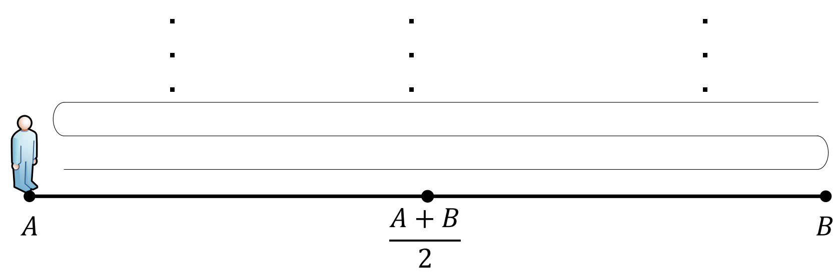
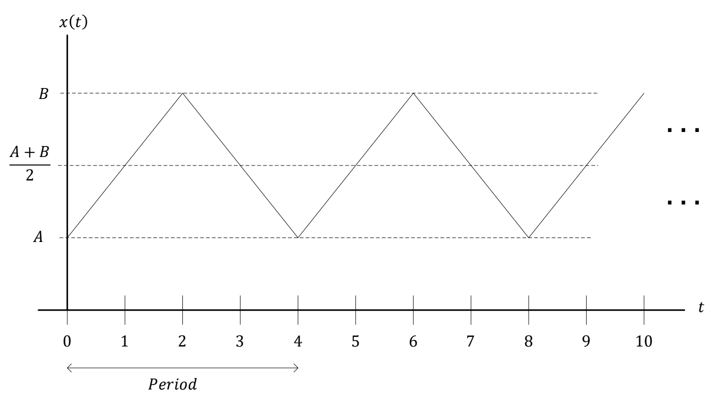
Physics 1 Mechanics - Simple Harmonic Motion

Periodic motion, (also called hormonic motion), is motion that repeats itself over regular intervals. Take the example shown below of a person running back and forth at a constant speed between points A and B. We can plot the position of this person as a function of time, $x(t)$, which is shown below. The time it takes for a periodic function to repeat is called the period, denoted by T. This time interval can be referred to as a *cycle*, giving T units of $sec/cycle$. In the example below, we have $T = 4\ sec/cycle$. The number of cycles per second is referred to as the frequency, denoted by f, and it is related to the period as

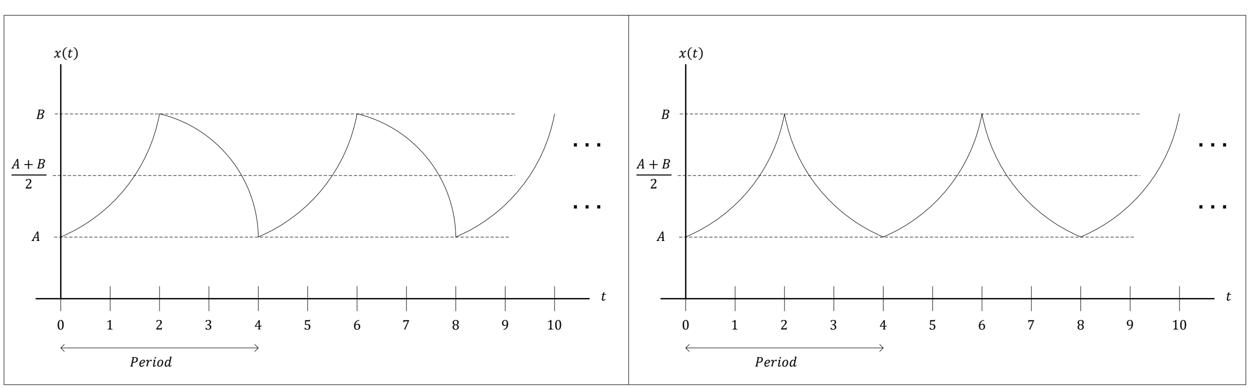
$$f = \frac{1}{T}$$

Which has units of $cycles/sec\ \ or\ Hertz\ (Hz)$.





This example gave one particular type of periodic motion, i.e., where the person maintains a constant speed. Of course, we can imagine all types of periodic motion, some of which are shown below.



