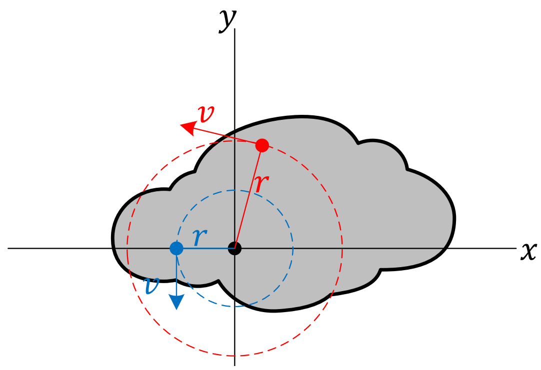
Physics 1 Mechanics - Angular Momentum

As we have seen in the past few lessons, many of the equations that govern rotational motion are direct analogues to those from translation motion. In this lesson we extend this to the concept of momentum. *Linear momentum* is a quantity associated with an object that is experiencing *translational motion*. Similarly, we can define the *angular momentum* as a quantity that is associated with an object that is experiencing *rotational motion*. Once again, the angular momentum equation is a direct analog to the linear momentum one a shown here.

Linear Momentum		***Angular Momentum***
$\boldsymbol{p} = m\boldsymbol{v}$	$\rightarrow$	$\boldsymbol{L} = I\boldsymbol{\omega}$

The angular momentum, $\boldsymbol{L}$, describes the angular momentum of a rigid body that is rotating about *any* fixed axis. As illustrated below, each point on the object moves in a circle.

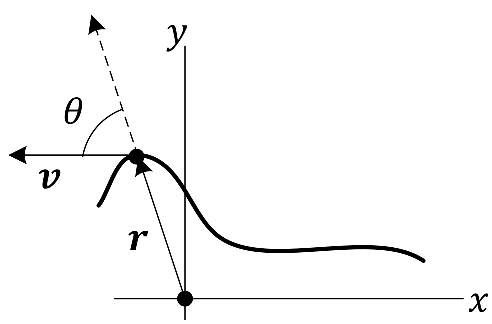


What if we instead had a point particle that is not necessarily moving in a perfect circle? Can we still define the angular momentum for this object? Let's investigate this by substituting the rotational inertia of a point particle, $I = mr^2$, into the above equation.

$$\boldsymbol{L}_{par} = I\boldsymbol{\omega}$$
$$\boldsymbol{L}_{par} = mr^2\frac{\boldsymbol{v}}{r}$$
$$\boldsymbol{L}_{par} = mr\boldsymbol{v}$$

Looking back at the rigid body in the above figure we notice that the velocity is always perpendicular to a line drawn from the axis of rotation to the point on the body. For a point object that is not necessarily moving in a circle we can use the vector cross product as shown below to maintain this same perpendicular relationship. Note that $\boldsymbol{r}$ is a vector from a point of rotation to the particle.

$$\boldsymbol{L}_{par} = m(\boldsymbol{r}\ x\ \boldsymbol{v}) \quad \rightarrow |\boldsymbol{L}_{par}| = mrv\,sin(\theta)$$



Conservation of Angular Momentum: The conservation of linear momentum is a very important law that provides us with significant information about an isolated system, i.e., one in which there are no external forces. Can we develop a similar law for the conservation of angular momentum? Recall that the conservation of linear momentum was developed based on the linear momentum form of Newton's 2nd law for translational motion.

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt}$$

Using this we derived the law of conservation of linear momentum as stated below.

<u>**Conservation of Linear Momentum**</u>

The total linear momentum of an isolated system, i.e., one in which there are no external forces, remains constant.

$$\sum_{k=1}^{N} \boldsymbol{p}_{k,i} = \sum_{k=1}^{N} \boldsymbol{p}_{k,f}$$

The analog to the linear momentum form of Newton's 2nd law for rotational motion can be derived in much the same way that it was for translational motion.

$$\boldsymbol{\tau}_{net} = I\boldsymbol{\alpha} = I\frac{d\boldsymbol{\omega}}{dt} = \frac{d(I\boldsymbol{\omega})}{dt}$$

And with $\boldsymbol{L} = I\boldsymbol{\omega}$ we have:

$$\boldsymbol{\tau}_{net} = \frac{d\boldsymbol{L}}{dt}$$

The conservation of angular momentum follows from this equation in the same way it did for linear momentum. The law is stated below without proof.

