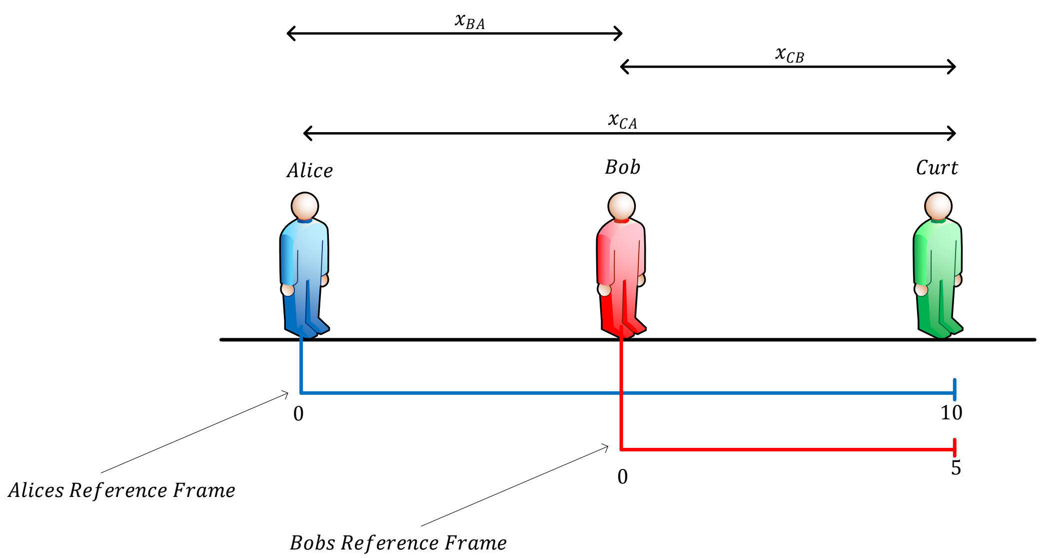
Physics 1 Mechanics - Relative Motion

In kinematics we studied the position, velocity, and acceleration of bodies. We developed equations that enabled us to find, for example, the position of a moving body at time t. When asked a question of this sort we arrived at a single answer, (e.g., The position of the train at time $t = 5$ sec is $17\ m$). If someone else arrived at a different answer we assumed only one of us was "correct". We now ask: Can there be more than one answer to the seemingly simple question of, "What is the position of body A?". The answer is... well yes… relatively speaking.

To explain how this could be true let's take a simple example of two people, Alice, and Bob, that are both asked to find the position of a third person, Curt. The figure below shows that Alice and Bob are not standing at the same location.



We start by asking Alice; "What is Curt's position?". She of course would answer, "$10\ m$". We then ask Bob the same question. His answer is $5\ m$. As we see from the figure, they both seem to be correct *from their own perspectives*. Curt does indeed have a position of $10\ m$ "as measured by" Alice, at the same time Curt has a position of $5\ m$ "as measured by" Bob. Therefore, we see that the position, as well as other kinematic quantities, need to be measured *relative* to something else. We call this "something else" the reference frame. Referring to the figure above, we can formalize the notation as follows:

$x_{CA}(t)$: Position of Curt *relative* to, (i.e., as measured by), Alice at time t.

$x_{CB}(t)$: Position of Curt *relative* to, (i.e., as measured by), Bob at time t.

$x_{BA}(t)$: Position of Bob *relative* to, (i.e., as measured by), Alice at time t.

