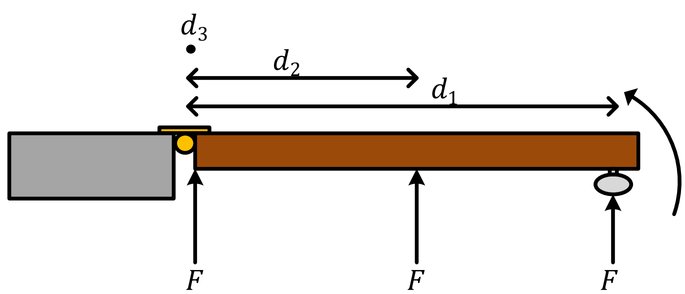
Physics 1 Mechanics - Torque

When studying translational motion, we saw that the net force on a body can cause a linear (or *translational*) acceleration, $\boldsymbol{a}$. Even further, through Newton's 2nd law we found that this acceleration is directly proportional to the net force acting on the body.

$$\boldsymbol{F}_{net} = m\boldsymbol{a}$$

We then studied rotational kinematics, we saw that a body can also undergo *rotational* motion, with an equivalent rotational acceleration, α. The question is: "*Is there a quantity equivalent to the net force from above that is proportional to this rotational acceleration?"*. In the next section, when we study rotational dynamics, we will find that the answer is *yes*, and that we refer to this quantity as *torque,* τ. In other words, a linear force, $\boldsymbol{F}$, is the quantity that accounts for the translational motion of an object, whereas torque, τ, is the quantity that accounts for the rotational motion (twisting) of an object. Before diving into the dynamics of rotational motion, this lesson will formally introduce the concept of torque.

We can introduce torque using the familiar action of a common door, for which the top view of is shown below.



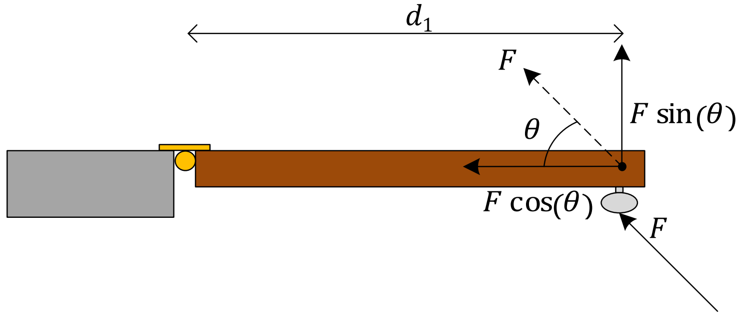
The door is hinged on the left side and will rotate about this hinge point when opening. The figure further shows an equal magnitude force acting on the door at three different distances from the hinge, with $d_3 = 0$. It might be obvious that the three forces will not cause the door to rotate with the same rotational acceleration. The force at distance d_1 will cause the greatest acceleration, while the force at distance d_3 will not cause the door to rotate at all. From this observation, we hypothesize that the quantity which causes the door to rotate depends not only on the magnitude of the force, but also on the distance this force is from the axis of rotation. Assuming the quantity that causes the rotation is directly proportional to both the applied force and the distance from the axis of rotation, we can write the following relationship.

$$Q = Fd_x$$

It is this quantity that we refer to as torque,τ_x, where the subscript x, reminds us that torque must be referenced to an axis of rotation.

$$\tau_x = Fd_x$$

Next, we can ask; *What if our force acted at an angle instead of perpendicular to the desired rotational motion as shown below?*



The force, $\boldsymbol{F}$, is shown acting on the doorknob, but at an angle, θ, with respect to the x-axis. If we proceed to decompose the force vector into its components, we can see that the horizontal component, $F\,cos(\theta)$, will not cause the door to rotate since it is directed towards the hinge. Only the vertical component, $F\,sin(\theta)$, will cause the door to rotate. Therefore, in this case, we should write the torque as follows.

$$\tau_1 = (F\sin(\theta))d_1$$

Which can be written more generally as follows:

$$\tau = F_{\perp}d$$

Where, $F_{\perp}$ represents the component of force that is perpendicular to the line joining the axis of rotation and the point of application of the force.