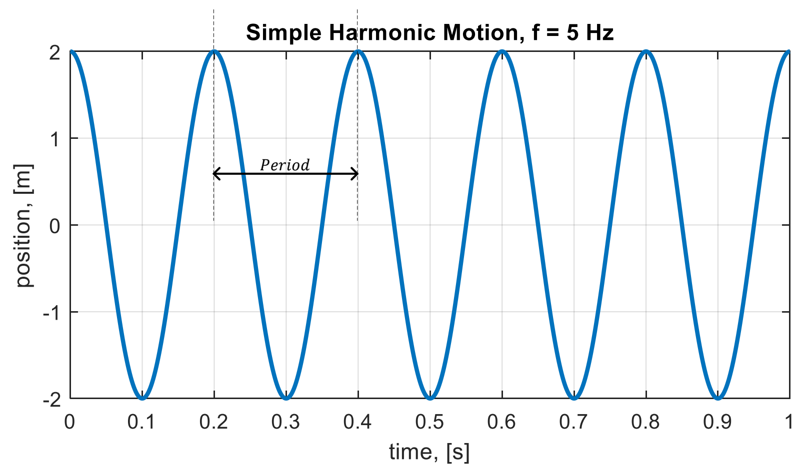
A special case of harmonic motion, which applies to many physical systems, is one in which the motion is *sinusoidal*. That is, the position follows a sine or cosine wave. We call this type of motion "*Simple Harmonic Motion*". The figure below shows a cosine function where x is the position with $[-2 \ \leq x \ \leq 2]$, the period is $0.2 \ sec/cycle$, and therefore the frequency is $5 \ Hz$. Sinusoids are defined on a unit circle with one cycle corresponding to one revolution around the circle, i.e., $2\pi \ radians$. Therefore, we define a radial frequency, ω, in radians per second as follows.

$$\omega = 2\pi f$$

The cosine function shown in the figure can be represented as

$$x(t) = 2\cos(\omega t)$$

With, $\omega = 2\pi(5) = \ 10\pi$.

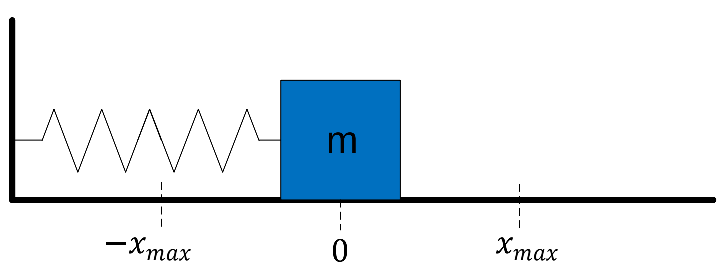


The most general way to represent a sinusoidal function, shown here with cosine, is as follows.

$$x(t) = x_{max}\cos(\omega t + \ \varphi)$$

Where, x_{max} is the maximum amplitude, ω is the radial frequency, and φ is a phase shift that can be used to account for a shifting of the function in time.

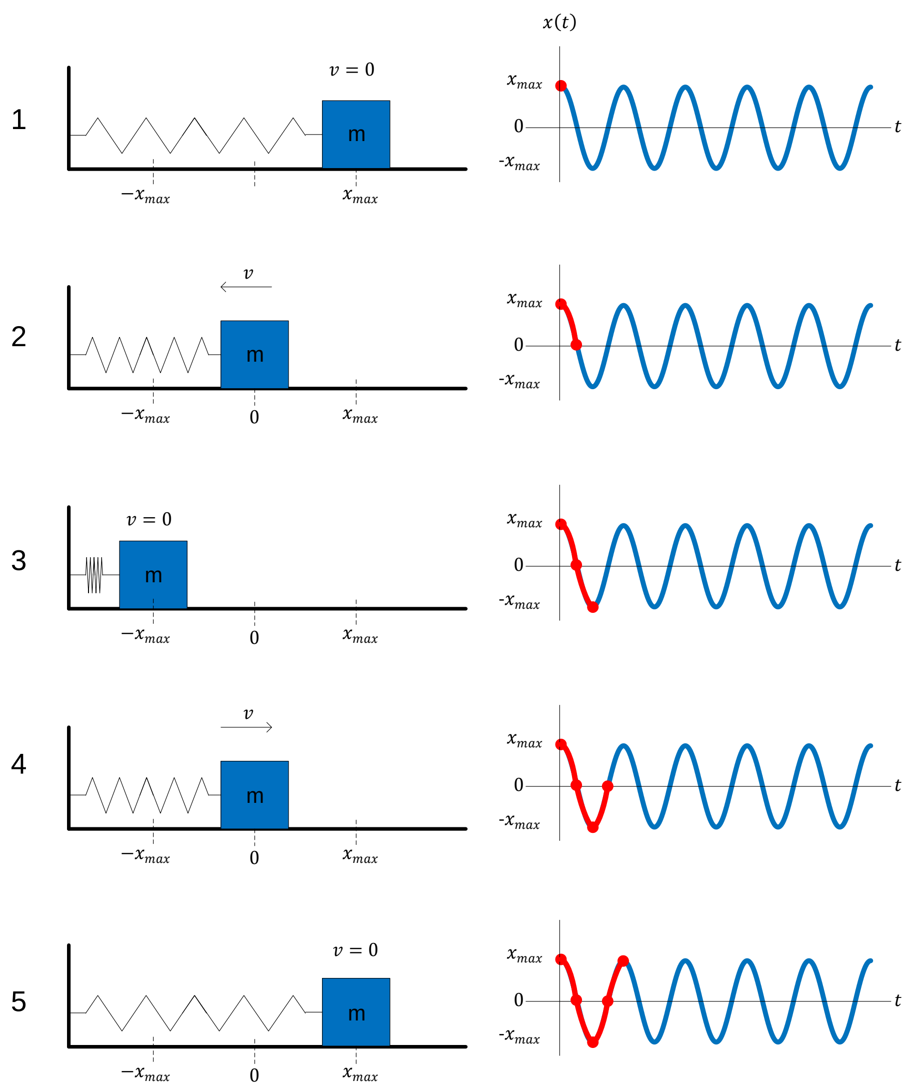
As mentioned, simple harmonic motion applies to many physical systems. The system we'll begin with consists of a mass on a frictionless surface attached to a spring, as shown below. The point along the table where the spring is neither stretched nor compressed is referred to as the equilibrium position, which we mark as the point $x = 0$. Furthermore, we assume the mass oscillates between $-x_{max}$ and $+x_{max}$.



