would expect the equation to match the expression for a person standing on the surface of the earth, and for the second case we would expect the force to be zero.

$r = R$

$$F_g = \left(\frac{GM_{total}m}{R^3}\right)R$$

$$F_g = \left(\frac{GM_{total}m}{R^2}\right)$$

$r = 0$

$$F_g = \left(\frac{GM_{total}m}{R^3}\right)\cdot 0$$

$$F_g = 0$$

Gravitational Potential Energy and Escape Velocity: In one of our previous lessons, we derived the following expression for the gravitational potential energy.

$$U(y) = mgy$$

Where, y is the height above the surface of the earth and $U(0) \stackrel{\text{def}}{=} 0$.

Recall however that the derivation assumed that the force of gravity was constant, $F_g = mg$. We now have a much more precise definition for the force of gravity as

$$F_g = G\frac{Mm}{r^2}$$

Furthermore, recall that in the previous derivation we chose the surface of the earth as the zero potential energy position, i.e., $U(0) \stackrel{\text{def}}{=} 0$. Using our new understanding of the gravitational force it makes more sense to choose the zero potential energy position to be when the two objects are infinitely separated from each other. With these changes in mind, we'll start our derivation by finding the work done by gravity to move an object from an infinite distance to a distance R from the center of a large body (e.g., the earth).

$$W = \int_{\infty}^{R} \boldsymbol{F_g}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\infty}^{R} F_g(r)dr$$

Where, we removed the dot product since $\boldsymbol{F_g}$ is in the same direction as $\boldsymbol{r}$. Substituting for $F_g(r)$ and integrating we have

$$W = \int_{\infty}^{R} G\frac{Mm}{r^2}dr$$

$$W = GMm\int_{\infty}^{R} \frac{1}{r^2}dr$$

$$W = GMm\left(\frac{-1}{R} - \frac{-1}{\infty}\right)$$

$$W = \frac{-GMm}{R}$$

Next, since the force of gravity is a conservative force, we can relate the work done to the potential energy as follows.

$$\Delta U = -W$$

$$U(\infty) - U(R) = -\left(-\frac{GMm}{R}\right)$$

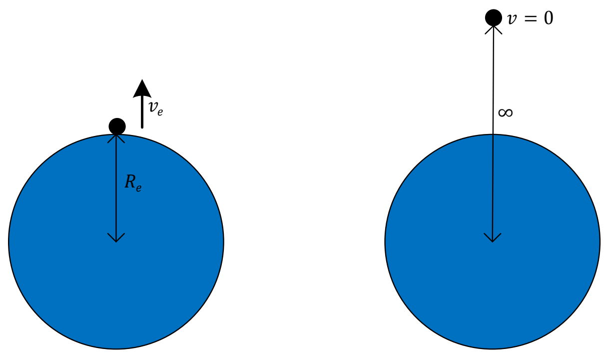
$$0 - U(R) = \frac{GMm}{R}$$

$$U(R) = -\frac{GMm}{R}$$

This gives us a general formula for the gravitational potential energy of an object of mass m, located a distance R from the center of another object of mass M.

Example 5 - Escape Velocity: If we launch an object vertically from the surface of the earth it will eventually reach a speed of zero and then begin falling again due to the force of gravity. The greater the initial launch speed, (kinetic energy), the further the object will travel before it momentarily stops and begins to fall. We can imagine that if we gave the object a large enough initial speed, that it would continue to move away from the earth forever. We refer to this initial speed as the *escape velocity*. Find the escape velocity for a 5000 kg mass object that is launched from the surface of the earth.

Solution 5: We can solve this problem using energy techniques. The figure below shows an initial time, *when the object is first launched from the surface of the earth*, and a final time, *when the object is at a distance of infinity and has a speed of 0*.



Initial Time *Final Time*

We can now use the conservation of energy principle to find the escape velocity, v_e.