And from inspection we can see:

$$x_{CA}(t) = x_{CB}(t) + x_{BA}(t)$$

We can take the derivative of the above equation to get a similar relationship for the velocities.

$$\frac{d}{dt}\left(x_{CA}(t)\right) = \frac{d}{dt}\left(x_{CB}(t)\right) + \frac{d}{dt}\left(x_{BA}(t)\right)$$
$$v_{CA}(t) = v_{CB}(t) + v_{BA}(t)$$

We can take the derivative again to get still another relationship for the accelerations.

$$\frac{d}{dt}\left(v_{CA}(t)\right) = \frac{d}{dt}\left(v_{CB}(t)\right) + \frac{d}{dt}\left(v_{BA}(t)\right)$$
$$a_{CA}(t) = a_{CB}(t) + a_{BA}(t)$$

However, for our studies we will consider only reference frames that move with constant velocity with respect to each other, (called inertial reference frames), therefore $a_{BA}(t) = 0$ and we have

$$a_{CA}(t) = a_{CB}(t)$$

Although, the above equations were developed for one dimensional motion they are also true for multi-dimensional motion where the position, velocity and acceleration are all vectors. Another helpful observation, which can be easily seen for the position but applies to velocity and acceleration as well, is the following.

$$x_{AB}(t) = -x_{BA}(t)$$

Finally, let's make one more observation which will help us when we do example problems. Re-writing the position equation below we see that the subscripts on the left-hand side are related to the subscripts on the right-hand side in the following way.

On the right-hand side the inner subscripts, (in red), match each other, and the outer subscripts, (in blue), match the left-hand side subscripts. This is true for all such relationships, and we can use the observation to ensure we set up the correct equations when solving relative motion problems.

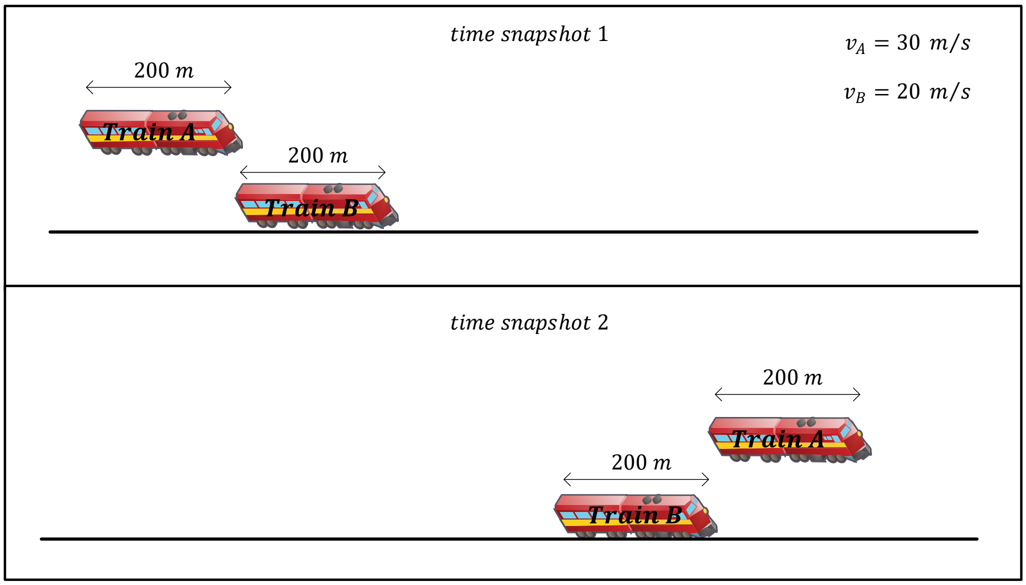
$$x_{CA}(t) = x_{CB}(t) + x_{BA}(t)$$

Let's do some examples to illustrate.

Example 1: (1D Relative Motion)

Two trains, A and B, are traveling on parallel tracks with constants speeds of 30 m/s and 20 m/s respectively. At a certain time, the front of train A just reaches the back of train B. If the trains are both 200 m long, how long does it take for train A to completely overtake train B, (i.e., the back of train A is just at the front of train B)?

Solution 1: We start by drawing a diagram of the scenario, shown below.



