Physics 1 Mechanics – Conservation of Energy

The conservation of energy principle is one of the most important (and useful) laws in physics. In previous lessons we said that all energy can be classified into two categories: kinetic energy and potential energy. We also related the change in these quantities to work as follows:

$$W = \Delta K \qquad\qquad W_c = -\Delta U$$

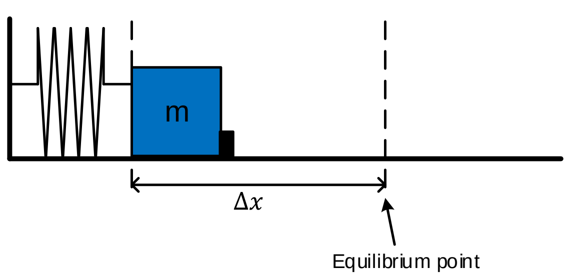
Where, we are reminded that the potential energy applies for conservative forces only. Therefore, if we assume a system with only conservative forces, we can equate the work from these equations to get the following fundamental result.

$$-\Delta U = \Delta K$$
$$U_i - U_f = K_f - K_i$$
$$U_i + K_i = U_f + K_f$$
$$E_{(mec)i} = E_{(mec)f}$$

This is a mathematical statement of *The Principle of the Conversation of Mechanical Energy*, where we define the mechanical energy of a system as the sum of the kinetic and potential energy due to conservative forces only.

The Principle of the Conservation of Mechanical Energy
The mechanical energy of a system is defined as the sum of the kinetic and potential energy. $$E_{(mec)} = U + K$$ The kinetic and potential energy may change, but the total mechanical energy remains constant in a closed system where only conservative forces are present. $$E_{(mec)i} = E_{(mec)f}$$ $$U_i + K_i = U_f + K_f$$

Let's explore this concept further with an example. Consider a block of mass m attached to a compressed spring being held by a small block.



The system consists of the block and the spring. In this initial configuration, there is only potential energy of the spring given as follows.

$$U_i = \frac{1}{2}k(\Delta x)^2$$

When the block is released the spring will begin to move towards its equilibrium position, transferring its potential energy to the block as kinetic energy. If the surface is frictionless the only force acting in the system is the *conservative* elastic force. Therefore, at any point in time the total mechanical energy will remain constant even as it is being transferred between the spring and the block. For example, when the block returns to the equilibrium position all of its potential energy will have been transferred to the block as kinetic energy. Assuming this is the final state we can write the following.

$$E_{(mec)i} = E_{(mec)f}$$

$$U_i + \cancel{K_i} = \cancel{U_f} + K_f$$

$$\frac{1}{2}k(\Delta x)^2 = \frac{1}{2}m(v)^2$$

Next, let's investigate what happens when the surface is ***not*** frictionless. In this case, the friction force, which is ***non-conservative***, will also perform work. This non-conservative work, W_{nc}, done by friction increases the thermal energy of the block and the surface. (*Note: thermal energy is a form of non-mechanical kinetic energy that accounts for the motion of the particles within an object*). For systems with only conservative forces the total energy of the system remains constant and usable as a form of mechanical energy. In other words, both the potential and kinetic energy may change within a process, however, the changes must offset each other so that no energy is lost or gained. This can be expressed as follows.

$$\Delta U + \Delta K = 0$$

When a non-conservative force is acting some energy is transferred to non-mechanical energy, e.g., thermal energy, due to the work done by the non-conservative forces, therefore we have:

$$\Delta U + \Delta K = W_{nc}$$

Using the definitions of ΔU and ΔK, we can re-write the above in a form that is commonly used to solve conservation of energy problems.

$$U_f - U_i + K_f - K_i = W_{nc}$$

$$U_i + K_i + W_{nc} = U_f + K_f$$

For the specific case above, where friction is the non-conservative force, we have $W_{nc} = -f\Delta x$.

