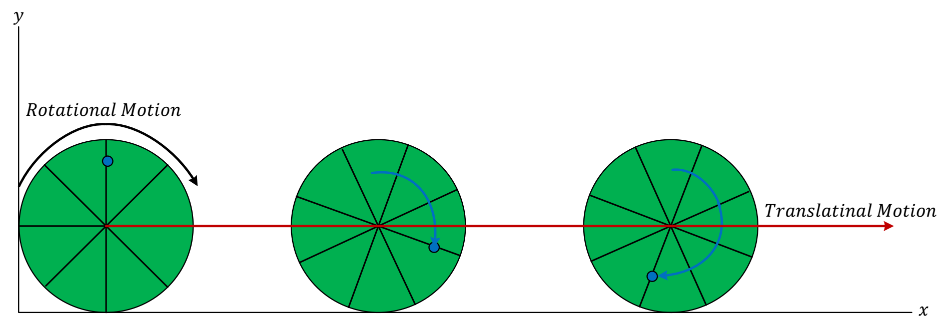
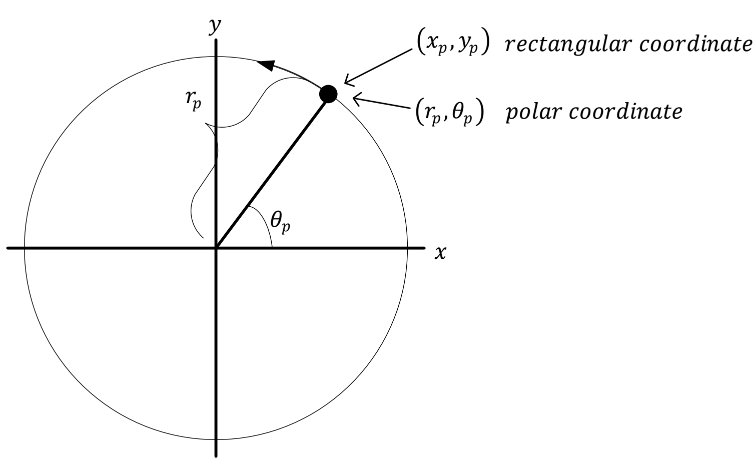
Physics 1 Mechanics - Rotational Kinematics

In the previous lesson on center of mass, we noted that a rigid body can undergo both translational and rotational motion. As we mentioned, pure translational motion is when the entire body moves from one location in space to another, whereas pure rotational motion is when all points move in circles around center point, or more specifically a line through the center, i.e., an axis of rotation.



When describing pure translational motion, we generally use a rectangular coordinate system, in which we locate points in $2D$ space by their x and y coordinates. An alternate coordinate system for $2D$ space is called a $polar\ coordinate\ system$. In a polar coordinate system, a point is identified by its straight-line distance, r, measured from the axis of rotation to the given point and by the angle, θ, measured between this straight line and some reference line, (usually the positive x-axis). When dealing with pure rotational motion we'll find it much more convenient to use the polar coordinate system. We can see why by examining the figure below, which shows a point in $2D$ space that is moving on a circular path. Any point can be located by its (x, y) coordinates, or equivalently by its (r, θ) coordinates. While listing the (x, y) coordinates along this circular path is possible, listing the (r, θ) coordinates is much simpler. Namely, the r coordinate would remain unchanged while the θ coordinate would simply change from 0 to 2π.



The rectangular coordinate system proved convenient when dealing with linear kinematics. For example, we could track a particle moving along a horizontal straight-line path using only its x coordinate, since the y coordinate is constant. In this case the kinematic quantities we used to describe the moving particle were its *position*, $x(t)$, *velocity*, $v_x(t)$, and *acceleration*, $a_x(t)$.

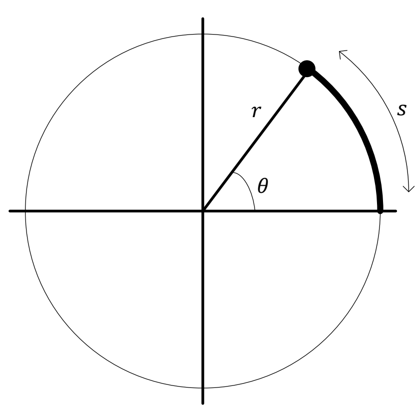
For a particle moving along a circular path, we will find it much more convenient to track a particle using polar coordinates. As an analogue to the linear example above, we could track a particle that is moving along a constant radius circular path using only its θ coordinate, since the r coordinate is constant. We can extend this analogy in order to build rotational kinematic quantities that are similar to the linear kinematics quantities from above. We do this by first substituting $\theta(t)$ for $x(t)$. The velocity would then refer to the rate of change of θ with respect to time, i.e., the angular velocity, $\omega(t)$. Finally, the rate of change of the angular velocity, referred to as the angular acceleration, is symbolized by $\alpha(t)$.