To analyze this system, let's look at various snapshots in time and plot a sinusoidal curve between these points. The figure below shows one complete cycle using snapshots $1-5$. After looking closely at how this system could map to simple harmonic motion, we'll use Newton's law to write down the governing equation and attempt to verify that a sinusoidal function can indeed satisfy the equation. The snapshots are briefly described below.

1. The mass is initially stretched and held at x_{max}, where $v = 0$.
2. The mass is released and it begins to slide to the left towards the equilibrium position.
3. The mass will continue through the equilibrium position and move towards $-x_{max}$, where it will again temporarily attain $v = 0$.
4. After reaching $-x_{max}$ the mass will change direction and move back toward the equilibrium position.
5. The mass will again pass the equilibrium position and move towards $+x_{max}$, where it will again attain $v = 0$.

The five snapshots show one complete cycle of motion for the mass, which we map to a cosine wave. Ignoring all frictional forces, this cycle will repeat, and the motion will continue forever.



Now let's look at the physics for the above mass-spring system. The only force on the mass is the spring force, which from Hook's law, is directly proportional to stretched/compressed distance, but points in the opposite direction. With this we can write Newton's 2nd law with a position and acceleration that are functions of time.

$$ma(t) = \sum F$$
$$ma(t) = -kx(t)$$

Using calculus, we can write the acceleration as the second derivative of position, which results in what is mathematically known as a differential equation.

$$m\frac{d^2x(t)}{dt^2} = -kx(t)$$
$$\boxed{\frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t)}$$

There are various techniques for solving differential equations, one of which can informally be called the "guess and check method". Notice that we need to find a function, $x(t)$, such that when you take the derivative twice you get the same function back multiplied by a constant. Do we know any functions that behave like this? Sinusoidal functions do! Let's guess that a generic cosine function, (sine function works just as well and I suggest you try it for yourself), can satisfy this differential equation. If you are not familiar with calculus you can skip ahead to the solution.

To proceed we substitute $x(t) = x_{max}\cos(\omega t + \varphi)$ into the differential equation and differentiate the left-hand side twice.

$$\frac{d^2}{dt^2}(x_{max}\,cos(\omega t + \varphi)) = -\frac{k}{m}(x_{max}\,cos(\omega t + \varphi))$$
$$\frac{d}{dt}\left(\frac{d}{dt}(x_{max}\,cos(\omega t + \varphi))\right) = -\frac{k}{m}(x_{max}\,cos(\omega t + \varphi))$$
$$\frac{d}{dt}(-\omega x_{max}\,sin(\omega t + \varphi)) = -\frac{k}{m}(x_{max}\,cos(\omega t + \varphi))$$
$$-\omega^2(x_{max}\,cos(\omega t + \varphi)) = -\frac{k}{m}(x_{max}\,cos(\omega t + \varphi))$$