With this we can provide the following definition of torque.

<u>Torque Definition 1</u>

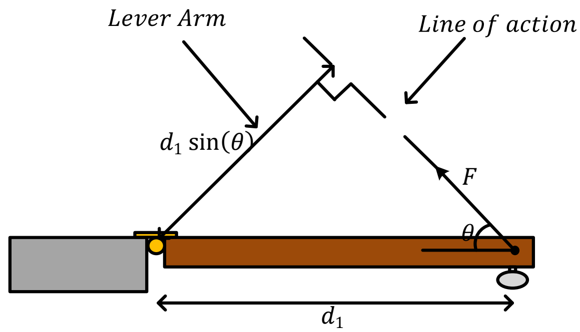
Let $\boldsymbol{d}$ represent a line that connects the axis of rotation to the point of application of a force and let d represents the distance of the line $\boldsymbol{d}$.

The torque can then be defined as:

- The amount of force that acts perpendicular the line, $\boldsymbol{d}$, multiplied by its distance, d.

$$\tau = F_{\perp}d$$

There is a second equivalent way to define torque, which we'll explain using the figure below.



For this definition we define a quantity called the *lever arm*. To find the lever arm we first draw a line extending the force vector, called the *line of action* of the force. We then draw the lever arm starting at the axis of rotation and extending until it perpendicularly meets this line of action. The torque is then computed as the magnitude of the acting force multiplied by the distance of the lever arm.

$$\tau_1 = F(d_1 \sin(\theta))$$

Which is equivalent to the first definition and can be similarly written as shown.

$$\tau = Fd_{\perp}$$

With this we can provide a second definition of torque.

Torque Definition 2

The ***line of action*** of a force is a line that extends along the direction of the force vector.
The ***lever arm*** is a line that starts at the axis of rotation and extends to perpendicularly meet the *line of action.*

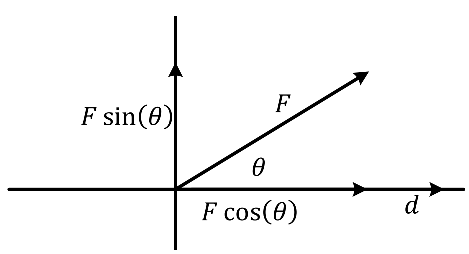
The torque can then be defined as:
- The magnitude of the applied force multiplied by the lever arm.

$$\tau = Fd_{\perp}$$

As it turns out there is yet another equivalent way to define torque. This definition relates to a mathematical operation between two vectors called the *cross-product.* To explain we first recall the dot product, which was also an operation defined between two vectors. The dot product was used when defining work to capture the amount of force directed parallel to the motion of an object.

$$W = \boldsymbol{F} \cdot \boldsymbol{d}$$
$$W = Fd\cos(\theta)$$

Where, θ is the angle between the two vectors.

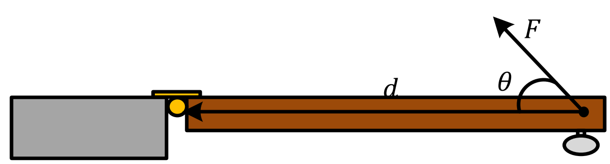


As an example, take a force acting on an object that is being made to move along the x-axis as shown above. The distance moved is represented by the vector, $\boldsymbol{d}$. The work done on that object considers only the component of force acting parallel to the motion, $F\,cos(\theta)$. Therefore, $W = F\,cos(\theta)\,d$, which is equivalent to the dot product as shown above.

We can now use this same logic to define the cross-product, which instead captures the *perpendicular* component of the force. With respect to torque we write the cross-product definition as follows:

$$\boldsymbol{\tau} = \boldsymbol{F} \times \boldsymbol{d}$$
$$|\boldsymbol{\tau}| = Fd\sin(\theta)$$

Where, θ is the angle between the force vector and a vector that represents the distance from point of application of the force to the axis of rotation as shown below. Note that $|\boldsymbol{\tau}|$ represents the magnitude of torque. In other words, the torque is a type of vector. We'll explain this further in the next section.



