Physics 1 Mechanics - Rotational Dynamics

Newton's 2nd law applies to linear (translational) motion. It says that the *linear acceleration* of an object is proportional to the *net force* acting on the object. The quantity that measures the objects ability to resist the *linear acceleration* is called the *linear inertia*, which is equal to the mass, m, of the object. The relationship is expressed as *Newton's 2nd law <for linear motion>*.

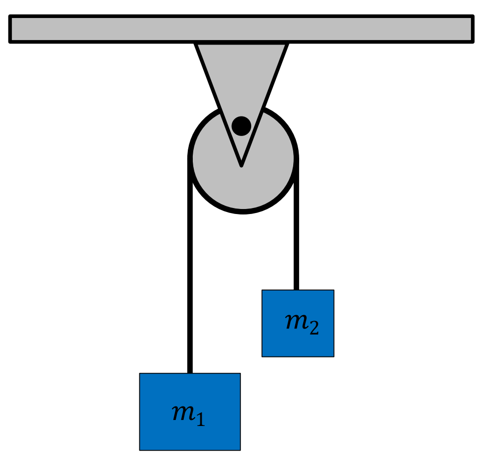
$$\boldsymbol{F}_{net} = m\boldsymbol{a}$$

As we know a body can also undergo *rotational motion* about an axis. In the previous lesson we found that there is an equivalent law for this type of motion, i.e., *Newton's 2nd law for rotational motion*. It says that the *rotational acceleration* of a rigid object is proportional to the *net torque* acting on the object. In this case, the quantity that measures the objects ability to resist the *rotational acceleration* is called the *rotational inertia*, I, which was introduced in the previous lesson.

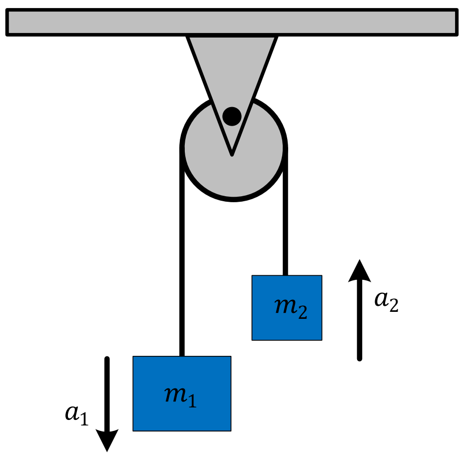
$$\boldsymbol{\tau}_{net} = I\boldsymbol{\alpha}$$

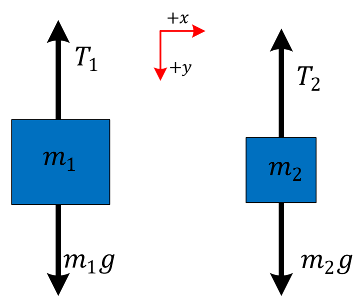
In this lesson we'll use this equation to analyze the rotational dynamics of systems.

Example 1: A pulley system is arranged as shown below with $m_1 = 20\ kg$ and $m_2 = 2\ kg$. Find the acceleration of the system with a massless pulley compared to a pulley with a mass of $8\ kg$ and radius of $0.25\ m$.



Solution1: In previous lessons we assumed all pulleys were massless, which allowed us to argue that the tension in the string is equal on both sides of the pulley. In this example we will show this result more explicitly. We'll start by drawing a free-body diagrams for the two blocks, each with a different tension. The rope, however, does not stretch and so the acceleration of each block is equal in magnitude but opposite in direction, i.e., $a_2 = -a_1$. Therefore, we can simplify the equations below by letting $a_1 = a$ so that $a_2 = -a$.