<u>**Conservation of Angular Momentum**</u>

The total angular momentum of an isolated rotating body, i.e., one in which there are no external torques, remains constant.

$$\sum_{k=1}^{N} \boldsymbol{L}_{k,i} = \sum_{k=1}^{N} \boldsymbol{L}_{k,f}$$

Consider a well-known example of a spinning skater. The skater starts out rotating at a relatively low speed with their arms extended. After some time, the skater pulls their arms in, thereby reducing the rotational inertia. Since the angular momentum must remain constant, the skaters rotational speed will increase. As a concrete example, let's assume the rotational inertia of the skater was reduced by a factor of $1/2$ after pulling in their arms. The new angular velocity is then found as follows:

$$I_{armsIn}\boldsymbol{\omega}_{armsIn} = I_{armsOut}\boldsymbol{\omega}_{armsOut}$$

$$\left(\frac{1}{2}I_{armsOut}\right)\boldsymbol{\omega}_{armsIn} = I_{armsOut}\boldsymbol{\omega}_{armsOut}$$

$$\boxed{\boldsymbol{\omega}_{armsIn} = 2\boldsymbol{\omega}_{armsOut}}$$

Example 1: A man stands in the center of a rotating platform. The rotational inertia of the system, consisting of the man and the platform, is $I_{mp} = 4.5\ kg \cdot m^2$. The man is then handed two $5\ kg$ bricks. He holds one in each hand with his arms extended and the platform begins to spin, (without friction), at an angular speed of $1.5\ rev/sec$. He then pulls his arms in so that the bricks are closer to his body. Find the new angular speed of the system. With his arms extended the bricks are $r_i = 0.76\ m$ from his body, and when he pulls them in the bricks are $r_f = 0.24\ m$ from his body.

Solution 1: This problem can be solved using the conservation of angular momentum. Treating the bricks as point object the initial angular momentum of the system is given by.

$$L_i = I_{i,s}\omega_i$$
$$L_i = \left(I_{mp} + 2I_{i,B}\right)\omega_i$$
$$L_i = \left(I_{mp} + 2m_B {r_i}^2\right)\omega_i$$

Similarly, the final angular momentum of the system is given by

$$L_f = I_{f,s}\omega_f$$
$$L_f = \left(I_{mp} + 2I_{f,B}\right)\omega_f$$
$$L_f = \left(I_{mp} + 2m_B {r_f}^2\right)\omega_f$$

We can now use the conservation of angular momentum to solve for the final angular speed.

$$L_f = L_i$$
$$\left(I_{mp} + 2m_B {r_f}^2\right)\omega_f = \left(I_{mp} + 2m_B {r_i}^2\right)\omega_i$$
$$\omega_f = \frac{\left(I_{mp} + 2m_B {r_i}^2\right)}{\left(I_{mp} + 2m_B {r_f}^2\right)}\omega_i$$
$$\omega_f = \frac{(4.5 + 2 \cdot 5 \cdot 0.75^2)}{(4.5 + 2 \cdot 5 \cdot 0.24^2)}\omega_i$$
$$\omega_f = 2\omega_i$$
$$\omega_f = 3\ rev/sec$$

Example 2: Neutron stars are believed to be formed from a larger star that has collapsed from its own gravitation. Suppose a star with the radius of our sun, ($6.96E^5\ km$), and twice the mass of our sun, ($2 \cdot 1.99E^{30}\ kg$), was rotating at $0.1\ rev/day$ before collapse. If it were to undergo gravitational collapse to a neutron star of radius $10\ km$, what would the angular speed be? How much work was done by gravity as the star collapsed?