The cross product can also be thought of in the same fashion as the dot product example from above, where the torque considers only the perpendicular component of the force, $F\,sin(\theta)$.

$$|\boldsymbol{\tau}| = (F\,sin(\theta))d$$

With this we can provide a third definition for the torque.

<u>Torque Definition 3</u>

Let the distance vector, $\boldsymbol{d}$, represents the distance the point of application of the force vector, $\boldsymbol{F}$ to from the axis of rotation.

The torque can then be defined as:

- The cross-product of the force and distance vector.

$$\boldsymbol{\tau} = \boldsymbol{F} \times \boldsymbol{d}$$
$$|\boldsymbol{\tau}| = Fd\sin(\theta)$$

Where, F is the magnitude of the force vector, d, is the magnitude of the distance vector, and θ is the angle between the two vectors.

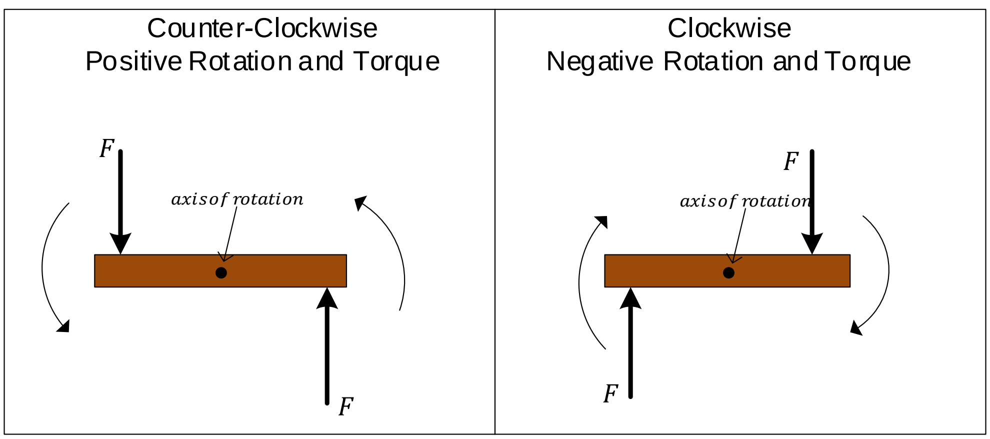
All definitions of torque are equivalent and therefore when solving problems one should choose the simplest form for problem currently being presented.

Direction of the Torque "Vector":

When defining torque through the cross-product we mentioned that torque is a type of vector. We can explain this concept by way of an example shown below.

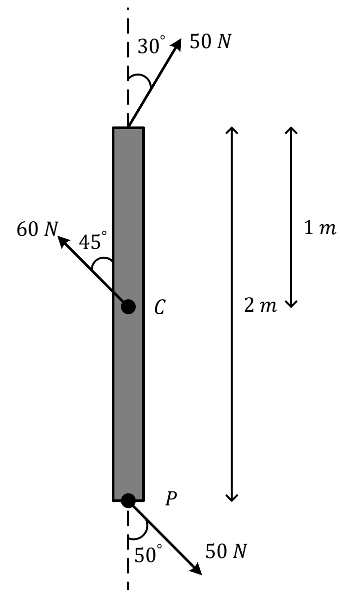
Assume a swinging door that is hinged at the center so that it can be made to swing in both directions, as shown in the figure below. Because of this it seems we need a way to distinguish these two different directions of rotation. As a matter of fact, this concept pertains to all the angular quantities we have so far discussed: angular velocity, ω, angular acceleration, α, and now torque, τ. When a quantity has both a magnitude and direction we generally call this a vector. The angular quantities do not strictly follow all the rules of vectors, but nonetheless we can treat them as a kind of *pseudo vector*. These "vectors" have two directions and the rule we use to keep track of these directions is called the right-hand rule. The right-hand rule can be stated as follows:

- When the fingers of the right hand are curled around the axis of rotation, the thumb points in the direction of the pseudo vector. The figure below shows examples of an object being rotating both counter-clockwise and clockwise. By convention, we let a counter-clockwise rotation refer to a positive vector, pointing out of the page, and clockwise rotation a negative vector, pointing into the page.