$$U_i + K_i - f\Delta x = U_f + K_f$$

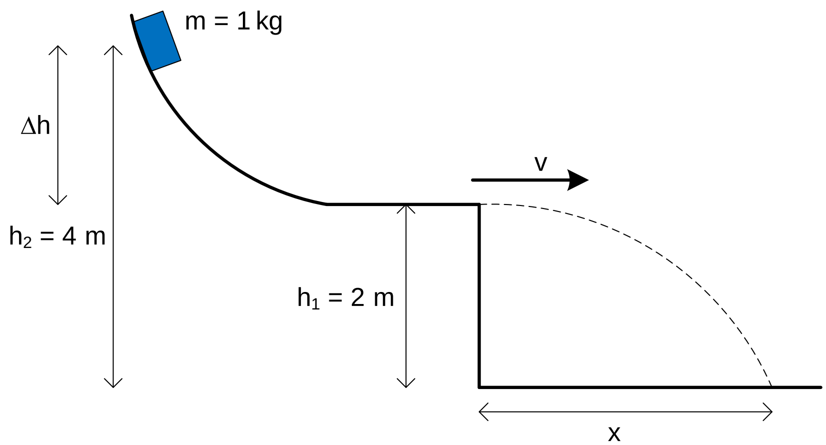
$$E_{(mec)i} - f\Delta x = E_{(mec)f}$$

Which clearly shows that the final mechanical energy is equal to the initial mechanical energy minus the thermal energy lost due to the non-conservative friction force.

Let's do some examples.

Example 1: A block starts at rest and slides down a frictionless track. It leaves the track horizontally, flies through the air, and subsequently strikes the ground.

a.) What is the speed of the block when it leave the ramp?
b.) What horizontal distance does the block travel in the air?
c.) What is the speed of the block when it hits the ground?



Solution 1a: We choose the time when the block is at the top of the ramp as the initial time and the instant it leaves the ramp as the final time. Then, since the track is frictionless, we can write the conservation of energy as follows.

$$U_i + K_i = U_f + K_f$$

The block is initially at rest, i.e., $K_i = 0$. Furthermore, if we designate the bottom of the ramp as the zero potential energy location, i.e., $U_G(y = h_1) = 0$, then $U_f = 0$. With this we can solve for the speed of the block as follows.

$$U_i = K_f$$

$$mg\Delta h = \frac{1}{2}mv^2$$

$$v = \sqrt{2g\Delta h}$$

$$v = \sqrt{2 \cdot 9.8 \cdot 2}$$

$$v = 6.3\ m/s$$

1b.) The horizontal distance the block travels before reaching the ground can be found using kinematics. We first find the time for the block to reach the ground using the y component.

$$y(t) = y(0) + v_y(0)t + \frac{1}{2}gt^2$$

$$0 = h_1 + 0 - \frac{g}{2}t^2$$

$$t = \sqrt{\frac{2h_1}{g}}$$

Then, since the velocity in the x direction is constant, we can use this time to find the horizontal distance traveled.

$$x = vt = v\sqrt{\frac{2h_1}{g}} = 6.3\sqrt{\frac{2 \cdot 2}{9.8}} = 4\ m$$

1c.) To find the speed of the block just before it hits the ground, we again use conservation of energy. In this case, we choose the ground to be at zero potential energy, i.e., $U_G(y = 0) = 0$, and we choose the initial time to be when the block is at the top of the ramp. In other words, all the potential energy the block has at the top of the ramp will be transferred to kinetic energy just before it hits the ground.

