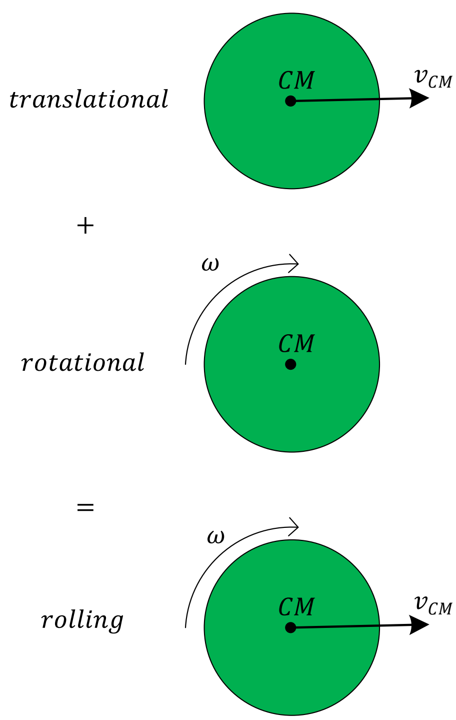
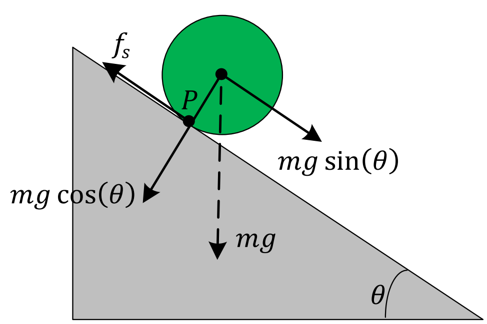
Physics 1 Mechanics - Rolling

When a body undergoes purely translational motion, we treat it as a point particle with all of its mass located at the center of mass. When a body undergoes purely rotational motion, we treat it as a rigid body that is made to rotate about a fixed axis. When a body simultaneously experiences both translational and rotational motion, we refer to this as *rolling motion.* Rolling motion is important to study for a myriad of reasons. The wheel and axle, arguably one of the most important inventions in history, is just one example that illustrates the importance of understanding the physics of rolling motion. In this lesson we will study *simple rolling*, that is rolling without slipping. The figure below illustrates rolling motion as a combination of translational and rotational motion.



To understand the forces responsible for rolling motion we'll look at the example of an object rolling down an incline, as shown below.



To analyze the translational motion, we consider all forces as if they act at the center of mass of the object. For this example, the gravitational force, $mg\sin(\theta)$, along with any kinetic friction force, would contribute to the translational motion of the object. To analyze the rotational motion, we first choose an axis of rotation, in this case the center of the object, and consider torques about this axis. We look at this more closely below.

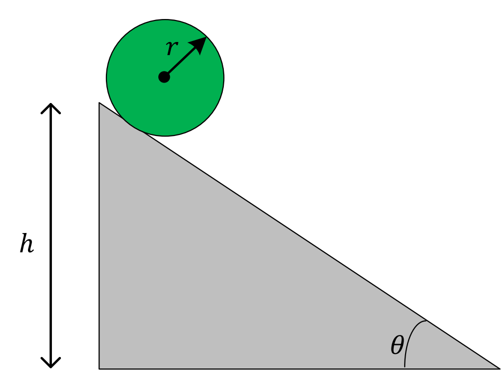
As can be seen from the figure, at any specific point in time the ball is in contact with the surface at a single point, P. At the point, P, assuming the object is at rest, a static friction force is directed up the incline, i.e., opposite to the intended direction of motion. In previous lessons, for this type of scenario, we generally considered two possible outcomes.

1. The gravitational force was smaller than the *maximum static friction force*, and the object remained at rest.
2. The gravitational force exceeded the *maximum static friction force*, and the object began to *slide* down the incline.

