Physics 1 Mechanics - Linear Momentum

Linear momentum is one of the many quantities that can be used to describe a moving body. Recall from the lesson on energy, another quantity associated with a moving body is kinetic energy, $K = \frac{1}{2}mv^2$. One reason why kinetic energy is such an important quantity is because it is conserved. As it turns there are other quantities that are also conserved, one of which is *linear momentum*. Just as the law of the conservation of energy provides a powerful tool to solve many interesting physics problems, so does the law of the conservation of linear momentum. Before we formally state this law, let's define linear momentum, which is best understood through Newton's 2nd law. Newton's 2nd law states that *the vector sum of the external forces, i.e., the net force, on an object is equal to the mass times the acceleration of that object*.

$$\boldsymbol{F}_{net} = m\boldsymbol{a}$$

And since $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$, we can write:

$$\boldsymbol{F}_{net} = m\frac{d\boldsymbol{v}}{dt} = \frac{d(m\boldsymbol{v})}{dt}$$

The quantity, $m\boldsymbol{v}$, is what physicist define as the linear momentum, $\boldsymbol{p}$, of a moving body.

$$\boldsymbol{p} = m\boldsymbol{v}$$

With this new definition, Newton's 2nd law can now be written as follows.

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt}$$

Which can then be restated as: "*The rate of change of momentum of an object is equal to the net force applied to the object*".

We can now use this new form of Newton's 2nd law to derive the law of conservation of momentum. We start by considering two colliding bodies. Before the collision, the bodies have momentum $\boldsymbol{p}_{1,i},\ \boldsymbol{p}_{2,i}$, and after they have momentum $\boldsymbol{p}_{1,f},\ \boldsymbol{p}_{2,f}$. If we consider only the force each body experiences from the other body during the brief time around the collision time we can write the force on one body from the other body as follows.

$$\boldsymbol{F}_{1,2} = \frac{d\boldsymbol{p}_1}{dt} \qquad\qquad \boldsymbol{F}_{2,1} = \frac{d\boldsymbol{p}_2}{dt}$$

Multiplying through by dt and integrating from t_i to t_f we have:

$$\int_{t_i}^{t_f} d\boldsymbol{p}_1 = \int_{t_i}^{t_f} \boldsymbol{F}_{1,2}\,dt \qquad\qquad \int_{t_i}^{t_f} d\boldsymbol{p}_2 = \int_{t_i}^{t_f} \boldsymbol{F}_{2,1}\,dt$$

$$\boldsymbol{p}_{1,f} - \boldsymbol{p}_{1,i} = \int_{t_i}^{t_f} \boldsymbol{F}_{1,2}\,dt \qquad\qquad \boldsymbol{p}_{2,f} - \boldsymbol{p}_{2,i} = \int_{t_i}^{t_f} \boldsymbol{F}_{2,1}\,dt$$

And from Newton's 3rd law we know that $\boldsymbol{F}_{1,2} = -\boldsymbol{F}_{2,1}$, which allows us to write the following:

$$-(\boldsymbol{p}_{1,f} - \boldsymbol{p}_{1,i}) = \int_{t_i}^{t_f} \boldsymbol{F}_{2,1} dt \qquad\qquad (\boldsymbol{p}_{2,f} - \boldsymbol{p}_{2,i}) = \int_{t_i}^{t_f} \boldsymbol{F}_{2,1}\, dt$$

Finally, equating these two integral expressions leads us to the fundamental result that the *total momentum of the system is constant.*

$$-(\boldsymbol{p}_{1,f} - \boldsymbol{p}_{1,i}) = (\boldsymbol{p}_{2,f} - \boldsymbol{p}_{2,i})$$
$$\boldsymbol{p}_{1,i} - \boldsymbol{p}_{1,f} = \boldsymbol{p}_{2,f} - \boldsymbol{p}_{2,i}$$
$$\boldsymbol{p}_{1,i} + \boldsymbol{p}_{2,i} = \boldsymbol{p}_{1,f} + \boldsymbol{p}_{2,f}$$
$$m_1\boldsymbol{v}_{1,i} + m_2\boldsymbol{v}_{2,i} = m_1\boldsymbol{v}_{1,f} + m_2\boldsymbol{v}_{2,f}$$

Although this derivation considered a system of only two bodies, the results apply equally to any number of bodies in an *isolated system*. We can now make a formal statement of the conservation of linear momentum.