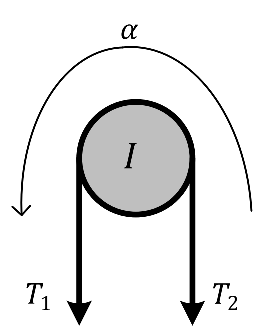


Since m_1 is larger than m_2 we assume the acceleration of the system is counterclockwise as shown in the figure on the left. Newton's 2nd law for each block is shown below.

$$\sum F_y = m_1 a_1$$
$$m_1 g - T_1 = m_1 a$$
$$T_1 = m_1 g - m_1 a$$

$$\sum F_y = m_2 a_2$$
$$m_2 g - T_2 = m_2(-a)$$
$$T_2 = m_2 g + m_2 a$$

Unlike earlier pulley problems where we initially assumed $T_1 = T_2$, we now have three unknowns, i.e., a, T_1, and T_2, but only two equations. For an additional equation, we draw a free-body diagram for the pulley itself and use *Newton's 2nd law for rotation*. We model the pulley as a solid disk with a rotational inertia, $I = 1/2\, MR^2$. Furthermore, we recall that rotational acceleration is related to linear acceleration as $\alpha = a/R$.



Newton's 2nd law for rotation is then written as follows

$$\sum \boldsymbol{\tau} = I\alpha$$

$$T_1R - T_2R = \frac{MR^2}{2}\alpha$$

Next, we relate the rotational acceleration to the linear acceleration of the blocks as $\alpha = a/R$.

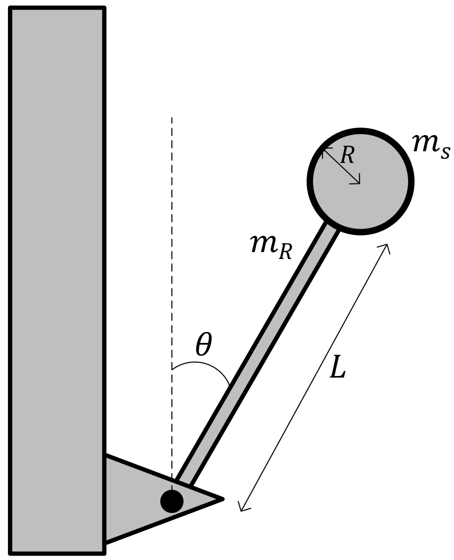
$$T_1R - T_2R = \frac{MR^2a}{2R}$$

$$T_1 - T_2 = \frac{Ma}{2}$$

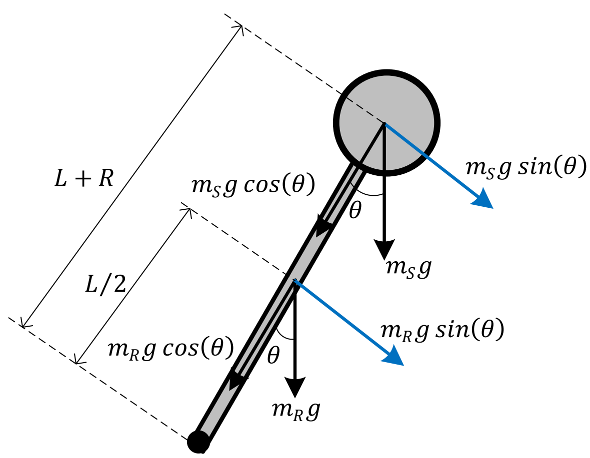
As you can now clearly see if we assume the pulley is massless, i.e., $M = 0$, we have $T_1 = T_2$. Comparing the acceleration of the system with and without a massless pulley, we find that the acceleration is less when the rotational inertia of the pulley is accounted for.