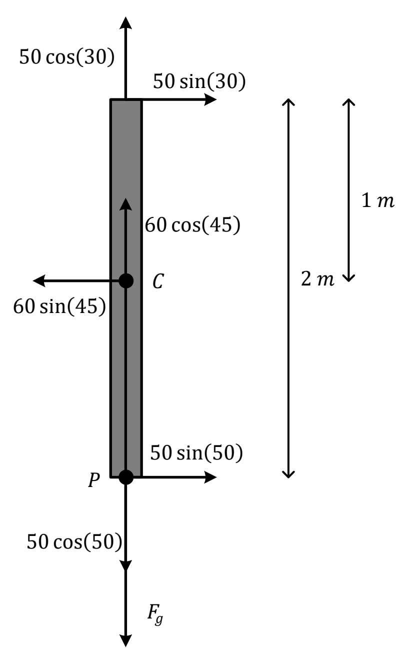


Example 1: A uniform rod that is being acted on by several forces is shown below.

a) Find the net torque on the rod using point C as the axis of rotation.
b) Find the net torque on the rod using point P as the axis of rotation.



Solution 1: We start by drawing a free-body diagram similar to the diagrams we used when we studied translational dynamics. As you will notice below, the main difference for a rotational free-body diagram is that the object is not considered as a point object and so the location of the forces with respect to the axis of rotation needs to be considered. Note that we also added the gravitational force associated with the rod itself.



When examining the free-body diagram we note that any force that acts through the axis of rotation will *not* contribute to the torque. Furthermore, the net torque will be different depending on the location of the axis of rotation.

1a: Axis of Rotation at Point C

The only force that contributes to a positive torque is the $50\ N$ force acting at the lower end of the rod. Conversely, the only force that contributes to a negative torque is the $50\ N$ force acting at the upper end of the rod. The net torque about C is then found as shown below.

$$\tau_{net,C} = 50\,sin(50^{\circ})\,(1) - 50\,sin(30^{\circ})\,(1)$$
$$\tau_{net,C} = 13.3\ N \cdot m$$

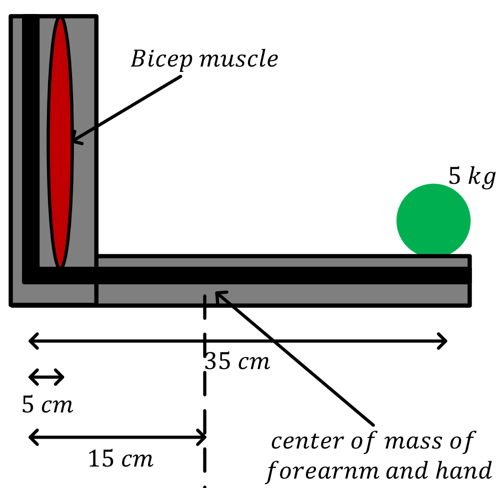
1b: Axis of Rotation at Point P

In this case, the only force that contributes to a positive torque is the $60\ N$ force acting at the center of the rod. The force that contributes to a negative torque is again the $50\ N$ force acting at the upper end of the rod. However, in this case, the distance to the axis point is $2\ m$. The net torque, which in this case is negative, is found as shown below.

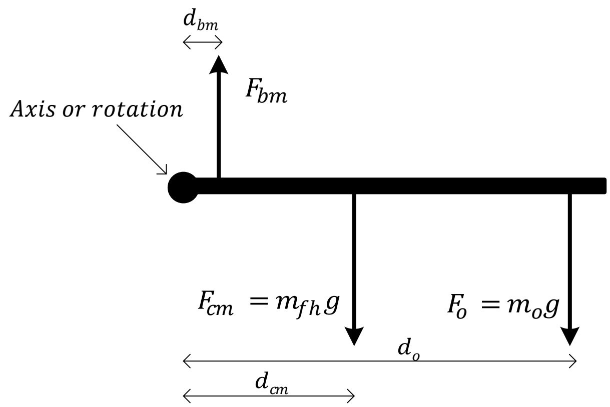
$$\tau_{net,P} = 60\,sin(45^{\circ})\,(1) - 50\,sin(30^{\circ})\,(2)$$
$$\tau_{net,P} = -7.6\ N \cdot m$$