As shown, our function satisfies the original differential equation if we let

$$\omega^2 = \frac{k}{m}$$
$$\omega = \sqrt{\frac{k}{m}}$$

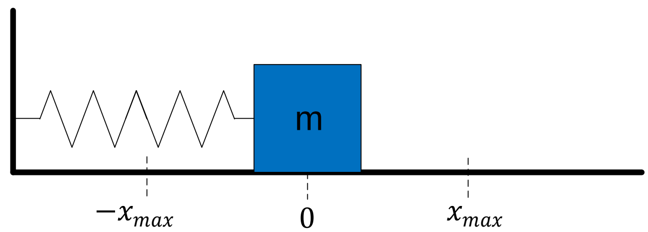
Note that the frequency of oscillation does not depend on the amplitude, but is only a function of the spring constant, k, and the attached mass, m. Furthermore, notice that we still have two undetermined constants, x_{max} and φ. As is the case for all differential equations these constants will depend on the initial conditions, which will become clear when we do example problems in the next section. Also note that once the position function is known, the velocity and acceleration equation follow from simple differentiation. Together we refer to these as the equations of simple harmonic motion.

Simple Harmonic Motion
Any physical system (e.g., A mass-spring system on a frictionless surface), that can be represented by a differential equation of the form. $$\frac{d^2x(t)}{dt^2} = -C\big(x(t)\big)$$ Will results in *simple harmonic motion.* The resulting equations of motion, i.e., position, velocity, and acceleration, are given as follows. $$x(t) = x_{max}\, cos(\omega t + \varphi)$$ $$x(t) = v_{max}\, sin(\omega t + \varphi)$$ $$x(t) = a_{max}\, cos(\omega t + \varphi)$$ Where $$v_{max} = -x_{max}\omega$$ $$a_{max} = v_{max}\omega = -x_{max}\omega^2$$ $$\omega = \sqrt{C}$$ The constants, x_{max} and φ, are found by using initial conditions, e.g., $x(0) = x_0, v(0) = v_0$.

Energy in Simple Harmonic Oscillators: Recall the fundamental conservation principle.

The Principle of the Conservation of Mechanical Energy
In a closed system when only conservative forces cause energy changes, the total mechanical energy remains constant. $$E_{(mec)} = U + K$$ $$U_i + K_i = U_f + K_f$$

We mention this because our spring and mass on a frictionless surface is indeed an example of a closed system where the above principle holds.



Let's look back at the 5 snapshots in time and consider the energy of our system. In snapshot 1 when the mass is held at $+x_{max}$ the only energy is potential energy in the spring. Note that this will also be the case in snapshots 3 and 5. In each of these three cases the spring is either stretched or compressed to its maximum distance, and the velocity of the mass is temporarily zero. For these points in time, we can write the energy of the system as follows.

$$E = U = \frac{1}{2}k{x_{max}}^2$$

Conversely in snapshots 2 and 4 the spring is at its equilibrium position and therefore has no potential energy. Since there is no friction all the potential energy from the spring has been transferred to kinetic energy of the mass. The velocity of the mass at these instants is maximum, and the energy can then be written as:

$$E = K = \frac{1}{2}m{v_{max}}^2$$

As the mass and the spring are in constant motion, the energy is constantly being transferred back and forth between potential energy in the spring and kinetic energy of the mass. More importantly, since the principle of the conservation of mechanical energy applies, we know that at any point in time the total energy of the system, which remains constant, is the sum of the potential energy of the spring and the kinetic energy of the mass.

$$E = U(t) + K(t)$$
$$E = \frac{1}{2}kx^2(t) + \frac{1}{2}mv^2(t)$$

To prove the energy is constant we can substitute $x(t)$ and $v(t) = \frac{d}{dt}\big(x(t)\big)$ in the above expression.