Note that the speeds for both trains are given relative to the earth, $v_A = v_{AE}$, $v_B = v_{BE}$. We can also define each of the train speeds relative to each other, (i.e., v_{AB}, v_{BA}). Based on what we learned in the introduction we can write the following relationship:

$$v_{AE} = v_{AB} + v_{BE}$$

Rearranging, we can find the speed of train A as seen by train B.

$$v_{AB} = v_{AE} - v_{BE}$$
$$v_{AB} = 30 - 20$$
$$v_{AB} = 10\ m/s$$

Using train B as the reference frame, i.e., assume train B is not moving, train A needs to travel 400 m from time snapshot 1 to time snapshot 2. The time over which this takes place can now be easily found using the reference frame of train B.

$$v_{AB} = \frac{\Delta x}{\Delta t}$$
$$\Delta t = \frac{\Delta x}{v_{AB}}$$
$$\Delta t = \frac{400}{10}$$
$$\Delta t = 40\ s$$

By computing the relative velocity of train A as measured by B, we were able to find more easily our desired Δt. To illustrate this point let's see how we would have solved this problem using our basic kinematic equations without the notion of relative velocity.

In this case we would start by writing the position equation for both trains as follows.

$$x_A(t) = x_A(0) + v_A t \qquad\qquad x_B(t) = x_B(0) + v_B t$$

Using a single reference frame, the earth, we have $x_A(0) = 0$, then $x_B(0) = 200$. Therefore, the equations become

$$x_A(t) = v_A t \qquad\qquad x_B(t) = 200 + v_B t$$

We can now restate the question as: "When is the position of train A, $x_A(t)$, 200 meters ahead of the position of train B, $x_B(t)$?", which we can mathematically state as:

$$x_A(t) = x_B(t) + 200$$

Substituting and solving for t we have

$$\begin{aligned} v_A t &= (200 + v_B t) + 200 \\ t(v_A - v_B) &= 400 \\ t &= \frac{400}{(v_A - v_B)} \end{aligned}$$

Noticing that $v_A = v_{AE}$ and $v_B = v_{BE}$, we see that this equation is identical to what we had using the first method above!

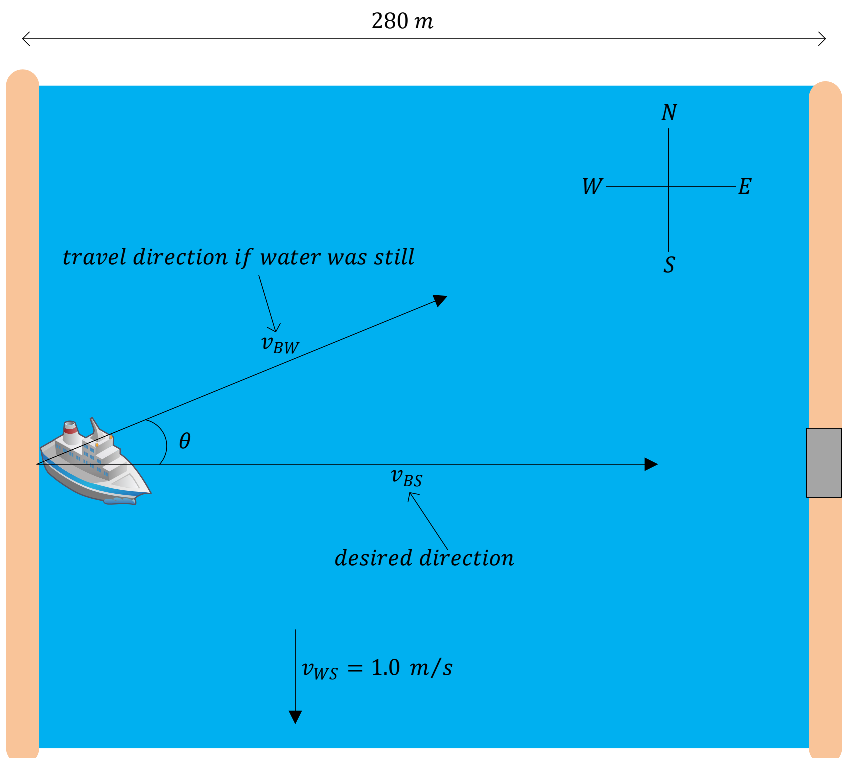
$$t = \frac{400}{10} = 40\ s$$

Therefore, both methods result in the same answer, but, as we'll see in the remaining problems, using the concept of relative motion in most cases makes the problem much easier to solve.