$$U_i = K_f$$

$$mgh_2 = \frac{1}{2}mv^2$$

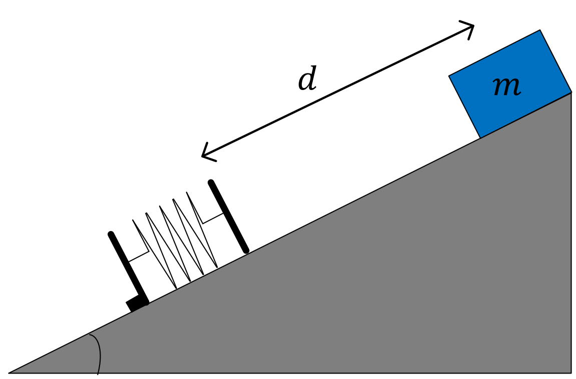
$$v = \sqrt{2gh_2}$$

$$v = \sqrt{2 \cdot 9.8 \cdot 4}$$

$$v = 8.9\ m/s$$

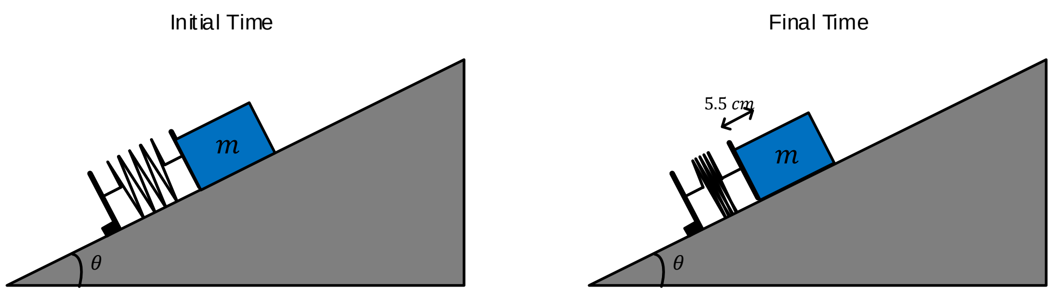
Example 2: A block of mass $m = 12\ kg$ is at rest at the top of a frictionless ramp with a 30° incline. There is a spring with a spring constant of $k = 13500\ N/m$ further down the incline. The block is released, and it comes to a momentary stop when the spring is compressed by $5.5\ cm$.

a.) What is the speed of the block when it first contacts the spring?
b.) How far up the ramp, as measured from the top of the uncompressed spring, was the block when it was released?

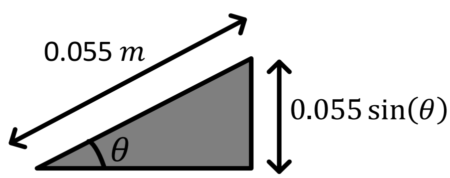


Solution 2a: For the block to compress the spring by $5.5\ cm$ it must have the appropriate kinetic energy when it reaches the spring. This kinetic energy is obtained as a result of the initial gravitational potential energy of the block. When the block reaches the spring, it will begin to slow down, transferring its kinetic energy to potential energy of the spring. It's important to note that the block also continues to lose gravitational potential energy while it is compressing the spring.

We'll set up the conservation of mechanical energy equation with the initial time being when the block first touches the spring and the final time being when the block comes to temporary rest.



If we designate the zero gravitational potential energy at the location where the spring is fully compressed, then the block has an initial height, h, of $0.055\sin(30)$.



We can now write the conservation of mechanical energy equation and solve for the speed of the block when it first comes into contact with the spring.

$$U_i + K_i = U_f + K_f$$

$$mgh + \frac{1}{2}mv^2 = \frac{1}{2}kx^2 + 0$$

$$mv^2 = kx^2 - 2mgh$$

$$v = \sqrt{\frac{kx^2 - 2mgh}{m}}$$

$$v = \sqrt{\frac{13500(0.055^2) - 2 \cdot 12 \cdot 9.8 \cdot 0.055\, sin(30)}{12}}$$

$$v = 1.7\ m/s$$

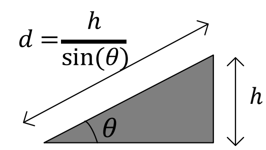
2b.) From part a.) we know the block attained a speed of $1.7\ m/s$ when it first contacted the spring. Since the block obtained this kinetic energy from its initial gravitational potential energy, we can find the height of the ramp using the conservation of energy equation again. In this case, we choose the initial time to be when the block is at the top of the incline and the final time when the block first touches the spring.

$$U_i + K_i = U_f + K_f$$

$$\cancel{m}gh + 0 = \frac{1}{2}\cancel{m}v^2 + 0$$

$$h = \frac{v^2}{2g}$$

This gives us the height from the uncompressed spring to the top of the ramp, but the question asks for the distance up the ramp. We can relate this distance to the height using the following right triangle.