Massless Pulley ***No Rotational Inertia***	***Pulley with Mass*** ***Rotational Inertia***
$T_1 = T_2$ $m_1g - m_1a = m_2g + m_2a$ $a(m_1 + m_2) = g(m_1 - m_2)$ $a = g\dfrac{(m_1 - m_2)}{(m_1 + m_2)}$ $a = 9.8\dfrac{(20 - 2)}{(20 + 2)}$ $a = 8.0\ m/s^2$	$T_1 - T_2 = \dfrac{1}{2}Ma$ $(m_1g - m_1a) - (m_2g + m_2a) = \dfrac{Ma}{2}$ $m_1a + m_2a + \dfrac{Ma}{2} = m_1g - m_2g$ $a\left(m_1 + m_2 + \dfrac{M}{2}\right) = g(m_1 - m_2)$ $a = g\dfrac{(m_1 - m_2)}{\left(m_1 + m_2 + \frac{M}{2}\right)}$ $a = 9.8\dfrac{(20 - 2)}{(20 + 2 + 4)}$ $a = 6.8\ m/s^2$

Example 2: Consider a thin $4\ m$ rod with a mass of $2\ kg$ pivoted at one end. A uniform spherical object with a mass of $2\ kg$ and radius of $0.25\ m$ is attached to the free end of the rod. Find an expression for the angular acceleration of the rod as a function of the angle. What is the value of the acceleration immediately after it released from $\theta = 37^{\circ}$?



Solution 2: We can start by drawing the free-body diagram, using the pivot point at the bottom of the rod as the axis of rotation.



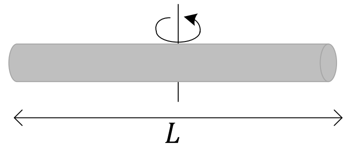
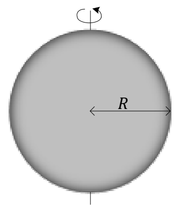
With the force components responsible for the torque shown in blue, we can write Newton's 2nd law for rotation as follows.

$$\sum \boldsymbol{\tau} = I\alpha$$

$$m_R g\ sin(\theta)\left(\frac{L}{2}\right) + m_s g\ sin(\theta)\ (L + R) = I\alpha$$

$$\alpha = \left(\frac{g\big(m_R L + 2m_s(L + R)\big)}{2I}\right) sin(\theta)$$

Next, we need to find the rotational inertia, I, of the combined rod and sphere. Since the axis of rotation is not located at the center of mass of either object, we need to use the parallel axis theorem. For the rotational inertia of each object about the center of mass we use the table in the appendix of the previous lesson.

	$I = \frac{1}{12}ML^2$
	$I = \frac{2}{5}MR^2$

Applying the parallel axis theorem and combing the inertias we find the following expression.

$$I = I_{rod} + I_{sphere}$$
$$I = \left(\frac{1}{12}m_R L^2 + m_R\left(\frac{L}{2}\right)^2\right) + \left(\frac{2}{5}m_s R^2 + m_s(L+R)^2\right)$$
$$I = \left(\frac{1}{3}m_R L^2\right) + \left(\frac{2}{5}m_s R^2 + m_s(L+R)^2\right)$$

Substituting this into our expression for the angular acceleration from above, we have

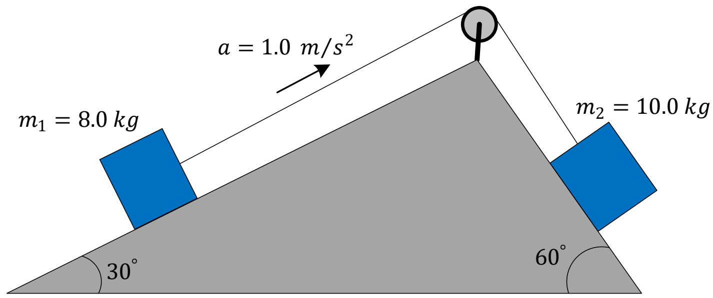
$$\alpha = \left(\frac{g\left(m_R L + 2m_s(L+R)\right)}{2\left(\left(\frac{1}{3}m_R L^2\right) + \left(\frac{2}{5}m_s R^2 + m_s(L+R)^2\right)\right)}\right) sin(\theta)$$

Finally, the angular acceleration as a function of the θ can be written as shown.