Solution 2: With no external torques acting on the star the angular momentum is conserved.

$$L_f = L_i$$

$$I_f\omega_f = I_i\omega_i$$

$$\omega_f = \frac{I_i}{I_f}\omega_i$$

$$\omega_f = \frac{\frac{1}{2}mR_i^{\,2}}{\frac{1}{2}mR_f^{\,2}}\omega_i$$

$$\omega_f = \frac{(6.96\ x\ 10^5\ )^2}{(10\ )^2}(0.1\ rev/day) = 4.84\ x\ 10^8\ rev/day$$

The work done by gravity can be determined from the change in the rotational kinetic energy.

$$W = \Delta K$$

$$W = K_f - K_i$$

$$W = \frac{1}{2}I_f\omega_f^{\,2} - \frac{1}{2}I_i\omega_i^{\,2}$$

$$W = \frac{1}{2}mR_f^{\,2}\omega_f^{\,2} - \frac{1}{2}mR_i^{\,2}\omega_f^{\,2}$$

$$W = \frac{1}{2}m\left(\left(R_f\omega_f\right)^2 - (R_i\omega_i)^2\right)$$

The angular speed should first be converted to radians per second.

$$\omega_f = \left(4.84\ x\ 10^8\ \frac{rev}{day}\right) \cdot \left(\frac{1\ day}{24 \cdot 3600\ sec}\right) \cdot \left(\frac{2\pi\ rad}{1\ rev}\right) = 4.57\ x\ 10^{11}\ rad/sec$$

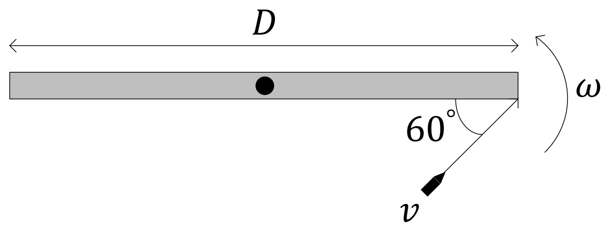
$$\omega_i = \left(0.1\ \frac{rev}{day}\right) \cdot \left(\frac{1\ day}{24 \cdot 3600\ sec}\right) \cdot \left(\frac{2\pi\ rad}{1\ rev}\right) = 94.25\ rad/sec$$

Now we can find the work done, which we can see is a very large number!

$$W = \frac{1}{2}2 \cdot 1.99E^{30}((10E^3 \cdot 4.57E^{11})^2 - (6.96E^8 \cdot 94.25)^2)$$

$$W = 4.16E^{61}\,J$$

Example 3: A uniform rod of length $0.5\ m$ and mass of $4\ kg$ can rotate in a horizontal plane about a vertical axis through its center. The rod is initially at rest when a bullet of mass $0.003\ kg$ is fired at an angle of 60°. If the bullet gets lodges in the rod and the angular velocity of the rod is $10\ rad/s$ immediately after the collision, find the speed of the bullet right before impact.



Solution 3: Taking the bullet and the rod as our system we use the conservation of angular momentum to write the following relationship.

$$L_{i,r} + L_{i,b} = L_{f,r} + L_{f,b}$$

Since the rod is initially at rest, $L_{i,r} = 0$. The axis of rotation for this system is located at the center of the rod. The bullet, which is treated as a point particle, is not *initially* traveling on a circular path with respect to this point. For this case, we use the cross product definition for the momentum of a point particle. However, after the bullet is lodged in the rod it begins to travel in a circle, i.e., $I = mR^2$. Using this we can solve for the initial speed of the bullet as shown below.

$$0 + L_{i,b} = \left(I_{f,r} + I_{f,b}\right)\omega_{f,rb}$$

$$m_b\left(\frac{D}{2}\right)v\,sin(60^{\circ}) = \left(\frac{1}{12}m_r D^2 + m_b\left(\frac{D}{2}\right)^2\right)\omega_{f,rb}$$

$$v = \frac{D^2\left(\frac{1}{12}m_r + \frac{1}{4}m_b\right)}{m_b\left(\frac{D}{2}\right)sin(60^{\circ})}\omega_{f,rb}$$

$$v = \frac{2D\left(\frac{1}{12}m_r + \frac{1}{4}m_b\right)}{m_b\,sin(60^{\circ})}\omega_{f,rb}$$