Example 2: (2D Relative Motion)

A $280\ m$ wide river is flowing to the south at a constant rate of $1.0\ m/s$. A boat, which travels at a constant rate of $4.0\ m/s$ in still water (i.e., with respect to the water), wishes to arrive at the other side directly across the shore. In what direction should the boat point to travel in a straight line to the desired point across the shore? How long does it take?

Solution 2: As usual we start by drawing a diagram.



We can look at this problem two different ways, which we illustrate below.

Method 1:

We can use our standard relative velocity equation, in vector form, where $\boldsymbol{v}_{BS}$ is the velocity of the boat with respect to the shore, $\boldsymbol{v}_{BW}$ is the velocity of the boat with respect to the water, and $\boldsymbol{v}_{WS}$ is the velocity of the water with respect to the shore. As shown in the figure, we would like $\boldsymbol{v}_{BS}$ to be directly in the east direction. Since the water velocity is directed south it should be obvious that we need to point our boat in the north-east direction at a certain angle. With that we write:

$$\boldsymbol{v}_{BS} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WS}$$

$$\boldsymbol{v}_{BW} = \boldsymbol{v}_{BS} - \boldsymbol{v}_{WS}$$

$$v_{BW}\langle cos(\theta), sin(\theta)\rangle = \langle v_{BS,x}, 0\rangle - \langle 0, -v_{WS,y}\rangle$$

$$4\langle cos(\theta), sin(\theta)\rangle = \langle v_{BS,x}, 0\rangle - \langle 0, -1\rangle$$

Treating the vector components separately we have the following two equations.

***x*- component equation**	***y*- component equation**
$4\,cos(\theta) = v_{BS,x}$	$4\,sin(\theta) = 1$

Solving the y equation first we have

$$\begin{aligned} 4\sin(\theta) &= 1 \\ \sin(\theta) &= \frac{1}{4} \\ \theta &= \sin^{-1}\left(\frac{1}{4}\right) \\ \theta &= 14.478^{\circ} \end{aligned}$$

With this we can solve for the $v_{BS,x}$ using the x-component equation.

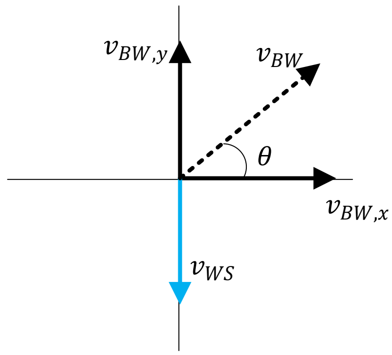
$$\begin{aligned} v_{BS,x} &= v_{BW}\cos(\theta) \\ v_{BS,x} &= 4\cos(14.478^{\circ}) \\ v_{BS,x} &= 3.87\ m/s \end{aligned}$$

Now since we know the speed of the boat with respect to the shore, $v_{BS,x}$, we can easily find the time to cross the river.

$$\begin{aligned} t &= \frac{D}{v_{BS,x}} \\ t &= \frac{280}{3.87} \\ t &= 72.3\ s \end{aligned}$$

Method 2:

An alternate way to look at the problem is the following. We know the boat can maintain a maximum speed of $4.0\,m/s$, regardless of the direction. If the boat directed all this speed towards the east it would end up moving to the southeast with respect to someone standing on the shore since the water is not still, (moving south $1.0\,m/s$). Therefore, we need to direct some of the boats speed to the north to counter the effect of the water. The resulting movement of the boat will be the sum of the two vectors, $\boldsymbol{v}_{BW}$ and $\boldsymbol{v}_{WS}$, as shown in figure below.