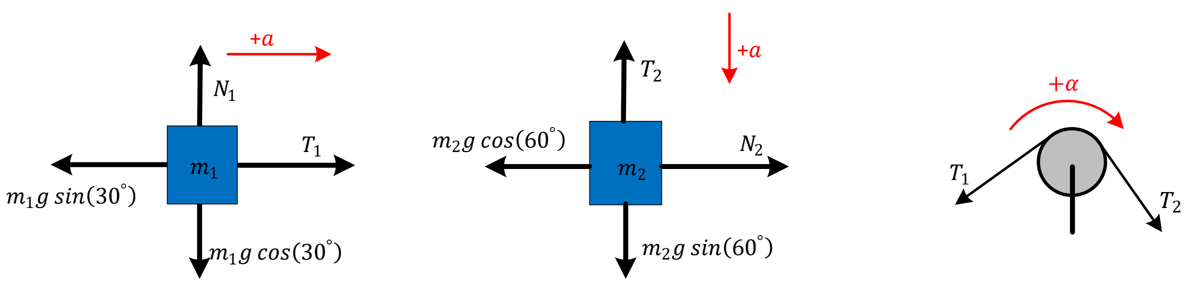
$$\alpha = \left(\frac{9.8\left(2\cdot 4 + 2\cdot 2(4.25)\right)}{2\left(\left(\frac{1}{3}\cdot 2\cdot 4^2\right) + \left(\frac{2}{5}\cdot 2\cdot 0.25^2 + 2(4.25)^2\right)\right)}\right) sin(\theta)$$
$$= 2.62\, sin(\theta)\, rad/s^2$$

Note that when $\theta = 0^{\circ}$, the rod does not rotate since the torque is zero. Furthermore, the largest angular acceleration occurs when $\theta = 90^{\circ}$, since this is the point when the force vectors have a single downward component. The angular acceleration when $\theta = 37^{\circ}$ is $1.57\ rad/s^2$.

Example 3: Two blocks are connected by a light string that passes over a pulley of radius 0.25 m and rotational inertia, I. The blocks move towards the right with an acceleration of 1.0 m/s^2 along frictionless inclines. Find the tension in the string and the rotational inertia of the pulley.



Solution 3: We can start by drawing a free-body diagram for the two blocks and the pulley.



Note, we tilt the axis along the incline and assign the acceleration to the positive direction. We can solve for the tensions using Newton's 2nd law for each block as shown below.

$$\sum F = m_1 a \qquad \sum F = m_2 a$$

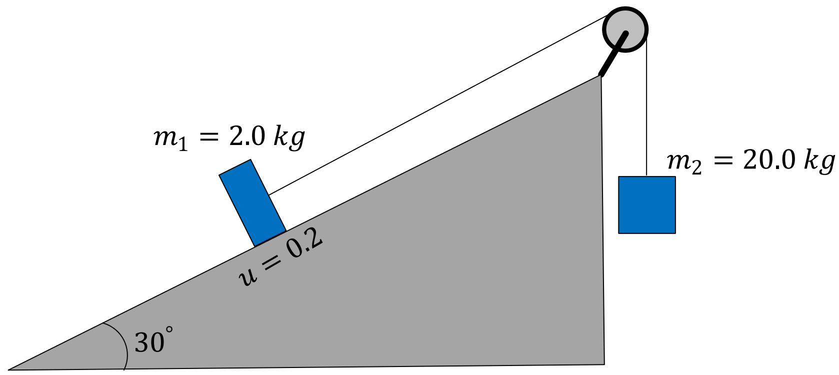
$$T_1 - m_1 g\,sin(30^{\circ}) = m_1 a$$
$$T_1 = m_1(g\,sin(30^{\circ}) + a)$$
$$T_1 = 8(9.8\,sin(30^{\circ}) + 1)$$
$$T_1 = 47.2\ N$$

$$m_2 g\,sin(60^{\circ}) - T_2 = m_2 a$$
$$T_2 = m_2(g\,sin(60^{\circ}) - a)$$
$$T_2 = 10(9.8\,sin(60^{\circ}) - 1)$$
$$T_2 = 74.9\ N$$