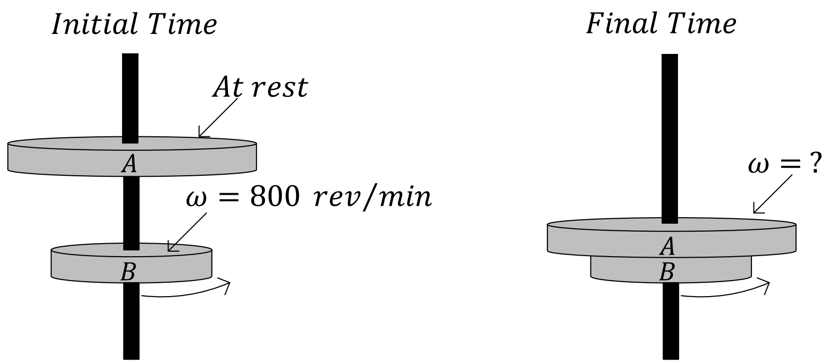
$$v = \frac{2\cdot 0.5\left(\frac{1}{12}\cdot 4 + \frac{1}{4}0.003\right)}{0.003\,sin(60^{\circ})}\omega_{f,rb}$$

$$v = 128.59\cdot\omega_{f,rb}$$

$$v = 1286\cdot m/s$$

Example 4: A wheel is rotating freely at an angular speed of $900\ rev/sec$ on a shaft whose rotational inertia is negligible. A second wheel, initially at rest and with twice the rotational inertia of the first, is suddenly coupled to the same shaft.

a) What is the angular speed of the after the second wheel is coupled to the first?
b) What fraction of the original kinetic energy is lost?



Solution 4a: The angular momentum of the system, i.e., both wheels, is conserved.

$$L_f = L_i$$

$$(I_A + I_B)\omega_{f,AB} = \left(I_A\omega_{i,A} + I_B\omega_{i,B}\right)$$

$$\omega_{f,AB} = \frac{\left(I_A\omega_{i,A} + I_B\omega_{i,B}\right)}{(I_A + I_B)}$$

Substituting $\omega_{i,B} = 0$, $\omega_{i,A} = 900\ rev/sec$, and $I_B = 2I_A$ we have the following

$$\omega_{f,AB} = \frac{(I_A \cdot 900 + 0)}{(I_A + 2I_A)}$$

$$\omega_{f,AB} = \frac{900}{3} = 300\ rev/min$$

b: The percentage change in the rotational kinetic energy is found as follows.

$$\Delta K_{frac} = \frac{K_f - K_i}{K_i}$$

$$\Delta K_{frac} = \frac{\frac{1}{2}\left(3I_A\left(\omega_{f,AB}\right)^2\right) - \frac{1}{2}\left(I_A\left(3\omega_{f,AB}\right)^2\right)}{\frac{1}{2}\left(I_A\left(3\omega_{f,AB}\right)^2\right)}$$

$$\Delta K_{frac} = \frac{(3) - (9)}{(9)} = -\frac{2}{3}$$

Therefore, about 67% of the initial kinetic energy is lost. The energy lost is due to the friction force between the two wheels, which allowed the two wheels to eventually reach the same angular speed.

Final Summary for Angular Momentum

Angular Momentum
The angular momentum of a rigid body rotating about a fixed axis is given as: $$\boldsymbol{L} = I\boldsymbol{\omega}$$ The angular momentum of a point particle rotating about a fixed axis is given as: $$\boldsymbol{L}_{par} = m(\boldsymbol{r}\ x\ \boldsymbol{v})$$ $$\boldsymbol{L}_{par} = mrv\sin(\theta)$$ For a particle in circular motion $\theta = 90^{\circ}$ and: $$\boldsymbol{L}_{par} = mrv$$
The Law of The Conservation of Angular Momentum
The total angular momentum of an isolated rotating body, i.e., one in which there are no external torques, remains constant. $$\boldsymbol{L}_i = \boldsymbol{L}_f$$ $$I_i\boldsymbol{\omega_i} = I_f\boldsymbol{\omega_f}$$

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