The water velocity, $\boldsymbol{v}_{WS}$, is shown by the blue vector is directed in the negative y direction and the velocity of the boat, v_{BW}, is shown as the dashed line vector, along with its x and y components. The sum of these vectors will be the resulting velocity of the boat. Since our desire is to have the boat move in the positive x direction only, we need $v_{BW,y}$ to be just large enough to cancel the effects of v_{WS}. When this occurs, we will be left with only the x-component of the boats velocity. Since $v_{BW,y} = v_{BW}\sin(\theta)$ we have

$$v_{BW}\,sin(\theta) = v_{WS}$$

$$\theta = sin^{-1}\left(\frac{v_{WS}}{v_{BW}}\right)$$

$$\theta = sin^{-1}\left(\frac{1}{4}\right)$$

$$\theta = 14.478^{\circ}$$

Which is the same answer we computed using method 1 above.

As you can see, of the $4.0\,m/s$ available to the boat, $4\,sin(14.478^{\circ}) = 1.0\ m/s$ was needed to cancel the effects of the water. Therefore, less speed was able to be used to move the boat in the positive x direction. This amount of course is given as $v_{BW,y} = v_{BW}\cos(\theta)$ shown below.

$$4\cos(14.5^{\circ}) = 3.87\ m/s$$

Using this speed, the time to cross the river, which was computed with method 1, is 72.3 s.

Final Summary for Relative Motion

Reference Frame: An established coordinate system that we use to measure the position of all objects. We assume this frame to be at rest when making these measurements.

Inertial Reference Frames: Reference frames that have constant velocity with respect to one another.

Relative Motion in Inertial Reference Frames

When two different inertial reference frames, $(A \text{ and } B)$ attempt to measure the position, velocity, and/or acceleration of an object, P, the following equations apply:

$$\boldsymbol{x}_{PA}(t) = \boldsymbol{x}_{PB}(t) + \boldsymbol{x}_{BA}(t)$$
$$\boldsymbol{v}_{PA}(t) = \boldsymbol{v}_{PB}(t) + \boldsymbol{v}_{BA}(t)$$
$$\boldsymbol{a}_{PA}(t) = \boldsymbol{a}_{PB}(t)$$

$\boldsymbol{x}_{PA}(t)$: Position of the object *relative* to, (i.e., as measured by), frame A at time t.
$\boldsymbol{x}_{BA}(t)$: Position of frame B *relative* to, (i.e., as measured by), frame A at time t.
$\boldsymbol{x}_{PB}(t)$: Position of the object *relative* to, (i.e., as measured by), frame B at time t.

Similar definitions apply to the velocity and acceleration.

Furthermore (for any two subscripts):

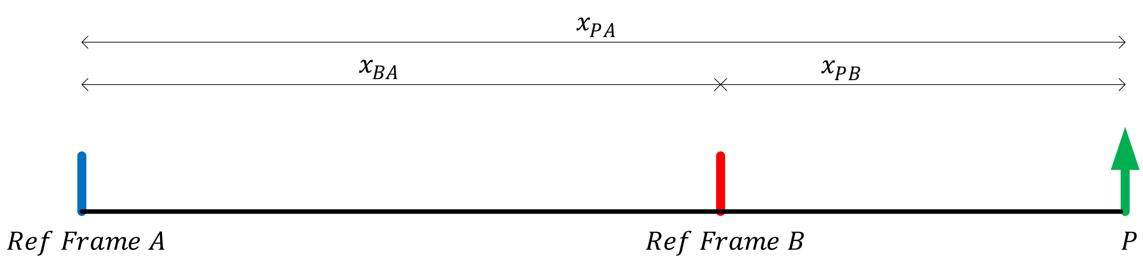
$$\boldsymbol{x}_{AB}(t) = -\boldsymbol{x}_{BA}(t)$$

We can use the following "subscript matching" to help us solve relative motion problems by making sure we set up the correct relationships.

$$x_{PA}(t) = x_{PB}(t) + x_{BA}(t)$$

On the right-hand side the inner subscripts, (in red), match each other, and the outer subscripts, (in blue), match the left-hand side subscripts.

The figure below can also be used to help visualize a given scenario.



By: ferrantetutoring