Therefore, the distance along the ramp above the spring where the block started, d, is

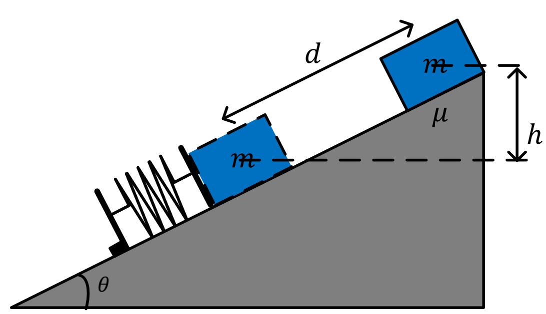
$$d = \frac{\frac{v^2}{2g}}{sin(\theta)}$$

$$d = \frac{1.7^2}{2 \cdot 9.8\, sin(30)}$$

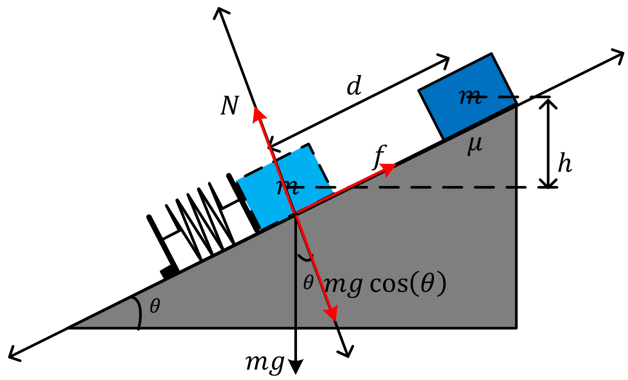
$$d = 0.295\ m$$

Example 3: Let's take the same scenario from example 2, but without the incline surface having a $u = 0.3$. We'll also assume the block is released from the same location, i.e., $d = 0.295\ m$.

a.) What is the speed of the block when it reaches the spring?
b.) How far does the spring compress before the block temporarily stops?



Solution 3a: Since the ramp is no longer frictionless, we need to use the more general version of the conservation of energy equation, which includes a term for non-conservative work. A new figure showing the initial and final times as well as frictional force is shown below.



When writing the conservation of energy equation, we use the same initial and final times that were used in 2a - the initial time is right before the block is released and the final time is the moment the block first touches the spring. The difference in this case is that the block will lose energy from friction while sliding down the ramp. As we know, the force of friction is given as

$$f = \mu N$$

Since the block is not accelerating perpendicular to the ramp, we can see from the figure that the normal force is equal to $mg\cos(\theta)$. The non-conservative work is then given as

$$W_{nc} = -\mu N d$$
$$W_{nc} = -\mu mg\, cos(\theta)\, d$$

The conservation of energy equation can now be used to find the speed of the block as shown.

$$U_i + K_i + W_{nc} = U_f + K_f$$
$$mgh + 0 - \mu mg\, cos(\theta)\, d = 0 + \frac{1}{2}mv^2$$
$$\frac{1}{2}v^2 = gh - \mu g\, cos(\theta)\, d$$
$$v = \sqrt{2(gh - \mu g\, cos(\theta)\, d)}$$