Conservation of Linear Momentum

The total momentum of an *isolated* system, (one in which there are no external forces), remains constant.

$$\sum_{k=1}^{N} \boldsymbol{p}_{k,i} = \sum_{k=1}^{N} \boldsymbol{p}_{k,f}$$

The law of the conservation of linear momentum can be used to determine much about the motion of a system of objects after a collision occurs, assuming any external forces are non-existent, or their effects can be ignored. In any collision involving macroscopic objects there will be at least some kinetic energy "lost" in the form of things like deformation, heat, and sound. Collisions are generally classified as elastic or inelastic based on whether kinetic energy is conserved or not.

- ***Elastic Collision:*** A collision in which kinetic energy *is conserved*.
  - This is an ideal scenario, which is very close to true when subatomic particles collide. In addition, we generally use the macroscopic example of billiard balls colliding since very little deformation occurs because of the hardness of the balls.
- ***Inelastic Collision***: A collision in which kinetic energy is *not conserved*.
  - Macroscopic collisions are almost always inelastic because of deformation of the colliding bodies. However, if the objects are such that the deformation is comparatively small we can assume the collision is elastic with minimal loss in accuracy.
  - ***Completely Inelastic Collision:*** A collision in which the objects stick together, which results in the maximum amount of kinetic energy lost, e.g., a baseball "colliding" with a baseball glove.

To summarize, for a system of objects involved in a collision where we assume no external forces, the linear momentum is *always* conserved.

$$\sum_{k=1}^{N} m_k \boldsymbol{v_{k,i}} = \sum_{k=1}^{N} m_k \boldsymbol{v_{k,f}}$$

For an *elastic collision,* the kinetic energy is also conserved.

$$\sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,i}}^2 = \sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,f}}^2$$

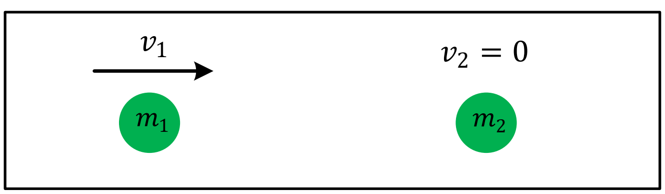
For an *inelastic collision*, the kinetic energy is *not conserved*.

$$\sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,i}}^2 = \sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,f}}^2 + thermal\ and\ other\ form\ of\ energy$$

Let's do some examples. The first example will examine an elastic head on collision, where we will derive some general expressions for different scenarios in order to gain some intuition.

Example 1: (Head-On Elastic Collision with One Object at Rest)

Assume an object of mass m_1 is moving with a speed of v_1 towards a target object of mass m_2, which is at rest. Derive a general solution for the two final velocities, assuming an elastic collision.



Solution 1: In an elastic collision both momentum and kinetic energy are conserved. Since m_2 initially at rest, we can write the conservation of momentum equation as follows.

$$m_1 v_{1,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

$$m_2 v_{2,f} = m_1 (v_{1,i} - v_{1,f})$$

And since kinetic energy is conserved, we can write

$$\frac{1}{2} m_1 {v_{1,i}}^2 = \frac{1}{2} m_1 {v_{1,f}}^2 + \frac{1}{2} m_2 {v_{2,f}}^2$$

$$m_2 {v_{2,f}}^2 = m_1 ({v_{1,i}}^2 - {v_{1,f}}^2)$$

$$m_2 {v_{2,f}}^2 = m_1 (v_{1,i} - v_{1,f})(v_{1,i} + v_{1,f})$$

Next, we divide the energy equation by the momentum equation to write expressions for the final velocities of both masses.