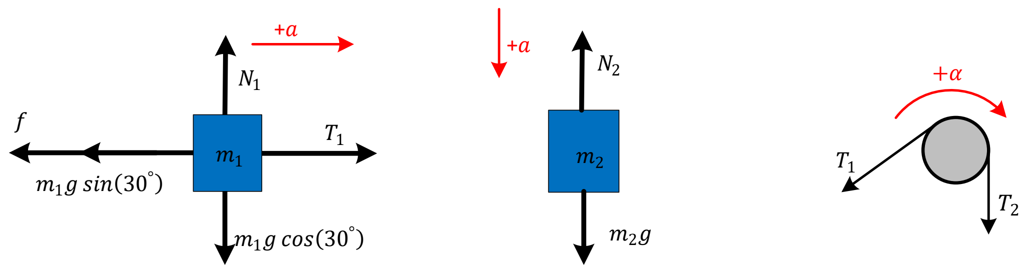
The rotational inertia of the pulley can now be found using Newton's 2nd law for rotation.

$$\tau_{net} = I\alpha$$
$$I = \frac{\tau_{net}}{a/R}$$
$$I = \frac{T_2 R - T_1 R}{a/R}$$
$$I = R^2\left(\frac{T_2 - T_1}{a}\right)$$
$$I = 0.25^2\left(\frac{74.9 - 47.2}{1}\right)$$
$$I = 1.7\ kg \cdot m^2$$

Example 4: Consider the setup shown below. The pulley has a mass of 8 kg and a radius of 0.25 m. The coefficient of friction between the block and the surface is 0.2. Find the acceleration of the blocks.



Solution 4: As in the previous example we start with free-body diagrams.



The friction force is given by $f = uN$, and since the forces that are perpendicular to the surface balance, the normal force is $m_1 g\ cos(30^{\circ})$. The friction force is then $f = u m_1 g\ cos(30^{\circ})$. We can now write Newton's 2nd law for both blocks.

$$\sum F_x = m_1 a$$

$$T_1 - m_1 g\ sin(30^{\circ}) - u m_1 g\ cos(30^{\circ}) = m_1 a$$

$$T_1 = m_1 g\ sin(30^{\circ}) + u m_1 g\ cos(30^{\circ}) + m_1 a$$

$$\sum F_y = m_2 a_2$$

$$m_2 g - T_2 = m_2 a$$

$$T_2 = m_2 g - m_2 a$$

As in example 1, the tensions are not equal, and we need to apply Newton's 2nd law for rotation to the pulley.

$$\sum \boldsymbol{\tau} = I\alpha$$

$$T_2 R - T_1 R = \frac{MR^2}{2}\left(\frac{a}{R}\right)$$

$$T_2 - T_1 = \frac{Ma}{2}$$

Finally, we can now solve for the acceleration by substituting for the tensions from the first two equations.

$$T_2 - T_1 = \frac{Ma}{2}$$

$$(m_2 g - m_2 a) - (m_1 g\, sin(30^{\circ}) + u m_1 g\, cos(30^{\circ}) + m_1 a) = \frac{Ma}{2}$$

$$a\left(m_1 + m_2 + \frac{M}{2}\right) = g(m_2 - m_1\, sin(30^{\circ}) - u m_1\, cos(30^{\circ}))$$

$$a = \frac{g(m_2 - m_1\, sin(30^{\circ}) - u m_1\, cos(30^{\circ}))}{\left(m_1 + m_2 + \frac{M}{2}\right)}$$

$$a = \frac{9.8(20 - 2\, sin(30^{\circ}) - 0.2 \cdot 2\, cos(30^{\circ}))}{\left(2 + 20 + \frac{8}{2}\right)}$$

$$a = 7\ m/s^2$$

The angular acceleration of the pulley is then given as

$$\alpha = \frac{a}{R} = \frac{7}{0.25} = 28\ rad/s^2$$

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