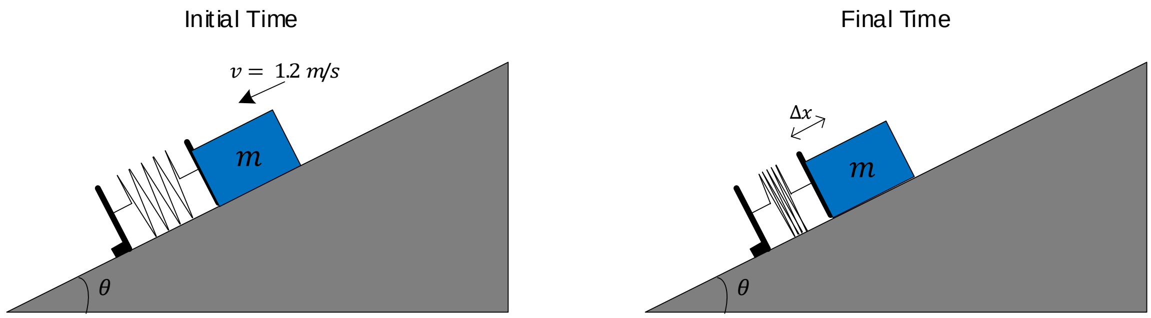
Referring to example 2b we can express h in terms of d as $h = d\, sin(\theta)$. Substituting we have

$$v = \sqrt{2gd(sin(\theta) - \mu\, cos(\theta))}$$
$$v = \sqrt{2 \cdot 9.8 \cdot 0.295(sin(30) - 0.3\, cos(30))}$$
$$v = 1.2\ m/s$$

Recall, in example 2 we found $v = 1.7\, m/s$. We obtain this same value if we set $\mu = 0$ in the above equation.

$$v = \sqrt{2gd(sin(\theta) - 0)} = \sqrt{2 \cdot 9.8 \cdot 0.295(sin(30))} = 1.7\ m/s$$

3b.) To see how far the spring is compressed we can refer to the diagrams in part 2a, which are copied here replacing $5.5\ cm$ with an unknown distance Δx and also noting the known initial velocity.



The conservation of energy equation will be similar to the one in 2a except for the added non-conservative work component as shown below. Furthermore, to get obtain an equation with Δx as the only variable we replace the height difference that results from the spring compression using $\Delta h = \Delta x\, sin(\theta)$.

$$U_i + K_i + W_{nc} = U_f + K_f$$

$$mg\Delta x\, sin(\theta) + \frac{1}{2}mv^2 - \mu mg\, cos(\theta)\, \Delta x = \frac{1}{2}k\Delta x^2 + 0$$

We can rearrange the equation to be a quadratic in Δx as follows.

$$\frac{1}{2}k\Delta x^2 + (\mu mg\, cos(\theta) - mg\, sin(\theta))\Delta x - \frac{1}{2}mv^2 = 0$$

$$6750\Delta x^2 + 12 \cdot 9.8(\, 0.3\, cos(30) - sin(30))\Delta x - 8.64 = 0$$

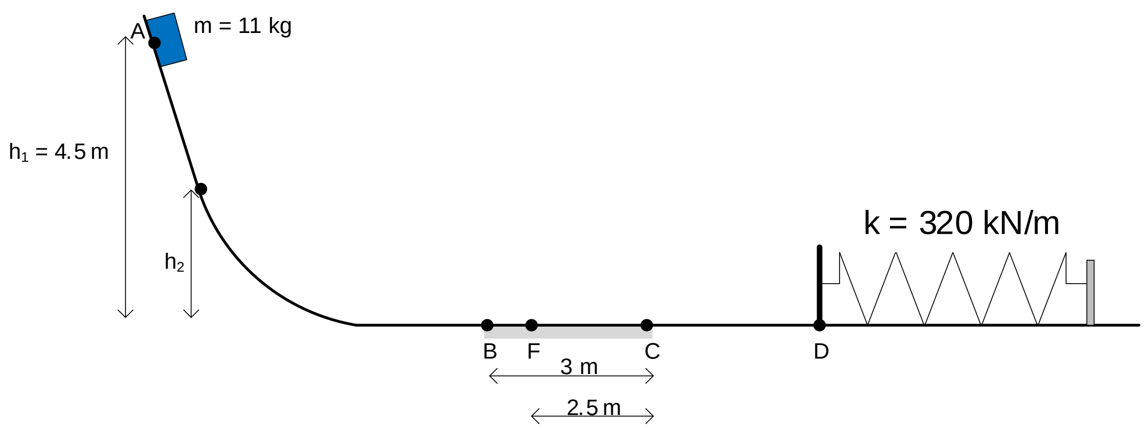
$$(6750)\Delta x^2 - (28.25)\Delta x - 8.64 = 0$$

Using the quadratic formula and eliminating the negative term we find $\Delta x = 3.8\ cm$.