$$\frac{m_2 {v_{2,f}}^2 = m_1 (v_{1,i} - v_{1,f})(v_{1,i} + v_{1,f})}{m_2 v_{2,f} = m_1 (v_{1,i} - v_{1,f})}$$

For which we obtain the following two expressions for the final velocities of both masses.

$$v_{1,f} = (v_{2,f} - v_{1,i}) \qquad\qquad v_{2,f} = (v_{1,i} + v_{1,f})$$

Finally, plugging these two equations into the original momentum equation we can derive relationships between the final velocities of both masses in terms of the initial velocity and the masses.

$$m_2 (v_{1,i} + v_{1,f}) = m_1 (v_{1,i} - v_{1,f})$$

$$m_2 v_{1,i} + m_2 v_{1,f} = m_1 v_{1,i} - m_1 v_{1,f}$$

$$v_{1,f} (m_2 + m_1) = v_{1,i} (m_1 - m_2)$$

$$\boxed{v_{1,f} = v_{1,i} \left[\frac{(m_1 - m_2)}{(m_2 + m_1)} \right]}$$

$$m_2 v_{2,f} = m_1 \left(v_{1,i} - (v_{2,f} - v_{1,i}) \right)$$

$$m_2 v_{2,f} = m_1 v_{1,i} - m_1 v_{2,f} + m_1 v_{1,i}$$

$$v_{2,f} (m_2 + m_1) = 2 m_1 v_{1,i}$$

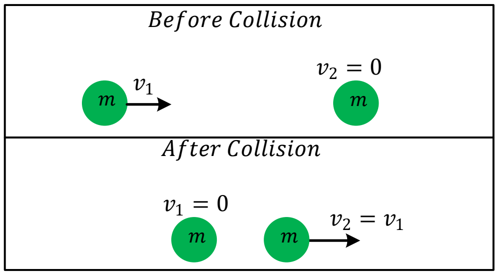
$$\boxed{v_{2,f} = v_{1,i} \left[\frac{2 m_1}{(m_2 + m_1)} \right]}$$

We can use these equations to obtain a level of intuition by looking at some specific cases shown below.

Case 1: $(m_1 = m_2)$ - All momentum is transferred to the target object. A good visual example is that of colliding billiard balls without spin.

$$v_{1,f} = v_{1,i}\frac{(m - m)}{(m + m)} \qquad v_{2,f} = v_{1,i}\frac{2m}{(2m)}$$

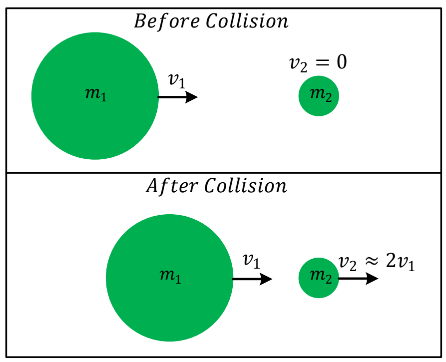
$$v_{1,f} = 0 \qquad v_{2,f} = v_{1,i}$$



Case 2: $(m_1 \gg m_2)$ - The larger mass speed is unchanged while the smaller mass is made to moves at twice the speed of the incoming large mass. A good visual example is that of a bowling ball hitting bowling pins.

$$v_{1,f} \approx v_{1,i}\frac{(m_1 - 0)}{(0 + m_1)} \qquad v_{2,f} \approx v_{1,i}\frac{2m_1}{(0 + m_1)}$$

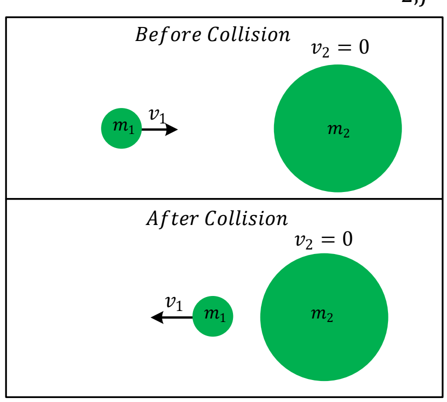
$$v_{1,f} \approx v_{1,i} \qquad v_{2,f} \approx 2v_{1,i}$$