	Linear	***Correspondence***	***Rotational / Angular***
Position	$x(t)$	$x \leftrightarrow \theta$	$\theta(t)$
Velocity	$v_x(t) = \dfrac{dx(t)}{dt}$	$v \leftrightarrow \omega$	$\omega(t) = \dfrac{d\theta(t)}{dt}$
Acceleration	$a_x(t) = \dfrac{dv_x(t)}{dt}$	$a \leftrightarrow \alpha$	$\alpha(t) = \dfrac{d\omega(t)}{dt}$

Using the correspondence from the above table we can directly map the linear kinematic equations we developed in a previous lesson to their rotational/angular counterparts.

Linear Kinematic Equations	***Angular Kinematic Equations***
$x_f = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v_f = v_0 + at$	$\omega_f = \omega_0 + \alpha t$
${v_f}^2 = {v_0}^2 + 2a(x_f - x_0)$	${\omega_f}^2 = {\omega_0}^2 + 2\alpha(\theta_f - \theta_0)$
$x_f = x_0 + \frac{1}{2}(v_f + v_0)t$	$\theta_f = \theta_0 + \frac{1}{2}(\omega_f + \omega_0)t$

Next, in order to develop mathematical relationships between the linear and angular quantities we must first recall the difference between distance and displacement. While distance refers to the total length along a path, displacement is a measure of a straight-line path between the starting and ending position. In linear kinematics, we focused on displacement, however for motion on a circular path we are more concerned with distance. To make the distinction clear we will use the symbol, s, to track the distance a particle travels along a circular path. We will generate relationships between the angular and linear quantities using the figure below.



We start by writing a proportion that equates the ratio of the number of degrees swept and the total number of degrees around a circle to the ratio of the distance traveled and the total distance around the circle.

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$
$$s = r\theta$$

Where, s refers to the distance measured along the circular path, and θ refer to the angle swept along this path.

The remaining relationships are obtained by differentiating with respect to time.

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$
$$v_t = r\omega$$

Where, v_t, refers to the tangential velocity, and ω refers to the angular velocity.

Differentiating again gives us a relationship between the tangential acceleration, a_t, and the angular acceleration, α.

$$\frac{dv_t}{dt} = r\frac{d\omega}{dt}$$
$$a_t = r\alpha$$

However, as you may recall when we introduced circular motion, we discovered that when a particle moves along a circular path it may have radial accelerations, a_r, as well as the tangential acceleration given above. The tangential component accounts for the change in the tangential velocity, while the radial component account for the change in direction as the particle traverses the circle. The radial acceleration was given as:

$$a_r = \frac{{v_t}^2}{r}$$

Substituting the tangential velocity for angular velocity we have:

$$a_r = \frac{r^2\omega^2}{r}$$
$$a_r = r\omega^2$$

The mathematical relationships between linear and angular quantities are summarized here.

Distance	***Velocity***	***Acceleration***	
$x = r\theta$	$v_t = r\omega$	$a_t = r\alpha$	$a_r = \frac{v^2}{r} = r\omega^2$

Example 1: A high speed sander has a disk of radius 2.5 cm and rotates about its axis at a constant rate of $1485\ rev/min$.

a) What is the angular speed of the disk?
b) What is the linear speed for a point on the edge of the disk and a point 1.5 cm from the center?
c) What is the total distance traveled in 2.5 s by a point on the on the edge of the disk and a point 1.5 cm from the center?
d) What is the linear acceleration for a point on the edge of the disk and a point 1.5 cm from the center?

Solution 1a: The angular speed is generally required in radians per second. We perform the conversion similar to any other unit conversion below.

$$\omega = 1485\ \frac{\cancel{rev}}{\cancel{min}} \cdot \frac{1\ \cancel{min}}{60\ sec} \cdot \frac{2\pi\ rad}{1\ \cancel{rev}} = 155.5\ rad/sec$$

1b: We use the following equation to convert from angular speed to linear speed.

$$v_t = r\omega$$

Note that the conversion is proportional to the radius, r, i.e., a point on the edge of a circle will travel a longer distance that a particle closer to the center in the same amount of time. At the extreme, a particle at the center of the circle, i.e., $r = 0$, will have a linear speed of 0.