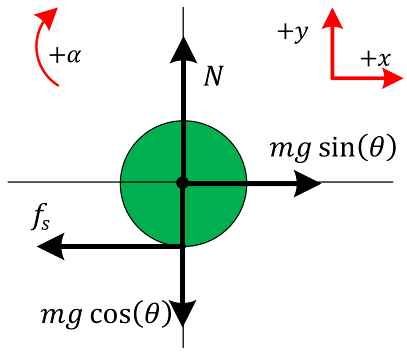
A third option, which we have not yet considered, is that the object instead begins to rotate. This motion is made possible based on the geometric shape of the object. For the example of a spherical ball, the static friction force will tend to cause a torque about the center of mass of the ball. If we couple this rotation with the translational motion caused by the gravitational force acting on the center of mass, the ball will undergo a *rolling motion*. Note, if the static friction force is not strong enough, i.e., a very smooth surface, the ball may begin to slide down the hill without rotating, or with intermittent rotation. We refer to this as *rolling with slipping*. As you can imagine this type of motion is much more complicated to analyze. In this lesson we consider only *simple rolling*, i.e., rolling without slipping. Note, kinetic friction is not relevant for rolling without slipping because at each instant of time the point P does not move parallel to the surface.

Let's look at an example that illustrates the analysis of the rolling motion described above.

Example 1: An object (drawn as a sphere) is held at rest at the top of an incline. Once released the object begins to roll down incline without slipping. Find the speed of the rolling object when it gets to the bottom of the incline.



Solution 1 (Force Analysis): We start by drawing a free-body diagram with the coordinate axes aligned with the incline. For the translational motion we choose down the incline as the positive direction, and for the rotational motion we choose clockwise rotation as positive. The gravitational force is shown to act at the center of mass and will cause the translational motion of the object. The static friction force, which acts on the edge of the object, causes a torque, and the subsequent rotational motion.



Newton's 2nd law for translational motion is shown below.

Newton's 2nd Law for Translation Motion.	
$\sum F = ma$ $mg\,sin(\theta) - f_s = ma$ $f_s = mg\,sin(\theta) - ma$	$\sum F_y = 0$ $N - mg\,cos(\theta) = 0$ $N = mg\,cos(\theta)$

Next, we write Newton's 2nd law for rotational motion, using the center of mass as the axis of rotation. Since gravity and the normal force act through the axis they do not create a torque.

Newton's 2nd Law for Rotational Motion.
$\sum \tau = I_{cm}\alpha$ $f_s r = I_{cm}\frac{a}{r}$ $f_s = I_{cm}\frac{a}{r^2}$

We can now solve for the acceleration, a, by substituting f_s from Newton's 2nd law for translational motion.

$$I_{cm}\frac{a}{r^2} = mg\,sin(\theta) - ma$$

$$a\left(\frac{I_{cm}}{r^2} + m\right) = mg\,sin(\theta)$$

$$a = \frac{mg\,sin(\theta)}{\left(\frac{I_{cm}}{r^2} + m\right)}$$

Next, we use kinematics to find the velocity at the bottom of the incline.

$$v_f^{\,2} = v_i^{\,2} + 2a(\Delta x)$$

The initial velocity, v_i, is zero, and the distance traveled along the ramp is given as

$$\Delta x = \frac{h}{sin(\theta)}$$

Finally, we can substitute for the acceleration and solve for the final velocity as shown below.

$$v_f^{\,2} = 0 + 2\left(\frac{mg\,sin(\theta)}{\left(m + \frac{I_{cm}}{r^2}\right)}\right)\left(\frac{h}{sin(\theta)}\right)$$

$$v_f = \sqrt{\frac{2mgh}{\left(m + \frac{I_{cm}}{r^2}\right)}}$$

We can rewrite the equation once more by multiplying through by $\frac{1/\sqrt{m}}{1/\sqrt{m}}$.

$$v_f = \sqrt{\frac{2gh}{\left(\frac{I_{cm}}{mr^2} + 1\right)}}$$

By writing the equation in this form, we can gain additional insight into the effect the rotational inertia has on the speed of the object. The rotational inertia of different rolling objects is shown below.