Case 3: $(m_1 \ll m_2)$ - The larger mass is not moved, while the smaller mass is forced back in the opposite direction with same speed. A good visual example is that of a tennis ball being thrown against a wall.

$$v_{1,f} \approx v_{1,i}\frac{(\cancel{m_1} - m_2)}{(m_2 \cancel{+ m_1})} \qquad v_{2,f} \approx v_{1,i}\frac{2m_1}{(m_2 \cancel{+ m_1})}$$

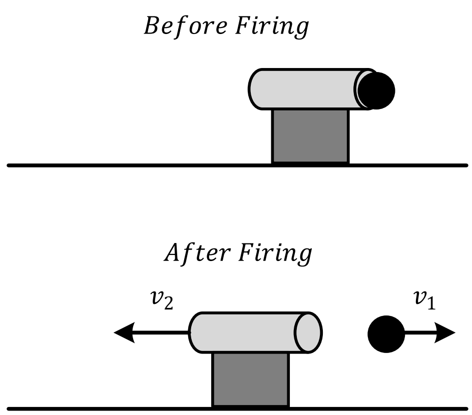
$$v_{1,f} \approx -v_{1,i} \qquad v_{2,f} \approx 0$$



Example 2: (Gun Recoil)

A cannonball of mass $m_1 = 10\ kg$ is fired from a cannon of mass $m_2 = 200\ kg$ at a speed of $80\ m/s$, while sitting on a frictionless surface. What is the velocity of the cannon after it fires?

Solution 2: The sketch below shows the cannon-cannon ball system before and after the firing.



In this case, we essentially have a collision in reverse, i.e., an explosion. We can solve for the final velocity of the cannon using the conservation of momentum.

$$m_1 \boldsymbol{v}_{1,i} + m_2 \boldsymbol{v}_{2,i} = m_1 \boldsymbol{v}_{1,f} + m_2 \boldsymbol{v}_{2,f}$$

$$0 + 0 = m_1 \boldsymbol{v}_{1,f} + m_2 \boldsymbol{v}_{2,f}$$

$$\boldsymbol{v}_{2,f} = -\frac{m_1}{m_2} \boldsymbol{v}_{1,f}$$

$$\boldsymbol{v}_{2,f} = -\frac{10}{100} 80$$

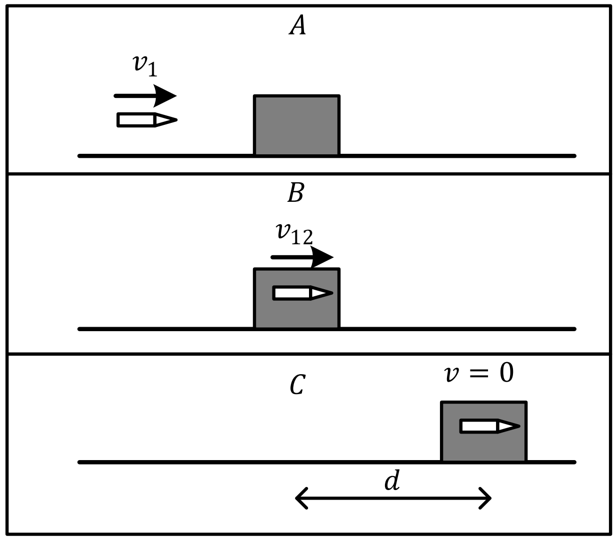
$$\boldsymbol{v}_{2,f} = -8\ {}^{m}/_{s}$$

To preserve the momentum of the system, the velocity if the cannon is to the left. This makes intuitive sense if we think of the recoil feeling when a gun is fired.