Example 2: The sketch below shows the arm of someone holding a $5\ kg$ object. The sketch shows that the length from the elbow joint to the object in the hand is $35\ cm$, the center of mass of the forearm and the hand is $15\ cm$ from the elbow joint, and the bicep muscle is attached $5\ cm$ from the elbow joint. The mass of the forearm and hand combined is $2\ kg$.

a) Determine the amount of force required by the bicep muscle to hold the $5\ kg$ object steady.
b) How would the required force change if the bicep muscle were attached $6\ cm$ from the elbow joint?



Solution 2: Using the elbow joint as the axis of rotation we start by drawing the free-body diagram as shown below.



The net torque with respect to the given axis of rotation can be written as follows.

$$\tau_{net} = F_{bm}d_{bm} - m_{fh}gd_{cm} - m_o g d_o$$

If the object is held steady the net torque must be zero. Therefore, we can write an expression for the required bicep force as shown below.

$$0 = F_{bm}d_{bm} - m_{fh}gd_{cm} - m_o g d_o$$

$$F_{bm} = \frac{m_{fh}gd_{cm} + m_o g d_o}{d_{bm}}$$

We can now solve for the required force using both $d_{bm} = 5\ cm$ and $d_{bm} = 6\ cm$.

5 cm Bicep Attachment Point	**6 cm Bicep Attachment Point**
$F_{bm} = \frac{2 \cdot 9.8 \cdot 0.15 + 5 \cdot 9.8 \cdot 0.35}{0.05}$ $F_{bm} = 402\ N$	$F_{bm} = \frac{22 \cdot 9.8 \cdot 0.15 + 5 \cdot 9.8 \cdot 0.35}{0.06}$ $F_{bm} = 334\ N$

What we find is that even a slight increase in the bicep attachment point results in much less force required by the muscle. This can give the advantage needed to a competitive athlete. To make the point clearer, let's find the amount of mass an athlete with a 6 cm bicep attachment point could hold if they used the same 402 N bicep force that an athlete with a 5 cm attachment point needed to hold a 5 kg object.

$$0 = F_{bm}d_{bm} - m_{fh}gd_{cm} - m_o g d_o$$

$$m_o = \frac{F_{bm}d_{bm} - m_{fh}gd_{cm}}{gd_o}$$

$$m_o = \frac{402 \cdot 0.06 - 2 \cdot 9.8 \cdot 0.15}{9.8 \cdot 0.35}$$

$$m_o = 6.2\ kg$$

That's a 124% increase in mass for only a 1 cm difference in the bicep attachment point!

Final Summary for Torque

Torque
Torque can be generally defined as a quantity that measures the amount of force that accounts for the rotational motion (twisting) of an object.
Mathematically torque can be defined in three different ways. Each definition is equivalent and the one used should be the chosen based on the problem being solved.
Torque Definition 1 Let $\boldsymbol{d}$ represent a line that connects the axis of rotation to the point of application of a force and let d represents the distance of the line $\boldsymbol{d}$. The torque can then be defined as: • The amount of force that acts perpendicular the line, $\boldsymbol{d}$, multiplied by its distance, d. $\tau = F_{\perp} d$
Torque Definition 2 The ***line of action*** of a force is a line that extends along the direction of the force vector. The ***lever arm*** is a line that starts at the axis of rotation and extends to perpendicularly meet the *line of action.* The torque can then be defined as: • The magnitude of the applied force multiplied by the lever arm. $\tau = F d_{\perp}$
Torque Definition 3 Let the distance vector, $\boldsymbol{d}$, represents the distance the point of application of the force vector, $\boldsymbol{F}$ to from the axis of rotation. The torque can then be defined as: • The cross-product of the force and distance vector. $\boldsymbol{\tau} = \boldsymbol{F} \times \boldsymbol{d}$ $\lvert\boldsymbol{\tau}\rvert = F d \sin(\theta)$ Where, F is the magnitude of the force vector, d, is the magnitude of the distance vector, and θ is the angle between the two vectors.

By: ferrantetutoring