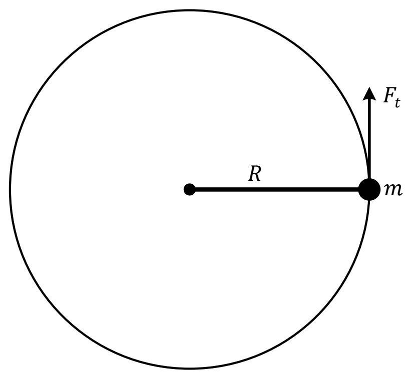
# Physics 1 Mechanics - Rotational Inertia

When studying translational dynamics, we saw that the net force applied to a body is proportional to the *linear acceleration* of that body. The proportionality constant was said to be the mass of the body. Using this understanding, physicists often refer to the mass of a body as the *linear inertia* of the body. Therefore, we can define the *linear inertia* as a measure of the resistance of a body to *linearly accelerate* under the influence of a *net force*.

$$\boldsymbol{F}_{net} \propto \boldsymbol{a} \quad \rightarrow \quad \boldsymbol{F}_{net} = m\boldsymbol{a}$$

In this lesson we'll discover a similar property of matter referred to as *rotational inertia*. Rotational inertia is similar to linear inertia in the sense that it is a measure of a resistance. In this case, it is a measure of the resistance of a body to *rotationally accelerate* under the influence of a *net torque*. We'll derive this quantity using the example below. Consider a point mass connected to a massless rod of radius, $R$.




When a tangential force, $F_t$, is applied to the mass, it will tend to travel along the circular path because of the centripetal force being supplied by the tension in the rod. As it travels around the circle the mass is undergoing tangential acceleration, from the applied force, and radial acceleration, from the tension in the rod. Focusing on the tangential acceleration we can use Newton's 2nd law to write the following relationship.

$$F_t = ma_t$$

Next, we recall from our study of rotational kinematics that the tangential acceleration is related to the angular acceleration as $a_t = R\alpha$. Therefore, we can write

$$F_t = mR\alpha$$

Additionally, from our lesson on torque we know that this tangential force causes a torque given by $\tau = F_t R$, since it is directed perpendicular to the rod at all times. Using this relationship, we can rewrite the equation again as follows.

$$\frac{\tau}{R} = mR\alpha$$
$$\tau = mR^2\alpha$$

This is the relationship we set out to find! It says that the *torque* is proportional to the *angular acceleration*. In this case the proportionality constant is given by $mR^2$, where $R$ is the distance from the mass to an axis or rotation. Following the example from linear motion, we can refer to this quantity, $I = mR^2$, as the *rotational inertia* of a body.

$$\tau \propto \alpha \quad \rightarrow \quad \tau = I\alpha$$

Although this relationship was derived for a single point mass, it can be easily extended for multiple point masses located at different distances from the axis of rotation as follows

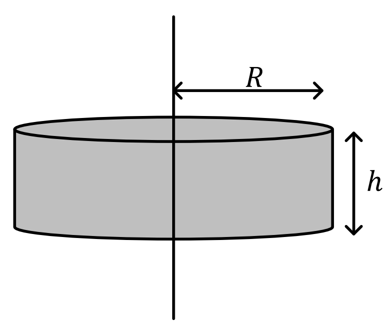
$$\tau_{net} = \sum_{i=1}^{N} \left(m_i {R_i}^2\right)\alpha = I\alpha$$

The above equation is the rotational equivalent of Newton's 2nd Law, where we define $I$ as the *rotational inertia* (or moment of inertia), which for a system of particles is given as below.

$$I = \sum_{i=1}^{N} m_i {R_i}^2$$

Using this equation to find the rotational inertia of a system of particles can be used to build an intuition for the quantity we refer to as rotational inertia. However, we will mostly be interested in the analysis of continuous bodies. Computing the rotational inertia for continuous bodies requires calculus and can often be quite difficult - or even impossible to compute analytically. However, the calculus can be made easier, (but not simple), for 'regularly' shaped bodies. For illustrative purposes we'll show the calculus for one of the simpler cases. However, when doing rotational dynamics problems, we will generally be given the expression for the rotational inertia for the body being studied – see the appendix at the end of the lesson.