Thin Hoop	$I_{cm} = 1mr^2$
Hollow Sphere	$I_{cm} = \frac{2}{3}mr^2$
Solid Cylinder	$I_{cm} = \frac{1}{2}mr^2$
Solid Sphere	$I_{cm} = \frac{2}{5}mr^2$

Each of these formulas is proportional to mr^2, and therefore can be expressed as shown below.

$$I_{cm} = Cmr^2, \quad 0 < C \leq 1$$

Where, a greater C value indicates a larger rotational inertia, assuming m and r are constant.

Substituting this expression, we can write the velocity equation as follows.

$$v_f = \sqrt{\frac{2gh}{\left(\frac{Cmr^2}{mr^2} + 1\right)}} = \sqrt{\left(\frac{1}{C+1}\right)2gh}$$

We can also go one step further and substitute for $D = \frac{1}{(C+1)}$.

$$v_f = \sqrt{D(2gh)}$$

Mapping from C to D we list the final velocities for each object below.

Object	***Thin Hoop***	***Hollow Sphere***	***Solid Cylinder***	***Solid Sphere***
C	1	$\frac{2}{3} = 0.\bar{6}$	$\frac{1}{2} = 0.5$	$\frac{2}{5} = 0.4$
$D = \frac{1}{C+1}$	$\frac{1}{2} = 0.5$	$\frac{3}{5} = 0.6$	$\frac{2}{3} = 0.\bar{6}$	$\frac{5}{7} \cong 0.7$
v_f	$\sqrt{gh}$	$\sqrt{1.2gh}$	$\sqrt{1.\bar{3}gh}$	$\sqrt{1.4gh}$

The table shows that as the rotational inertia of the objects decrease, (moving from left to right in the table), the final velocities increase.

What if the surface was frictionless? Returning to Newton's 2^{nd} law from above, we see that the object would slide down the incline, experiencing only translational acceleration caused directly by the gravitational force.

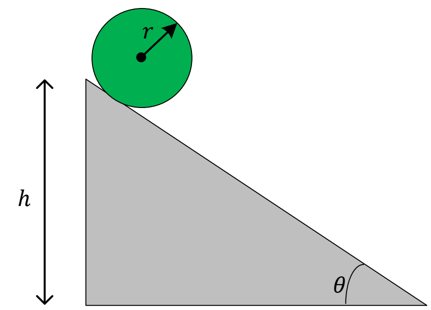
$$a_x = g\,sin(\theta)$$

And using the same kinematic equation we find the final velocity in this case is given by

$$v_f = \sqrt{2gh}$$

Which you'll notice is larger than any of the values from the table above. It's important to note that this is *not* due to any loss in energy from friction because it is static friction that is causing the rotation, which does not dissipate energy. The decrease in speed is instead caused by the fact that some of the initial gravitational potential energy is converted to rotational kinetic energy, leaving less energy for translational kinetic energy. We can make this concept clearer by solving the same problem using energy techniques.

Solution 1: (Energy Analysis): The incline is shown again for illustration.



In this case we can use the conservation of energy to directly find the speed at the bottom of the incline. We take the initial time to be when the object is at rest at the top of the incline and the final time to be when the object reaches the bottom of the incline. If we assume the object is rolling it has both translational and rotational kinetic energy.

$$K_{t,f} + K_{r,f} + U_f = K_{t,i} + K_{r,i} + U_i$$

$$\frac{1}{2}m{v_f}^2 + \frac{1}{2}I_{cm}\omega^2 + 0 = 0 + 0 + mgh$$

$$m{v_f}^2 + I_{cm}\frac{{v_f}^2}{r^2} = 2mgh$$

$${v_f}^2\left(m + \frac{I_{cm}}{r^2}\right) = 2mgh$$

$$v_f = \sqrt{\frac{2mgh}{\left(m + \frac{I_{cm}}{r^2}\right)}}$$

which we once again rewrite as follows.