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m {v_e}^2 + U(R) = 0 + U(\infty)$$

$$\frac{1}{2} m {v_e}^2 + \frac{-GmM_E}{R_E} = 0 + 0$$

$$v_e = \sqrt{\frac{2GM_E}{R_E}}$$

Of course, this equation can be used for any planet by replacing the mass and radius. Substituting the values for earth we find

$$v_e = \sqrt{\frac{2 \cdot 6.67E^{-11} \cdot 5.98E^{24}}{6.37E^6}}$$

$$v_e = 11190.7 \frac{\cancel{m}}{\cancel{s}} \cdot \frac{2.24\, mph}{1 \frac{\cancel{m}}{\cancel{s}}} \approx 25{,}000\, mph\,!$$

Satellites: Objects that revolve around another, usually larger, object can be referred to as satellites. The planets in our solar system, which are revolving around the sun, are satellites. The moon, which revolves around the earth, can also be called a satellite. We have also developed technology to the point where we can launch our own man-made, sometimes referred to as, "artificial satellites". These satellites are used for such things as weather monitoring and communications. We'll soon see from Kepler's Laws, that a satellites orbital paths are elliptical. However, we can approximate the paths as circular to simplify the analysis.

An orbiting satellite has both potential energy and kinetic energy. By assuming a circular orbit, we fix the distance between the satellite and the objects it orbits. Using the results from above, the gravitational potential energy of a satellite of mass m that revolves around an object of mass M at a constant distance of R is given as

$$U = -\frac{GMm}{R}$$

Furthermore, if we assume the satellite is undergoing uniform circular motion, we can also determine its kinetic energy. The centripetal force keeping the satellite in uniform circular motion is the gravitational force. With this, we can use Newton's 2nd law to find the speed of a satellite in circular orbit.

$$F_g = ma_c$$

$$\frac{GMm}{R^2} = m\frac{v^2}{R}$$

$$v = \sqrt{\frac{GM}{R}}$$

The kinetic energy is then given as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{GMm}{R}\right)$$

Finally, we can find the total energy associated with an object of mass m revolving in a circular orbit around a second object of mass M at a distance of R as shown below.

$$E = K + U$$

$$E = \frac{1}{2}\left(\frac{GMm}{R}\right) + \left(-\frac{GMm}{R}\right)$$

$$E = -\frac{GMm}{2R}$$

Example 6: Compare the escape velocity of the Moon, the Earth, and Jupiter.

Solution 6: We derived the equation for the escape velocity above as

$$v_e = \sqrt{\frac{2GM}{R}}$$

Where, M and R are the mass and radius of the planet for which we are "escaping". We also computed the escape velocity for the earth above as approximately $25{,}000\ mph$. Using the known the mass and radius of the Moon and Jupiter the escape velocities for these objects are computed below.

Moon	***Jupiter***
$v_e = \sqrt{\frac{2GM_M}{R_M}}$ $v_e = \sqrt{\frac{2 \cdot 6.67E^{-11} \cdot 0.07346E^{24}}{1.736E^6}}$ $v_e = 2376 \cdot \left(\frac{2.24}{1}\right) \approx 5{,}322\ mph$	$v_e = \sqrt{\frac{2GM_J}{R_J}}$ $v_e = \sqrt{\frac{2 \cdot 6.67E^{-11} \cdot 1898.19E^{24}}{66.854E^6}}$ $v_e = 61544 \cdot \left(\frac{2.24}{1}\right) \approx 137{,}860\ mph$

Example 7: Suppose a $10\ kg$ object is launched vertically from a planet with a mass of $4.6E^{23}\ kg$ and a radius of $3E^6\ m$ with an initial speed of $3E^3\ m/s$.

a. What would the speed of the object be when it is $4E^6\ m$ from the center of the planet.
b. If the desire were to have the object reach a maximum distance of twice this distance, what should the launch speed be?

Solution 7a: To find the speed of the object when the object is $4E^6\ m$ from the center of the planet, we can use the conservation of energy principle.