Example 3: (Inelastic Collision – Measuring the Speed of a Bullet)

A $10\ g$ bullet is fired at a $0.5\ kg$ block. The bullet gets lodged in the block. The block and bullet then slide across the surface, which has $\mu_k = 0.3$, for $d = 2\ m$.

a.) What was the speed of the bullet upon entering the block?
b.) How much energy was lost from the bullet being lodged into the block?



Solution 3a: Since the bullet and the block stick together this is a *completely inelastic collision*, and kinetic energy. To aid in solving the problem we divide the process into the following two time instants.

- $A \rightarrow B$: The brief time around the collision before the bullet-block system begins to slide across the floor.
  - Momentum is conserved during this time. Energy is expended in order for the bullet to travel into the block. Since the objects "stick" together the collision causes the maximum amount of energy to be lost. i.e., completely inelastic collision.
- $B \rightarrow C$: The time after the collision where the block begins to slide, and the external friction force is introduced.
  - The kinetic energy that remains from the collision process is now part of the bullet-block system. As the bullet-block system slides across the floor this energy dissipates, until all is lost when the system comes to rest.

Starting with $A \rightarrow B$, we use the conservation of momentum to write an expression for the velocity of the bullet as a function of the velocity of the bullet-block system immediately after the collision.

$$m_1 v_{1,A} = (m_1 + m_2) v_{12,B}$$

$$v_{1,A} = \left[\frac{(m_1 + m_2)}{m_1}\right] v_{12,B}$$

Next, we look at $B \rightarrow C$, where the kinetic of the bullet-block system is lost due to the work done by friction. For this we can use the work-kinetic energy principal, which we learned in a previous lesson, to find the initial velocity of the bullet-block system immediately after the collision.

$$\Delta K = W_{net}$$

$$K_C - K_B = fd$$

$$0 - \frac{1}{2}(m_1 + m_2){v_{12,B}}^2 = -u(m_1 + m_2)gd$$

$$v_{12,B} = \sqrt{2ugd}$$

Finally, we substitute this expression into the momentum equation from above and solve for the speed of the bullet.

$$\boldsymbol{v}_{1,A} = \left[\frac{(m_1 + m_2)}{m_1}\right]\sqrt{2\mu_k gd}$$

$$\boldsymbol{v}_{1,A} = \left[\frac{(0.01 + 0.5)}{0.01}\right]\sqrt{2 * 0.3 * 9.8 * 2}$$

$$\boldsymbol{v}_{1,A} = 174.9\ {}^{m}/_{s}$$

3b: To find the loss of energy due to the bullet entering the block we compute the difference in the initial kinetic energy of the bullet and the kinetic energy of the bullet-block system immediately after the collision.

$$E_L = K_A - K_B$$

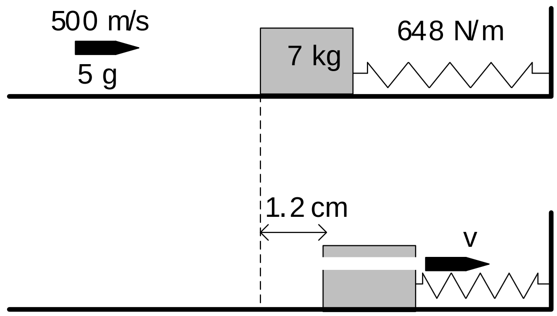
$$E_L = \frac{1}{2}m_1{v_{1,A}}^2 - \frac{1}{2}(m_1 + m_2){v_{12,B}}^2$$

$$E_L = \frac{1}{2}m_1{v_{1,A}}^2 - \frac{1}{2}(m_1 + m_2)\left(\sqrt{2\mu_k gd}\right)^2$$

$$E_L = \frac{1}{2}\left((0.01 \cdot 175.9^2) - \left((0.01 + 0.5)2 \cdot 0.3 \cdot 9.8 \cdot 2\right)\right)$$