$$\boxed{v_f = \sqrt{\frac{2gh}{\left(\frac{I_{cm}}{mr^2} + 1\right)}}}$$

Furthermore, we see that if the object were to slide down the incline without rotating, we have

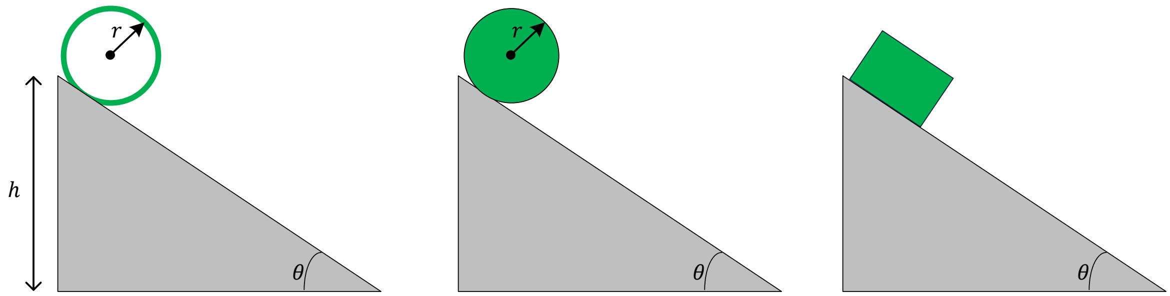
$$K_{t,f} + K_{r,f} + U_f = K_{t,i} + K_{r,i} + U_i$$

$$\frac{1}{2}m{v_f}^2 + 0 + 0 = 0 + 0 + mgh$$

$$\boxed{v_f = \sqrt{2gh}}$$

Of course, these results exactly match what we obtained using a force based analysis.

Example 2: Three objects; a thin hoop, and a solid sphere, and a box, are held at rest at the top of a $50\ m$ high incline. The mass of all three objects is $m = 5\ kg$. The radius of the hoop and the sphere is $r = 0.5\ m$. The angle of inclination for the ramp is 30°. The coefficient of kinetic friction between the objects and the surface is 0.4. Compare the time it takes for each of these three objects to arrive at the bottom of the incline.



Solution 2: The time to reach the bottom of the incline can be found if we know the acceleration using the following kinematic equation.

$$\Delta x = v_i t + \frac{1}{2} a t^2$$

Where, Δx is the incline distance, i.e., $\Delta x = \frac{h}{sin(\theta)}$. Furthermore, since $v_i = 0$ we have

$$t = \sqrt{\frac{2\Delta x}{a}} = \sqrt{\frac{2h}{a\, sin(\theta)}}$$

For the thin hoop and the sphere, we can use the same force analysis from above to find the acceleration, which is given as

$$a = \frac{mg\, sin(\theta)}{\left(\frac{I_{cm}}{r^2} + m\right)}$$

The rotational inertia of the hoop and the sphere can be written as $I_x = C_x m r^2$, where $C_{hoop} = 1.0$ and $C_{sphere} = 2/5$.

$$a_x = \frac{mg\, sin(\theta)}{\left(\frac{C_x m r^2}{r^2} + m\right)} = \frac{g\, sin(\theta)}{(C_x + 1)}$$

For the sliding box we can use the translation motion equations from example 1, substituting the static friction force with the kinetic friction force, i.e., $f_k = uN = umg\, cos(\theta)$.

$$a_{box} = g\, sin(\theta) - ug\, cos(\theta)$$

Finally, we can find the time for each object to reach the bottom of the incline by substituting the various acceleration expressions into the kinematic time equation from above. This is done in the table below.