**Rotational Inertia of a Solid Cylinder (Illustration using Calculus):** A pulley is a common body used in rotational dynamic problems. Recall that the pulleys we worked with previously were considered massless. This assumption allowed us to ignore any rotational inertia, i.e., the tendency of the pulley to resist a change in its rotational motion. To account for this resistance, we need to find an expression for the rotational inertia of a pulley. We'll do this by modeling the pulley as a solid disk with an axis of rotation that runs through the center as shown below.




$axis\ of\ rotation$ .

The rotational inertia for a system of discrete particles was given above as follows.

$$I = \sum_{i=1}^{N} m_i {R_i}^2$$

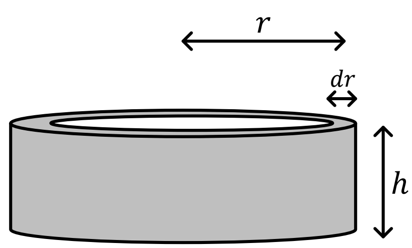
For a solid body we can apply calculus using the following substitutions.

$$\sum \cdot \to \int \cdot \qquad\qquad m_i \to dm$$

The rotational inertia of a solid body can then be expressed as follows

$$I = \int r^2 \, dm$$

To begin, we assume the disk has a uniform mass density per unit volume, $\rho$. Therefore, we can express the infinitesimal mass as $dm = \rho dV$. Furthermore, using the figure below we can express the infinitesimal volume as $dV = 2\pi rhdr$.




Substituting, we obtain an integral equation with the radius as the integration variable.

$$I = \int r^2 \rho dV$$

$$I = \int r^2 \rho 2\pi rhdr$$

$$I = \int (\rho 2\pi h) r^3 dr$$

Next, we apply limits of integration from 0 to $R$ and evaluate.

$$I = \rho 2\pi h \int_{r=0}^{R} r^3 \, dr = \left(\rho 2\pi h \frac{1}{4} r^4\right)\Big|_0^R = \frac{1}{2}\rho\pi h R^4$$

We can simplify this expression even more by using the following definition for the total mass.

$$M = \rho V = \rho \pi R^2 h$$

Finally, the rotational inertia, $I$, of a solid disk whose axis of rotation runs through the center is given as

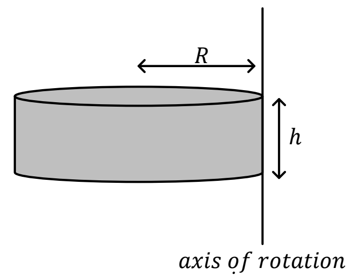
$$I = \frac{1}{2} M R^2$$

**Parallel Axis Theorem:** The parallel axis theorem allows us to find the rotational inertia of an object with respect to a certain axis of rotation when the rotational inertia using the center of mass of the object as the axis of rotation is known. The theorem is given below without proof.

$$I = I_{CM} + Mx^2$$

Where $M$ is the total mass of the object and $x$ is the distance from the center of mass axis of rotation to a new *parallel* axis of rotation.

We'll use the solid cylinder from the example above to illustrate the theorem. The figure below shows the axis or rotation placed at the edge of the cylinder.