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 + \left(-\frac{GMm}{R_f}\right) = \frac{1}{2}mv_i^2 + \left(-\frac{GMm}{R_i}\right)$$

$$v_f = \sqrt{v_i^2 + 2GM\left(\frac{1}{R_f} - \frac{1}{R_i}\right)}$$

$$v_f = \sqrt{(3E^3)^2 + 2 \cdot 6.67E^{-11} \cdot 4.6E^{23}\left(\frac{1}{4E^6} - \frac{1}{3E^6}\right)}$$

$$v_f \cong 2E^6\ m/s$$

b. In this case, we would like to find the launch speed for the object to reach a distance of $8E^6\ m$. The conservation of energy can be used with a final velocity of $0\ m/s$.

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_i{}^2 + \left(-\frac{GMm}{R_i}\right) = 0 + \left(-\frac{GMm}{R_f}\right)$$

$$v_i = \sqrt{2GM\left(\frac{1}{R_i} - \frac{1}{R_f}\right)}$$

$$v_i = \sqrt{2 \cdot 6.67E^{-11} \cdot 4.6E^{23}\left(\frac{1}{3E^6} - \frac{1}{8E^6}\right)}$$

$$v_i = 3.6E^6\ m/s$$

Example 8: Geostationary satellites have orbits located above the equator at a distance of approximately $42{,}000\ km$ from the center of the earth. This distance is determined based on the desire to have the satellite remain over a fixed location on the earth. What is the orbital speed of a geostationary satellite? What is the energy required to place a $100\ kg$ geostationary satellite into its orbit?

Solution 8: If we assume the satellite to be in uniform circular motion, with the earths gravitational force providing the centripetal force, we can find the orbital speed with Newton's 2nd law as shown below.

$$F_g = ma_c$$

$$\frac{GMm}{d^2} = m\frac{v^2}{d}$$

$$v = \sqrt{\frac{GM}{d}}$$

$$v = \sqrt{\frac{6.67E^{-11} \cdot 5.98E^{24}}{4.2E^7}}$$

$$v = 3081.7\ m/s \cong 7000\ mph$$

The energy required to launch the satellite can be determined by looking at the change in energy of the satellite from launch time to when it is in orbit. Note that before launch the satellite has a speed that coincides with the rotational speed of the earth, v_{rot}, given as follows.

$$v_E = \frac{\Delta D}{\Delta t} = \frac{2\pi \cdot 6.37E^6}{24 \cdot 3600} = 463 \ m/s$$

The total energy of the satellite in orbit is then given by

$$E_f = K_f + U_f$$
$$E_f = \frac{1}{2} m v_{\mathrm{orb}}^2 + \left(-\frac{GMm}{d}\right)$$

And the total energy of the satellite before launch is given by

$$E_i = K_i + U_i$$
$$E_i = \frac{1}{2} m v_{rot}^2 + \left(-\frac{GMm}{R_E}\right)$$

The change in energy can then be written as follows.

$$\Delta E = E_f - E_i$$
$$\Delta E = \frac{1}{2} m v_{orb}^2 - \frac{GMm}{d} - \frac{1}{2} m v_{rot}^2 + \frac{GMm}{R_E}$$
$$\Delta E = \frac{1}{2} m (v_{orb}^2 - v_{rot}^2) + GMm\left(\frac{1}{R_E} + \frac{1}{d}\right)$$

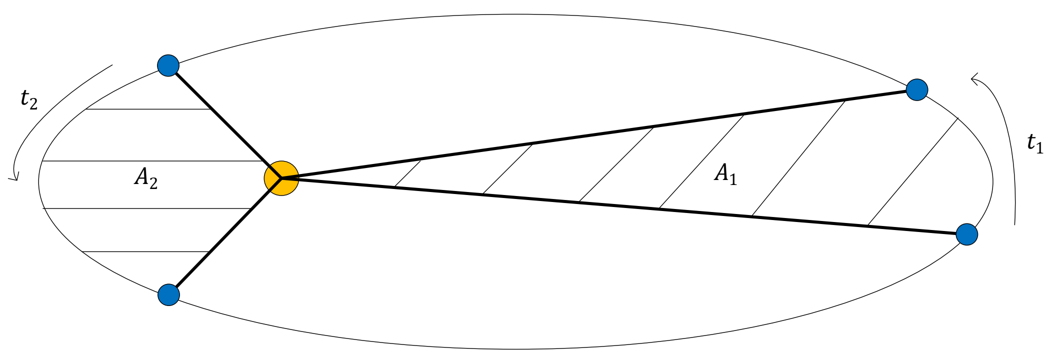
Substituting, we find the energy required as