Thin Hoop	***Solid Sphere***	***Box***
$\sqrt{\frac{2h}{\left(\frac{g\,sin(\theta)}{(C_{hoop}+1)}\right)sin(\theta)}}$ $\sqrt{\frac{2h(C_{hoop}+1)}{g\,sin^2(\theta)}}$ $\sqrt{\frac{100(1+1)}{2.45}}$	$\sqrt{\frac{2h}{\frac{g\,sin(\theta)}{(C_{sphere}+1)}sin(\theta)}}$ $\sqrt{\frac{2h(C_{sphere}+1)}{g\,sin^2(\theta)}}$ $\sqrt{\frac{100(0.4+1)}{2.45}}$	$\sqrt{\frac{2h}{(gsin(\theta)-u\,gcos(\theta))\,sin(\theta)}}$ $\sqrt{\frac{2h}{g\,sin^2(\theta)-ug\,cos(\theta)\,sin(\theta)}}$ $\sqrt{\frac{100}{2.45-(0.4\cdot 4.24)}}$
$\Delta t_{hoop} = 9.04\ s$	$\Delta t_{sphere} = 7.56\ s$	$\Delta t_{box} = 11.52\ s$

One key observation with regard to the hoop and the sphere is that since $I_{hoop} > I_{sphere}$ it takes longer for the hoop to get to the bottom of the hill. As mentioned earlier, this has to do with the fact that more of the initial potential energy needs to go into rotational kinetic energy for the hoop compared to the sphere.

With regard to the sliding box, you may recall we stated earlier that it should be the fastest, i.e., take the least amount of time to get to the bottom of the hill. However, we find that it actually takes the longest. This is due to the fact that, in this case, the surface is not frictionless. We can see that the energy lost due to friction is even greater than the energy that was converted to rotational energy of the hoop and the sphere. If we instead assume a frictionless surface, i.e., $u = 0$, the time is indeed the shortest.

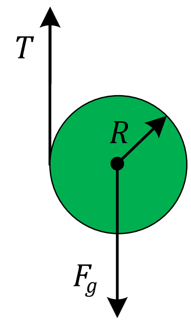
$$\Delta t_{box,frictionless} = \sqrt{\frac{2h}{g\,sin^2(\theta) - ug\,cos(\theta)\,sin(\theta)}}$$

$$\Delta t_{box,frictionless} = \sqrt{\frac{2h}{g\,sin^2(\theta) - 0}}$$

$$\Delta t_{box,frictionless} = \sqrt{\frac{100}{2.45 - 0}}$$

$$\Delta t_{box,frictionless} = 6.39\ s$$

Example 3: String is wrapped around a uniform solid cylinder, (something like a yo-yo), of mass M and radius R. If the cylinder starts from rest find the acceleration and the tension in the string.



Solution 3: We start with Newton's 2nd law for translation motion.

$$\sum F_y = ma$$
$$mg - T = ma$$
$$a = g - \frac{T}{m}$$

The cylinder also undergoes rotational motion caused by the tension. Therefore, we can also write Newton's 2nd law for rotational motion.

$$\sum \tau = I\alpha$$
$$TR = \left(\frac{1}{2}mR^2\right)\frac{a}{R}$$
$$T = \frac{1}{2}ma$$

Substituting this tension expression into the first equation we find the acceleration is given as

$$a = g - \left(\frac{1}{2}ma\right)\frac{1}{m}$$
$$a = g - \left(\frac{1}{2}a\right)$$
$$a + \frac{1}{2}a = g$$
$$a = \frac{2}{3}g$$

Note, the acceleration is less that what it would be, $a = g$, if the cylinder were simply dropped. This makes sense since the tension works against the gravitational force. The tension in the string can be found by substituting the acceleration in the tension equation from above.

$$T = \frac{1}{2}m\left(\frac{2}{3}g\right) = \frac{1}{3}mg$$

Final Summary for Rolling Motion

Rolling Motion:

- When a rigid body simultaneously experiences translational motion and rotational motion.
- Is made possible by the geometric shape of the object.
- The axis of rotation is generally the center of mass of the object.
- For an object rolling down an incline, the static friction force causes a torque, which in turn causes the rotational motion.
  - If the static friction force is not adequate *rolling with slipping* may occur. Rolling without slipping is refereed to as *simple rolling*.
- Rolling objects contain both translation and rotational kinetic energy.

$$K_{Total} = E_{Translational} + E_{Rotational}$$

$$K_{Total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

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