Compare this value to the value in example 2, without friction, where the spring was compressed $5.5\ cm$. Note also that if we use the speed from example 2, $v = 1.7$ and a $u = 0$, in the quadratic equation above we would indeed find $\Delta x = 5.5\ cm$, as expected.

Example 4: A block is released from point A on a track ABCD as shown in the figure. The track is frictionless except for a portion BC which has a coefficient of friction u. The block travels down the track and hits the spring with a spring constant k.

a.) If the spring compresses by 5 cm, find u.
b.) Using the value of u found in a.) how far will the block travel back up the hill, i.e., $h_2 = ?$
c.) What would be the value of u if the block hit the spring and only traveled back to point F?



Solution 4a: The block starts at point A with gravitational potential energy mgh_1 and arrives at point B with all of the potential energy transferred to kinetic energy. As it slides across the track some energy is lost to friction as it passes over BC. The block then hits the spring at D and transfers the remaining part of its kinetic energy to the spring, compressing it by $5\ cm$. With this instant in time being the final state, and the top of the ramp as the initial state, we can use the principal of conservation of energy to write the following.

$$U_i + K_i + W_{nc} = U_f + K_f$$

$$mgh_1 - umgd_{BC} = \frac{1}{2}kx^2$$

Next, we solve for u by rearranging and substituting known values.

$$umgd_{BC} = \frac{mgh_1 - \frac{1}{2}kx^2}{mgd_{BC}}$$

$$u = \frac{11 \cdot 9.8 \cdot 4.5 - 0.5 \cdot 320E^3(0.05)^2}{11 \cdot 9.8 \cdot 3}$$

$$u = 0.26$$

4b: When the spring is fully compressed, all of the energy of the system, less the amount lost to friction, exists in the form of elastic potential energy of the spring. As the spring returns back to its equilibrium position, it transfers all of this energy back to the block. Then, as the block begins to move back towards the hill, it once again passes through BC and loses some if its energy. Finally, what energy remains is transferred to gravitational potential energy, allowing the block to reach a height of h_2. The conservation of energy equation for the process is as follows.

$$U_i + K_i + W_{nc} = U_f + K_f$$
$$\frac{1}{2}kx^2 - umgd_{BC} = mgh_2$$

In this case, we solve for h_2 by again rearranging and substituting known values.

$$h_2 = \frac{\frac{1}{2}kx^2 - umgd_{BC}}{mg}$$
$$h_2 = \frac{0.5 \cdot 320E^3(0.05)^2 - 0.26 \cdot 11 \cdot 9.8 \cdot 3}{11 \cdot 9.8}$$
$$h_2 = 2.9\ m$$

4c: In this case, the block only makes it to the point F, i.e., 2.5 m on the way back. Since the spring returns all of the energy is gets from the block as it returns to it equilibrium position, we can ignore its part. Instead, we can assume the initial energy at the top of the hill is totally lost to friction. The distance the block travels in the friction patch is 3 m on the way forward and 2.5 m on the way back. Therefore, we have

$$umg(d_{BC} + d_{BF}) = mgh_1$$
$$u = \frac{h_1}{(d_{BC} + d_{BF})}$$
$$u = \frac{4.5}{3 + 2.5}$$
$$u = 0.82$$

Additional Insight: To gain some additional insight let's look more closely at the simple case of an object being dropped from a tall building.

An object being dropped from a building of height h will have an initial energy of $E = mgh$. As the object falls it will lose potential energy and gain kinetic energy, given by $\frac{1}{2}mv^2$.