Point on edge of Circle	***Point at 1.5 cm from the Center***
$v_t = 0.025 \cdot 155.5 = 3.9\ m/s$	$v_t = 0.015 \cdot 155.5 = 2.3\ m/s$

1c: The distance traveled is also proportional to the radius.

$$x = r\theta$$

However, we first need to find the degrees swept out in a certain time, which can be computed with one of the angular kinematic equations as follows.

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\theta = \omega t$$

The distance traveled by the two different points in 2.5 s is then computed as shown.

Point on edge of Circle	***Point at 1.5 cm from the Center***
$x = r\omega t$ $x = 0.025 \cdot 155.5 \cdot 2.5 = 9.7\ m$	$x = r\omega t$ $x = 0.015 \cdot 155.5 \cdot 2.5 = 5.8\ m$

1d: The acceleration has two components, tangential and radial. As mentioned, the tangential acceleration accounts for the change in angular speed, ω, and so since the angular speed is constant, the tangential acceleration component is zero.

$$a_t = r\alpha = r \cdot 0 = 0$$

The radial component of acceleration, however, accounts for the change in direction and is given as follows.

$$a_r = r\omega^2$$

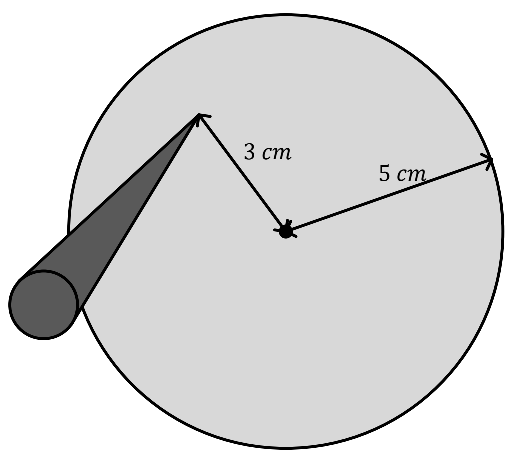
This quantity is again proportional to the radius and is computed for each point below.

Point on edge of Circle	***Point at $\mathbf{1.5}$ cm from the Center***
$a_r = r\omega^2$ $a_r = 0.025 \cdot 155.5^2$ $a_r = 604.5\ m/s^2$	$a_r = r\omega^2$ $a_r = 0.015 \cdot 155.5^2$ $a_r = 362.7\ m/s^2$

Example 2:

A hard drive with a radius of 5 cm can rotate anywhere from $5000 - 10000\ rpm$. A single bit requires 5 μm of length.

a) How many bits per second are being read when the reading head is 5 cm from the center and the disk is spinning at 7500 rpm?
b) How fast does the disk need to spin to maintain the same reading rate (in bits per second) when the reading head is 3 cm from the center? Is this speed supported by the disk? If not, what is the maximum bit rate when the head is at this distance?



Solution 1a: Finding the number of bits per second that are read is the equivalent to finding the number of 5 μm lengths that pass the read head per second, since each bit occupies 5 μm. We first need to convert the angular speed to linear speed.

$$v = r\omega$$
$$v = 0.05 \cdot \left(7500 \frac{\cancel{rev}}{\cancel{min}} \cdot \frac{1 \cancel{min}}{60\ sec} \cdot \frac{2\pi\ rad}{1 \cancel{rev}}\right)$$
$$v = 39.27\ m/s$$

Next, we convert the meters per second to bits per second, using the fact that 1 bit is read every 5 μm.

$$R_b = 39.27 \frac{\cancel{m}}{s} \cdot \left(\frac{1\ bit}{5.0 \cdot 10^{-6}\ m}\right)$$
$$R_b = 7{,}853{,}982\ bits/sec$$

1b: If the angular speed remains unchanged, then the linear speed will decrease as the reading head moves towards the center of the disk because it passes over less linear distance during the same time. To maintain the same linear speed, (and hence the same reading bit rate), the angular speed will need to increase.

$$\omega = \frac{v}{r}$$
$$\omega = \frac{39.27}{0.03}$$
$$\omega = 1309\ rad/sec$$

To see if this rate is supported by our disk, we need to convert the angular speed in radians per second to revolutions per minute.

$$\omega_{rpm} = 1309 \frac{\cancel{rad}}{\cancel{s}} \cdot \left(\frac{1\ rev}{2\pi \cancel{rad}} \cdot \frac{60 \cancel{s}}{1\ min}\right)$$
$$\omega_{rpm} = 12{,}500\ rpm$$

Since $12{,}500 > 10{,}000$ the disk cannot support this bit rate when the head is 3 cm from the center of the disk.