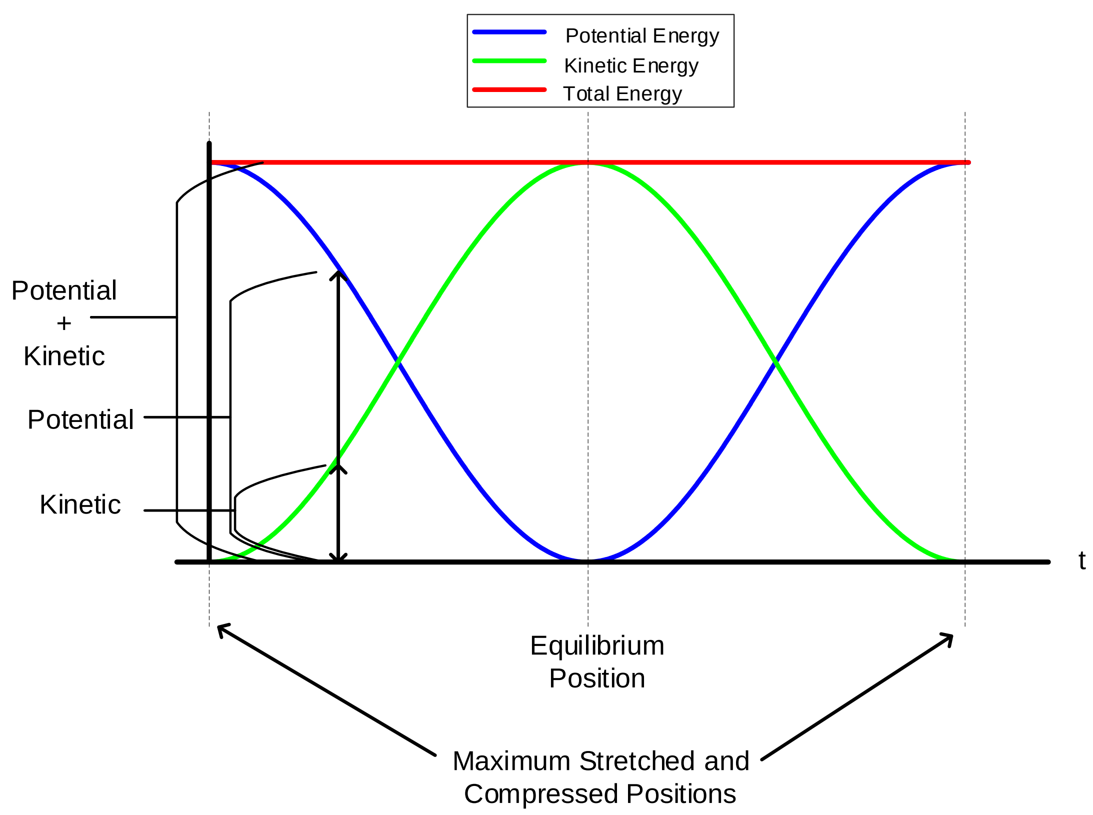
$$E = \frac{1}{2}k(x_{max}\,cos(\omega t + \varphi))^2 + \frac{1}{2}m(\omega x_{max}\,sin(\omega t + \varphi))^2$$
$$E = \frac{1}{2}kx_{max}{}^2\,cos^2(\omega t + \varphi) + \frac{1}{2}m\omega^2 x_{max}{}^2\,sin^2(\omega t + \varphi)$$
$$E = \frac{1}{2}x_{max}{}^2(k\,cos^2(\omega t + \varphi) + m\omega^2\,sin^2(\omega t + \varphi))$$

Next, we substitute $\omega^2 = \frac{k}{m}$ and use the trigonometric identity: $cos^2(A) + sin^2(A) = 1$

$$E = \frac{1}{2}x_{max}{}^2\left(k\,cos^2(\omega t + \varphi) + m\left(\frac{k}{m}\right)sin^2(\omega t + \varphi)\right)$$
$$E = \frac{1}{2}kx_{max}{}^2(cos^2(\omega t + \varphi) + sin^2(\omega t + \varphi))$$
$$E = \frac{1}{2}kx_{max}{}^2$$

Therefore, the total energy of the system is indeed constant.

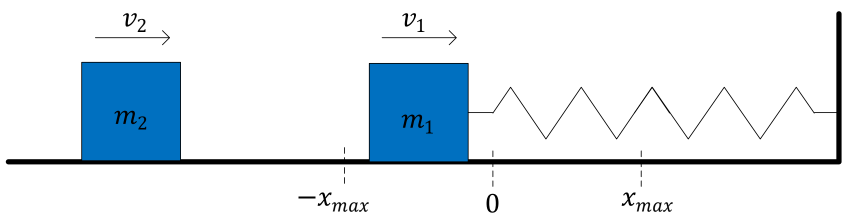
Below is a plot of the potential, kinetic, and total energy for one half of a cycle. The figure points out the equilibrium positions, when the velocity, (i.e., kinetic energy), is maximum, and the positions of maximum stretch and compression of the spring, i.e., when the potential energy is maximum. As expected, the total energy is constant for all time. Even further, we see that the potential and kinetic energy always sum to the total energy at all points in time.



Example 1 - Horizonal Mass and Spring:

Two blocks are in motion on a frictionless surface. The first block has a mass $m_1 = 10\ kg$ and is attached to a spring with spring constant $k = 490\ \ N/m$. The block, which was initially compressed a distance of $0.5\ m$, is experiencing *simple harmonic motion.* The second block with a mass of $m_2 = 5\ kg$ is traveling towards the first block at a speed of $v_2 = 5\ m/s$. At the exact moment when the first block is at its equilibrium position and moving to the right, the two blocks collide. The blocks then stick together, i.e., a completely inelastic collision.

a. Find the position, velocity, and acceleration equations for the first block before the collision.
b. Find the new position, velocity, and acceleration equations after the collision.