Since the new axis of rotation is oriented parallel to the center of mass axis used previously, we can use the parallel axis theorem to find the new rotational inertia. The rotational inertia with respect to the center of mass axis of rotation was found above as

$$I_{CM} = \frac{1}{2}MR^2$$

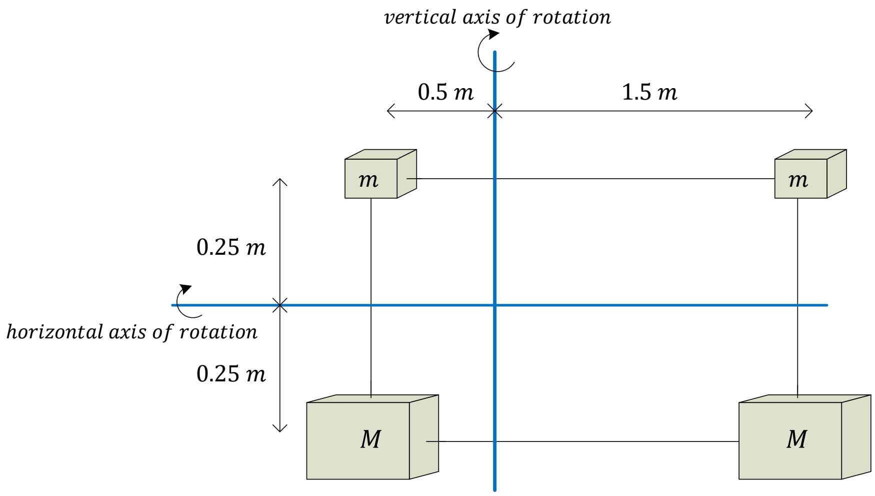
The rotational inertia about the new axis of rotation is the given as follows.

$$I = I_{CM} + Mx^2$$

$$I = \frac{1}{2}MR^2 + MR^2$$

$$I = \frac{3}{2}MR^2$$

**Example 1:** The structure below consists of four point objects, $m = 1.8\ kg, M = 3.1\ kg$, connected by massless rigid wires. Find the rotational inertia about the vertical and horizontal axis of rotation as shown in the figure.



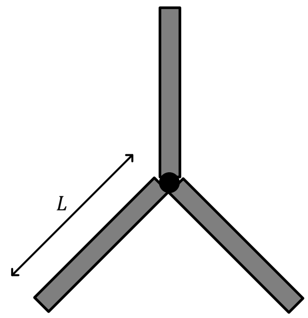

**Solution 1:** The summation formula can be used for each axis of rotation as shown.

| ***Rotational Inertia About the Vertical Axis*** |
|---|
| $I_V = \sum_{i=1}^{4} m_i {R_i}^2$<br>$= (m \cdot 1.5^2) + (M \cdot 1.5^2) + (m \cdot 0.5^2) + (M \cdot 0.5^2)$<br>$= 1.5^2(m + M) + 0.5^2(m + M)$<br>$= (1.5^2 + 0.5^2)(m + M)$<br>$= (1.5^2 + 0.5^2)(1.8 + 3.1)$<br>$= 12.25\ kg \cdot m^2$ |
| ***Rotational Inertia About the Horizontal Axis*** |
| $I_H = \sum_{i=1}^{4} m_i {R_i}^2$<br>$= (m \cdot 0.25^2) + (m \cdot 0.25^2) + (M \cdot 0.25^2) + (M \cdot 0.25^2)$<br>$= 0.25^2(2m + 2M)$<br>$= 0.25^2(2 \cdot 1.8 + 2 \cdot 3.1)$<br>$= 0.6125\ kg \cdot m^2$ |

The rotational inertia is significantly less when it is with respect to the horizontal axis. This is due to the fact that the mass is concentrated closer to the axis in the horizontal case. Because of this it would require less effort, in the form of torque, to rotate this structure about the horizontal axis compared to the vertical axis.

**Example 2:** A helicopter rotor, which consists of three identical equally spaced thin rods, is shown below. Each rod has a length of $3.75\ m$ and mass of $160\ kg$.

a) Find the rotational inertia of the helicopter rotor.
b) How much torque does the motor need to supply to bring the blades to a speed of $5\ rev/sec$ in $8\ sec$?