Ignoring drag force the total energy content of the ball-earth system must always remain the same (that's of course until the object lands and converts all this energy into various forms such as thermal, sound, etc.). Therefore, as the object is in free fall the following equation must be satisfied.

$$U(y) = K(y)$$
$$mgy = \frac{1}{2}mv^2$$

And for an object in free fall, we know from kinematics that the position and speed of the object can be written as functions of time as:

$$y(t) = h - \frac{1}{2}gt^2$$
$$v(t) = gt$$

Using the position equation, $y(t)$, we can find the time when the object hits the ground as:

$$0 = h - \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{2h}{g}}$$

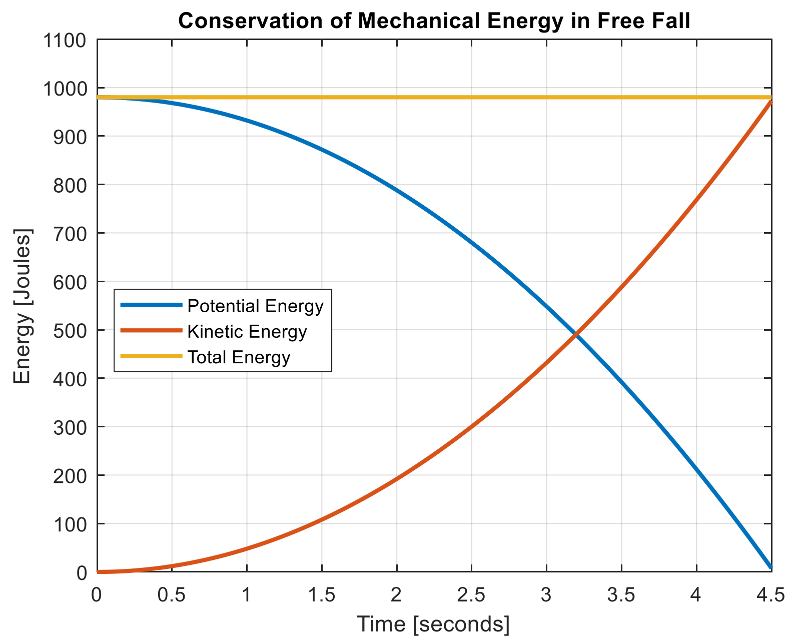
Substituting the position and speed equations into the expressions for potential and kinetic energy we can write them as functions of time.

$$U\big(y(t)\big) = mg\left(h - \frac{1}{2}gt^2\right) = U(t)$$
$$K\big(v(t)\big) = \frac{1}{2}m(gt)^2 = K(t)$$

The following equation must be satisfied for all time the ball is in free-fall.

$$E(t) = U(t) + K(t) = mgh$$

For illustration, the figure below shows plots of $U(t)$, $K(t)$, and $E(t)$ for a 1 kg mass object being dropped from a 100 m tall building.



Plot Highlights:

- The total energy is constant for all times and equal to $mgh = 1 \cdot 9.8 \cdot 100 = 980\ J$.
- The kinetic energy starts at zero and ends with exactly $980\ J$, right before impact.
- The potential energy starts with $980\ J$, and ends with $0\ J$, right before impact.
- The rate of change for both the potential and kinetic energy over time is non-linear. The rate increases as time increases.

Final Summary for Conservation of Energy

The Principle of the Conservation of Mechanical Energy

The mechanical energy of a system is defined as the sum of the kinetic and potential energy.

$$E_{(mec)} = U + K$$

The kinetic and potential energy may change, but the total mechanical energy remains constant in a closed system where only conservative forces are present.

$$\Delta U + \Delta K = 0$$

$$U_i + K_i = U_f + K_f$$

The Principle of the Conservation of Energy

When a non-conservative force acts to cause an energy transfer, this energy is not recoverable, and the total energy of the system no longer remains constant.

$$\Delta U + \Delta K = W_{nc}$$

Where, W_{nc} is the work performed by the non-conservative force.

In this case, we can write the conservation of energy as follows.

$$\left(U_f - U_i\right) + \left(K_f - K_i\right) = W_{nc}$$

$$U_i + K_i + W_{nc} = U_f + K_f$$

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