Solution 1a – Before Collision:

Before the collision, the first mass is undergoing simple harmonic motion with the following generic position function.

$$x(t) = \ x_{max}\, cos(\omega t + \ \varphi)$$

Next, we can differentiate to find general expressions for the velocity and acceleration.

$$v(t) = \frac{dx(t)}{dt} = -x_{max}\omega\, sin(\omega t + \ \varphi) = -v_{max}\, sin(\omega t + \ \varphi)$$

$$a(t) = \frac{dv(t)}{dt} = -x_{max}\omega^2\, cos(\omega t + \ \varphi) = -a_{max}\, cos(\omega t + \ \varphi)$$

Where, $v_{max} = x_{max}\omega$ and $a_{max} = x_{max}\omega^2$.

Since the block started at the maximum compressed position, i.e., $x_{max} = 0.5$, then $\varphi = 0$. Furthermore, the radial frequency is determined based on the mass and the spring constant.

$$\omega = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{490}{10}} = 7\ \ rad/sec$$

With this, the final expressions are given as follows.

$$x(t) = \ 0.5\, cos(7t)$$

$$v(t) = \ \ -3.5\, sin(7t)$$

$$a(t) = \ -24.5\, cos(7t)$$

1b. Before the collision, the energy in the system, which remains constant, is given as

$$E = \frac{1}{2}k{x_{max}}^2 = \frac{1}{2}490(0.5)^2 = 61.25\,J$$

When the collision occurs, energy is added to the system and so the maximum amplitude should increase. On the other hand, since the mass is increased the radial frequency should decrease. The new radial frequency is computed as shown.

$$\omega = \sqrt{\frac{k}{m_1 + m_2}} = \sqrt{\frac{490}{10+5}} = 5.72\ rad/sec$$

Next, since energy is added to the system as a result of a collision, we can use the conservation of momentum to find to find the new maximum velocity. At the moment of impact, the mass is located at the equilibrium position and therefore it has a velocity of 3.5 m/s. The new maximum velocity is found with the conservation of momentum as shown.

$$p_f = p_i$$

$$\boldsymbol{v_f}(m_1 + m_2) = m_1\boldsymbol{v_{1.i}} + m_2\boldsymbol{v_{2.i}}$$

$$\boldsymbol{v_f} = \frac{m_1\boldsymbol{v_{1,i}} + m_2\boldsymbol{v_{2.i}}}{(m_1 + m_2)}$$

$$\boldsymbol{v_f} = \frac{10 \cdot 3.5 + 5 \cdot 5}{(10+5)}$$

$$\boldsymbol{v_f} = 4\ m/s$$

The collision added energy such that the maximum velocity is now $v_{max} = 4\ m/s$. From the generic velocity and acceleration equations in part a, we have the following relationships.

$$x_{max} = \frac{v_{max}}{\omega}$$

$$x_{max} = \frac{4}{5.72}$$

$$x_{max} = 0.7\ m$$

$$a_{max} = x_{max}\omega^2$$

$$a_{max} = \frac{4}{5.72}(5.72)^2$$

$$a_{max} = 22.9\ m/s^2$$

Finally, we can write the three new motion equations. We will assume the phase is still zero since we can simply re-define the time axis and allow time zero to correspond to the maximum positive position.