$$\Delta E = \frac{1}{2} \cdot 100(3081.7^2 - 463^2) + 6.67E^{-11} \cdot 5.98E^{24} \cdot 100\left(\frac{1}{6.37E^6} - \frac{1}{4.2E^7}\right)$$
$$\Delta E \cong 5.8E^9 J$$

Kepler's Laws: Tyco Brahe was a great astronomer who complied a tremendous amount of data regarding the motion of the planets in our solar system. Johannes Kepler used this data to derive three *empirical* laws for planetary motion. As the laws were directly obtained from the data, without any clear theoretical understanding of why they were true, they were assumed to apply to our solar system only. Newton later showed that by using his laws of motion, as well as his law of universal gravitation, one can find these same three laws. In fact, Newton used this as a verification of his theories. Newton's theoretical explanation allowed the extension of Kepler's three laws so that they could then be applied to the entire universe. The three laws, as stated by Kepler, are provided below.

Kepler's First Law: Law of Orbits
All planets move in elliptical orbits, with the sun at one of the foci.

Kepler's Second Law: Law of Areas
All planets sweep equal amounts of area (shown in the figure below) in equal amounts of time.



With respect to the figure the second law says that if $t_1 = t_2$, then $A_1 = A_2$. The figure also highlights the fact that planets travel faster when they are closer to the sun.

Kepler's Third Law: Law of Periods
For any two planets the ratio of the periods squared is equal to the ratio of the semimajor axis cubed. $$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^3$$ Note: The semimajor axis is the planets average distance from the sun.

Rearranging the ratio above we find the following ratio.

$$\frac{{T_1}^2}{{S_1}^3} = \frac{{T_2}^2}{{S_2}^3}$$

What this says is that the ratio of the period squared divided by the semimajor axis cubed is a *constant* for all planets. Using Newton's law of universal gravitation, we can show this to be true for circular orbits, where $S_x = R_x$.

First, we note that the distance traveled by a planet in a circular orbit of radius R is given by

$$D = 2\pi R$$

Next, we can use the equation derived above for the orbital speed of satellites.

$$v = \sqrt{\frac{GM}{R}}$$

The period is given by the distance traveled divided by the speed.

$$T = \frac{D}{v} = \left(\frac{2\pi R}{\sqrt{\frac{GM}{R}}}\right)$$

Finally, if we square the period and rearrange, we can solve for T^2/R^3.

$$T^2 = \left(\frac{4\pi^2 R^2}{\frac{GM}{R}}\right)$$

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

$$\frac{T^2}{R^3} = \frac{4\pi^2}{GM}$$

Where, M is the mass of the object for which the planet if orbiting, in our case the sun. Although we used a circular orbit this expression also holds for elliptical orbits with $R = S$.

Example 9: Determine the mass of the Earth using the known period and distance of the Moon.

Solution 9: Kepler's 3rd law states that the ratio of the period squared to the radius cubed of an object orbiting another is a constant. This constant, however, depends on the mass of the center object. In this case, the Moon is orbiting the Earth so we have

$$\frac{{T_m}^2}{{R_m}^3} = \frac{4\pi^2}{GM_E}$$

Solving this equation for M_E we have

$$M_E = \frac{{R_m}^3 4\pi^2}{G{T_m}^2}$$

The period of the Moon is $27.322\ days$. The average Earth Moon distance is $3.844E^8\ m$. Converting the period to seconds and substituting we find

$$M_E = \frac{(3.844E^8)^3 4\pi^2}{6.67E^{-11} \cdot (27.322 \cdot 86400)^2} \cong 6E^{24}\ kg$$

Which is very close to the accepted value of $5.98E^{24}\ kg$!