The maximum bit rate that can be supported at this distance is found as follows.

$$R_{b,max} = r\omega \cdot \left(\frac{1\ bit}{5.0 \cdot 10^{-6}\ m}\right)$$
$$R_{b,max} = 0.03 * 10{,}000 \frac{\cancel{rev}}{\cancel{min}} \cdot \left(\frac{1 \cancel{min}}{60\ s} \cdot \frac{2\pi\ rad}{1 \cancel{rev}}\right) \cdot \left(\frac{1\ bit}{5.0 \cdot 10^{-6}\ m}\right)$$
$$R_{b,max} = 6{,}283{,}185$$

Example 3: The tires of a car make 85 revolutions as the car reduces its speed at a constant rate from 90 km/h to 60 km/h. The tires have a radius of 0.5 m.

a) What was the angular acceleration?
b) If the car continues to decelerate at this rate, how much more time is required for the car to stop?

Solution 3a: We can use the third kinematic equation to find the angular acceleration as shown below.

$$\omega_f{}^2 = \omega_0{}^2 + 2\alpha(\theta_f - \theta_0)$$
$$\alpha = \frac{\omega_f{}^2 - \omega_0{}^2}{2(\Delta\theta)}$$

To solve this, we first need to convert the 85 revolutions of the tire into radian.

$$\Delta\theta = 85\, rev \cdot \left(\frac{2\pi\, rad}{rev}\right) = 170\pi\, rad$$

We also need to convert the given linear speeds to angular speeds using the known tire radius.

$$\omega_0 = \frac{v_0}{r}$$
$$\omega_0 = \frac{90\, \frac{\cancel{km}}{\cancel{hr}} \cdot \left(\frac{1000\, m}{1\, \cancel{km}} \cdot \frac{1\cancel{hr}}{3600\, s}\right)}{0.5\, m}$$
$$\omega_0 = 50\ rad/sec$$

$$\omega_f = \frac{60\, \frac{\cancel{km}}{\cancel{hr}} \cdot \left(\frac{1000\, m}{1\, \cancel{km}} \cdot \frac{1\cancel{hr}}{3600\, s}\right)}{0.5\, m}$$
$$\omega_f = 33.3\ rad/sec$$

Finally, we can solve for the angular acceleration.

$$\alpha = \frac{\omega_f{}^2 - \omega_0{}^2}{2(\Delta\theta)}$$
$$\alpha = \frac{(33.3)^2 - (50)^2}{2(170\pi)}$$
$$\alpha = -1.3\ rad/s^2$$

3b: We can use the second kinematic equation to solve for the additional time required to stop when starting from an angular speed of 33.3 rad/sec.

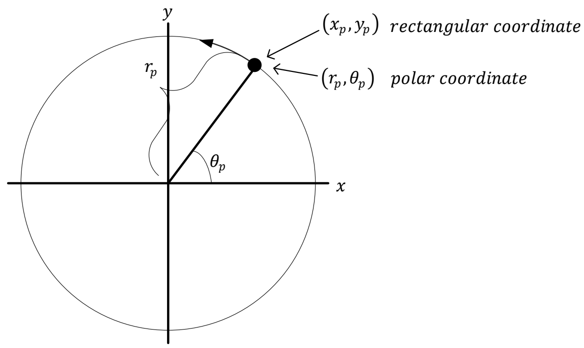
$$\omega_f = \omega_0 + \alpha t$$
$$t = \frac{\omega_f - \omega_0}{\alpha}$$
$$t = \frac{0 - 33.3}{-1.3}$$
$$t = 25.6\ s$$

Final Summary for Rotational Kinematics

2D Polar Coordinates

A point in $2D$ space is identified by a pair or polar coordinates, (r, θ).

- r: Straight line distance from the axis of rotation to the given point.
- θ: Angle between the straight line and some reference line, (usually the positive x-axis)



Rotational Kinematic Variables

The following variables are used to analyze the motion of a particle under rotational motion.

- $\theta(t)$: Distance traveled by a particle in angular units.
- $\omega(t) = \frac{d\theta(t)}{dt}$: Angular Velocity in $radians/time$
- $\alpha(t) = \frac{d\omega(t)}{dt}$: Angular Acceleration in $radians/time^2$

Angular Kinematic Equations

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega_f = \omega_0 + \alpha t$$

$$\omega_f{}^2 = \omega_0{}^2 + 2\alpha(\theta_f - \theta_0)$$

$$\theta_f = \theta_0 + \frac{1}{2}(\omega_f + \omega_0)t$$

Linear to Angular Variable Conversions

Distance	Velocity	Acceleration	
$x = r\theta$	$v_t = r\omega$	$a_t = r\alpha$	$a_r = \frac{v^2}{r} = rw^2$

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