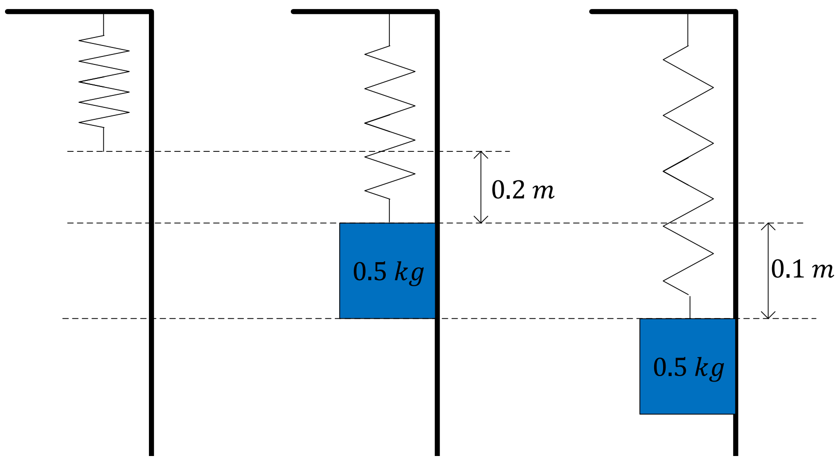
$$x(t) = 0.7\,cos(5.72t)$$

$$v(t) = -4\,sin(5.72t)$$

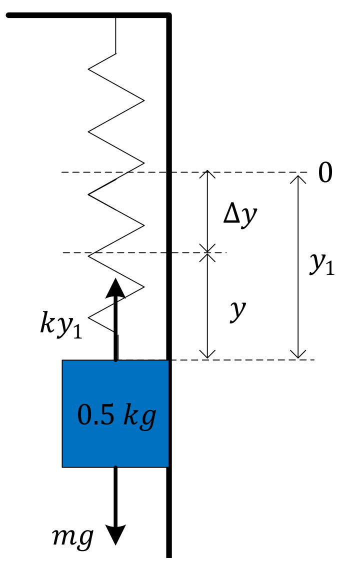
$$a(t) = -22.9\,cos(5.72t)$$

Example 2 – Vertical Mass and Spring:

A vertical spring stretches $0.2\ m$ when a $0.5\ kg$ mass is attached. The spring is then stretched an additional $0.1\ m$ before it is released and goes into motion. Determine whether the motion of this system is simple harmonic motion. If so, find the equations of motion.



Solution 2: Note that once the mass if attached to the spring there is essentially a new equilibrium position for the mass-spring system. We first redraw the system labeling the distance from the original equilibrium position y_1 and the position from the new equilibrium position as y. The forces on the spring, as well as Newton's 2nd law for the mass-spring is also shown.



$$ma(t) = \sum F$$

$$ma(t) = mg - ky_1(t)$$

$$ma(t) = mg - k(\Delta y + y(t))$$

$$ma(t) = mg - k\Delta y - ky(t)$$

At this point the equation doesn't look like it will necessarily result in simple harmonic motion. However, we can again write Newton's 2nd law for the time when the mass was initially hanging at rest.

$$\sum F = 0$$

$$mg - k\Delta y = 0$$

This equation allows us to remove the first two terms on the left-hand side from Newton's 2nd law equation when the mass is in motion.

$$
\begin{aligned}
ma(t) &= -ky(t) \\
m\frac{d}{dx}\big(y(t)\big) &= -ky(t) \\
\frac{d}{dx}\big(y(t)\big) &= -\frac{k}{m}y(t)
\end{aligned}
$$

Which is in now in the same form as the horizontal mass-spring system, and therefore results in the same *simple harmonic motion*. Note the y position is measured with respect to the new equilibrium position. The static equation also allows us to find the spring constant.

$$k = \frac{mg}{\Delta y}$$

The generic motion equations for an object in simple harmonic motion are given as follows.

$$y(t) = y_{max}\, cos(\omega t)$$

$$v(t) = -y_{max}\omega\, sin(\omega t)$$

$$a(t) = -y_{max}\omega^2\, cos(\omega t)$$

For our specific case we have $y_{max} = 0.1$. Furthermore, the radial frequency can be determined as follows.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{\Delta y}\cdot\frac{1}{m}} = \sqrt{g/\Delta y} = \sqrt{9.8/0.2} = 7\ \ rad/sec$$

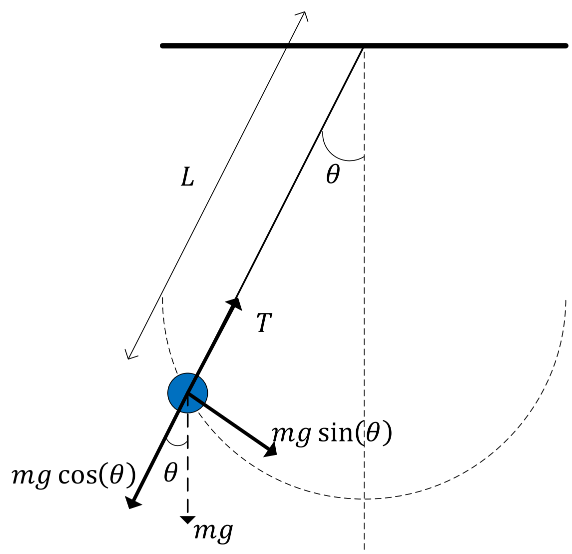
The final equations of motion for the current mass-spring system are given as follows.

$$y(t) = 0.1\, cos(7t)$$

$$v(t) = -0.7\, sin(7t)$$

$$a(t) = -4.9\, cos(7t)$$

Example 3 – Simple Pendulum:
A pendulum consists of an object suspended from a cord that doesn't stretch. If we start by initially lifting the object a certain distance along the radial line, as shown in the figure, and then release it we assume the pendulum will begin the swing back and forth. The pendulum system shown below consists of a mass, $m = 2kg$, hanging from a string of length, $L = 3m$. It is initially lifted such that the string makes an angle, $\theta = 0.25\ rad$, with the vertical. Ignoring air drag, does the pendulum experience simple harmonic motion? If so, find the equations of motion for this system.