**Solution 2a:** Just as we can compute the rotational inertia of a system of point objects by summing the inertia of each object so can we find the rotational inertia of a system of continuous bodies by summing the inertia of each body. It's important to note that in both cases the axis of rotation must be the same for all objects /bodies being summed. With this knowledge, the rotational inertia of the rotor is:

$$I_{rotor} = I_{rod1} + I_{rod2} + I_{rod3}$$
$$I_{rotor} = 3I_{rod}$$

From the table in the appendix, we find the rotational inertia of a uniform rod about its center of mass is given as follows.

$$I_{rod,cm} = \frac{1}{12}mL^2$$

In this example, the axis of rotation is at one end of each rod. Therefore, we use the parallel axis theorem to find the inertia of each rod.

$$I_{rod} = I_{rod,cm} + m\left(\frac{L}{2}\right)^2$$
$$I_{rod} = \frac{1}{12}mL^2 + \frac{3}{12}mL^2$$
$$I_{rod} = \frac{1}{3}mL^2$$

The rotational inertia of the rotor is then given as

$$I_{rotor} = 3I_{rod}$$
$$I_{rotor} = \cancel{3}\left(\frac{1}{\cancel{3}}mL^2\right)$$
$$I_{rotor} = 160(3.75)^2$$
$$I_{rotor} = 2250\ kg \cdot m^2$$

**b.)** To determine the torque required for a certain angular acceleration we can use Newton's 2nd law for rotational motion.

$$\tau_{net} = I\alpha$$

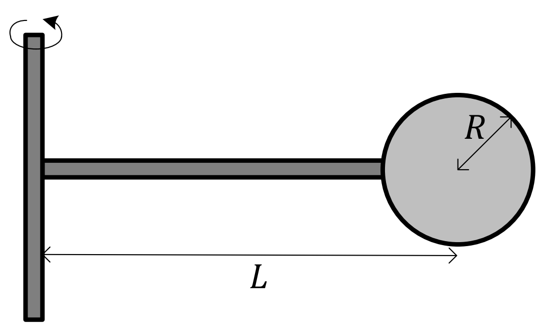
We first find the radial acceleration as follows.

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{5\frac{\cancel{rev}}{sec} \cdot 2\pi\frac{rad}{\cancel{rev}}}{8} = 3.93\ \frac{rad}{sec}$$

The torque required to cause the above rotation acceleration is then

$$\tau_{motor} = I_{rotor}\alpha$$
$$\tau_{motor} = 2250 \cdot 3.93$$
$$\tau_{motor} = 8843\ N \cdot m$$

**Example 3:** A solid sphere of radius $1\ m$ and mass $10\ kg$ is attached to a thin rod of length $5\ m$ and mass $5\ kg$ as shown below. The rod with the attached ball is rotated in a horizontal circle around a vertical pole. Find the rotational inertia of the horizontal rod and ball structure.




**Solution 3:** We can use the same technique of adding the inertias from example 2.

$$I = I_{rod} + I_{ball}$$

We also know from above that the inertia of a rod rotated about one of its ends is:

$$I_{rod} = \frac{1}{3}m_r L^2$$

The rotational inertia for a sphere about its center of mass is given in the appendix as