Example 10: The asteroid belt between Mars and Jupiter consists of many fragments, once hypothesized to be a planet. If the center of mass of the asteroid belt is about 3 times farther from the sun than the earth is, how long would it take a hypothetical planet to orbit the Sun?

Solution 10: We can relate the period and orbital distance of the earth to the period and orbital distance of the hypothetical planet as follows.

$$\frac{{T_x}^2}{{R_x}^3} = \frac{{T_E}^2}{{R_E}^3}$$

Using the fact that $R_x = 3R_E$ and letting $T_E = 1\ yr$ we can solve for the period, T_x, in earth years.

$$T_x = \sqrt{\frac{{T_E}^2}{{R_E}^3}(3R_E)^3}$$

$$T_x = \sqrt{27(1)^2}$$

$$T_x \cong\ 5.2\ years$$

Final Summary for Gravitation

Newton's Universal Law of Gravitation
Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them. The force is vector that acts along the line connecting the center of mass of the two objects. $$\boldsymbol{F}_g = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}$$ Where, G is the universal gravitational constant: $G = 6.67\ E^{-11}$.
Principle of Superposition for the Force of Gravity
For a collection of N objects, the force on the m^{th} object is computed as follows: $$\boldsymbol{F}_{m,net} = \sum_{\substack{n=1 \\ n\neq m}}^{N} \boldsymbol{F}_{m,n}$$ E.g.: $N = 4, m = 1$ $$\boldsymbol{F}_{1,net} = \boldsymbol{F}_{1,2} + \boldsymbol{F}_{1,3} + \boldsymbol{F}_{1,4}$$
Newton's Shell Theorem
1. A uniform shell of matter attracts *particles located outside the shell* as if all the matter were concentrated at the center of the shell.   2. A uniform shell of matter exerts no net force on a *particle located inside the shell*, regardless of its location.   Consequently, the force of gravity for an object of mass m located inside a uniform solid sphere at a distance r from its center is given as: $$F_g = \left(\frac{GM_{total}m}{R^3}\right)r$$ Where, M_{total} and R are the mass and radius of the uniform solid sphere.
Gravitational Potential energy
The gravitational potential energy of an object of mass m, located a distance R from the center of another object of mass M is given as: $$U(R) = -\frac{GMm}{R}$$ Where, we take the gravitational potential energy at $R = \infty$ to be zero.

Escape Velocity
The initial velocity required for an object to continue to move away from another object of mass M and radius R forever, theoretically coming to rest at infinity, is given as: $$v_{esc} = \sqrt{\frac{2GM}{R}}$$
Satellite Total Energy
A satellite has both kinetic and potential energy given by: $$K = \frac{GMm}{2R} \qquad U = -\frac{GMm}{R}$$ Therefore, the total energy, $E = K + U$, is given as: $$E = -\frac{GMm}{2R}$$ The speed of satellite in circular orbit is given as: $$v_{orb} = \sqrt{\frac{GM}{R}}$$
Kepler's First Law: Law of Orbits
All planets move in elliptical orbits, with the sun at one of the foci.
Kepler's Second Law: Law of Areas
All planets sweep equal amounts of area in equal amounts of time.
Kepler's Third Law: Law of Periods
For any two planets the ratio of the periods squared is equal to the ratio of the semimajor axis cubed. $$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{S_1}{S_2}\right)^3$$ Note 1: The semimajor axis is the planets average distance from the sun. We can approximate orbits as circular Note 2: We can approximate orbits as circular and let $R = S$. The above leads to the fact that the period squared divided by the semimajor axis, (radius), cubed is a *constant for all planets*, and is equal to: $$\frac{T^2}{S^3} = \frac{4\pi^2}{GM}$$ Where, M is the mass of the object for which the planet if orbiting, in our case the sun.

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