Solution 3: Notice that if the cord stays taut the object will oscillate on the circular arc shown. Recall for circular motion it was most convenient to use a coordinate system that aligned with the radial and tangential directions. The position of the object will remain L meters from the point of connection as measured in the radial direction, so it is in the tangential direction that we want to track the position via the angle the object makes with the vertical.

Examining the problem this way clearly shows the similarity to the mass and spring problem in the sense that the $mg\ sin(\theta)$ force acts as the restoring force in the same way the spring force does. We let the center vertical center line correspond to $\theta = 0$, which is the equilibrium position of the mass with respect to the angular position of the object. For this case we will use the rotational form of Newton's law since the restoring force acts to rotate the object. Note that the tension, as well is the other gravity component, act along the radial direction and do not cause any torque.

$$I\alpha = \tau_{net}$$

$$I\frac{d\theta^2(t)}{dt^2} = -mg\ sin(\theta)\ L$$

$$\frac{d\theta^2(t)}{dt^2} = -\left(\frac{mgL}{I}\right) sin(\theta)$$

The differential equation does not represent simple harmonic motion because of the $sin(\theta)$ term on the right-hand side. However, the following approximation holds well for small angles.

$$sin(\theta) \approx \theta, \qquad \text{e.g., } sin(0.25) = 0.2474$$

Using this approximation allows us to approximate the motion as simple harmonic motion.

$$\frac{d\theta^2(t)}{dt^2} = -\frac{mgL}{I}\,\theta(t)$$

In this case, the radial frequency is given as

$$\omega = \sqrt{\frac{mgL}{I}}$$

For a *simple pendulum* we treat the object as a point mass with rotational inertia, $I = mL^2$. The radial frequency can then be written as

$$\omega = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

Which as you can see is independent of the size of the mass!

The generic equations of motion are

$$\theta(t) = \theta_{max}\, cos(\omega t)$$

$$\omega(t) = -\theta_{max}\omega\, sin(\omega t)$$

$$\alpha(t) = -\theta_{max}\omega^2\, cos(\omega t)$$

In this example, $\theta_{max} = 0.25$, and

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{3}} = 1.8\ rad/sec$$

Finally, letting $\varphi = 0$, we can write the motion equations as follows.

$$\theta(t) = 0.25\, cos(1.8t)$$

$$\omega(t) = -0.45\, sin(1.8t)$$

$$\alpha(t) = -0.81\, cos(1.8t)$$

Final Summary for Simple Harmonic Motion

Periodic (Harmonic) Motion:

Motion that repeats itself over regular intervals.

Simple Harmonic Motion:

Periodic/Harmonic motion that is sinusoidal, i.e., follows a sine or cosine wave.

Definitions

Cycle: The unit used to represent one complete repeating interval.

Period, T: The length of time contained in one cycle, usually measured in $sec/cycle$.

Frequency, $f = \frac{1}{T}$: The number of cycles contained in one unit of time, usually measured in $cycles/sec = Hertz$.

Radial Frequency, $\omega = 2\pi f$: The number of radians contained in one unit of time, usually measured in $radians/sec$.

Dynamic of Simple Harmonic Motion

Any physical system (e.g., A mass-spring system on a frictionless surface), that can be represented by a differential equation of the form.

$$\frac{d^2x(t)}{dt^2} = -C\big(x(t)\big)$$

Will results in *simple harmonic motion.* The resulting equations of motion, i.e., position, velocity, and acceleration, are given as follows.

$$x(t) = x_{max}\, cos(\omega t + \ \varphi)$$

$$x(t) = v_{max}\, sin(\omega t + \ \varphi)$$

$$x(t) = a_{max}\, cos(\omega t + \ \varphi)$$

Where

$$v_{max} = -x_{max}\omega$$

$$a_{max} = v_{max}\omega = -x_{max}\omega^2$$

$$\omega = \sqrt{C}$$

The constants, x_{max} and φ, are found by using initial conditions, e.g., $x(0) = x_0, v(0) = v_0$.

Mass-Spring System: $\omega = \sqrt{k/m}$

Simple Pendulum: $\omega = \sqrt{g/L}$

Energy of Simple Harmonic Motion:
The total energy of a system undergoing simple harmonic motion is constant at all instants of time. The energy, however, is continually being transferred back and forth between potential and kinetic energy. $E = U(t) + K(t)$ For the mass and spring system on a frictionless surface: $E = \frac{1}{2}kx^2(t) + \frac{1}{2}mv^2(t) = \frac{1}{2}kx_{max}{}^2 = \frac{1}{2}kv_{max}{}^2$

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