$$I_{sphere,cm} = \frac{2}{5}m_s R^2$$

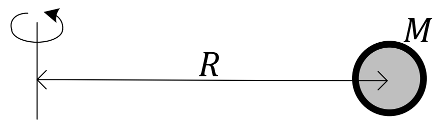
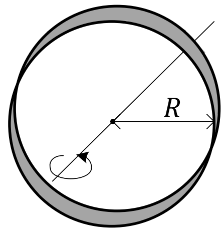
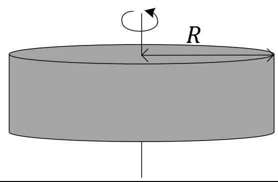
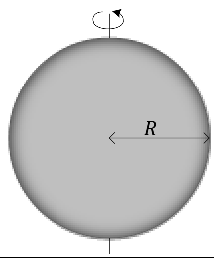
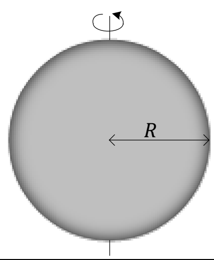
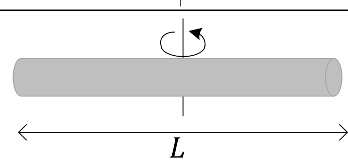
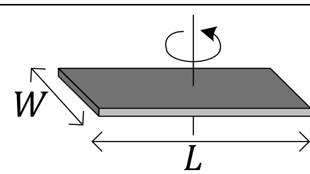
However, since the sphere is **not** rotated about its center of mass, we again use the parallel axis theorem to find its rotational inertia for this example.

$$I_{sphere} = I_{sphere,cm} + m_s L^2$$
$$I_{sphere} = \frac{2}{5} m_s R^2 + m_s L^2$$
$$I_{sphere} = m_s \left(\frac{2}{5} R^2 + L^2\right)$$

Finally, the rotational inertia of the ball rod system is found below.

$$I = I_{rod} + I_{ball}$$
$$I = \left(\frac{1}{3} m_r L^2\right) + m_s \left(\frac{2}{5} R^2 + L^2\right)$$
$$I = \left(\frac{1}{3} 5 \cdot 5^2\right) + 10 \left(\frac{2}{5} 1^2 + 5^2\right)$$
$$I = 295.7 \; kg \cdot m^2$$

**Appendix**

| Rotational Inertia of Various Objects of Uniform Composition About the Center of Mass | | |
|---|---|---|
| ***Object*** | | ***Rotational Inertia*** |
| Point Object |   | $I = MR^2$ |
| Thin Hoop |   | $I = MR^2$ |
| Solid Cylinder |   | $I = \frac{1}{2}MR^2$ |
| Solid Sphere |   | $I = \frac{2}{5}MR^2$ |
| This Hollow Sphere |   | $I = \frac{2}{3}MR^2$ |
| Thin Rod |   | $I = \frac{1}{12}ML^2$ |
| Thin Rectangular Plate |   | $I = \frac{1}{12}M(L^2 + W^2)$ |

# ***Final Summary for Rotational Inertia***

## Newton's Second Law for Rotational Motion

The net torque on a body is equal to the rotational inertia times the angular acceleration of that object.

$$\tau_{net} = I\alpha$$

Where, $I$ is the rotational inertia (moment of inertia), and $\alpha$ is the angular acceleration.

## ***Rotational Inertia***

The rotational inertia of an object is completely analogous to the linear inertia (mass).

**Definition**: It is a measure of the resistance of a body to *rotationally accelerate* under the influence of a *net torque.*

The rotational inertia of a system of $N$ point masses about a given axis of rotation is:

$$I = \sum_{i=1}^{N} m_i {R_i}^2$$

Where, $m_i$ is the mass of the $i^{th}$ object and $R_i$ is the perpendicular distance from the axis of rotation to the $i^{th}$point mass.

The rotational inertia for a solid body about a given axis of rotation is:

$$I = \int R^2 \, dm$$

Where, $dm$ is and infinitesimal mass element and $R$ is the distance from the axis of rotation to an infinitesimal mass element.

## ***Parallel Axis Theorem***

The rotational inertia about an axis that passes through the center of mass of an object is related to the rotational inertia about a parallel axis that is a distance $x$ away.

$$I = I_{CM} + Mx^2$$

Where, $M$ is the total mass of the object, $x$ is the distance between the two rotational axes, and $I_{CM}$ is the rotational inertia about an axis that passes through the center of mass of the object. Note: The new axis of rotation must be parallel to the one used for $I_{CM}$.

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