$$E_L = 151.7\,J$$

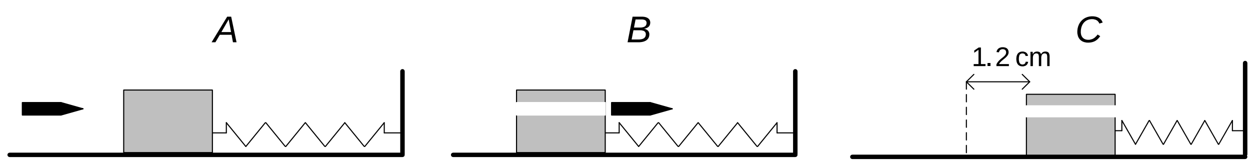
Example 4: A bullet of mass $5g$ moving with an initial velocity of $500\, m/s$ is fired into and *instantly* passes through a block of mass $7kg$, as shown in the figure below. The block, initially at rest on a frictionless surface, is connected to a spring with a force constant of $648\ N/m$.



a.) If the block moves a distance $1.2cm$ to the right after the bullet passed through it, find the speed at which the bullet emerges from the block.

b.) Find the energy lost in the collision.

Solution 4: The figure below shows the three sperate time instants we'll use to solve this problem.



State A: At this time instant the bullet is traveling towards the block, which is at rest.

State B: This state is the exact instant the bullet exits the block. At this time instant, the bullet and the block each have independent velocities.

State C: Finally, this state is when the block has fully compressed the spring and has therefore converted all of its kinetic energy to potential energy of the spring.

Part a:

Using $A \rightarrow B$, we can use the conservation of momentum to write the following.

$$m_B \boldsymbol{v}_{B,i} = m_B \boldsymbol{v}_{B,f} + m_{block} \boldsymbol{v}_{block,f}$$

Solving for the final velocity of the bullet we have

$$\boldsymbol{v}_{B,f} = \frac{m_B \boldsymbol{v}_{B,i} - m_{block} \boldsymbol{v}_{block,f}}{m_B}$$

Next, we can use the conservation of energy from $B \rightarrow C$ to find the velocity of the block immediately after the bullet emerges. The block must contain enough kinetic energy to compress the spring by the distance given. Therefore, we can find an expression for its velocity as follows.

$$\frac{1}{2}m_{block}\left(\boldsymbol{v}_{block,f}\right)^2 = \frac{1}{2}kx^2$$

$$\boldsymbol{v}_{block,f} = \sqrt{\frac{kx^2}{m_{block}}}$$

Substituting into the momentum equation from above, we can find the velocity of the bullet as it emerges from the block.

$$\boldsymbol{v}_{B,f} = \frac{m_B\boldsymbol{v}_{B,i} - m_{block}\sqrt{\frac{kx^2}{m_{block}}}}{m_B}$$

$$\boldsymbol{v}_{B,f} = \frac{0.005 \cdot 500 - 7\sqrt{\frac{648 \cdot 0.012^2}{7}}}{0.005}$$

$$\boldsymbol{v}_{B,f} = 338.4\ m/s$$

Part b:

To find the energy lost in this process we need to subtract the energy of the bullet after it emerges and the energy of the block from the initial energy of the bullet. Note the energy of the block after the bullet emerges is equivalent to the energy given to the spring.

$$E_L = E_{State\ A} - E_{end\ of\ State\ B}$$

$$E_L = \left(\frac{1}{2}m_B\left(\boldsymbol{v}_{B,i}\right)^2\right) - \left(\frac{1}{2}m_B\left(\boldsymbol{v}_{B,f}\right)^2 + \frac{1}{2}kx^2\right)$$

$$E_L = \frac{1}{2}(0.005 \cdot 500^2 - 0.005 \cdot 338.4^2 - 648 \cdot 0.012^2 -)$$

$$E_L \cong 339\ J$$

Final Summary for Linear Momentum

Linear Momentum

The linear momentum of an object, $\boldsymbol{p}$, is defined as the product of the mass and the velocity.

$$\boldsymbol{p} = m\boldsymbol{v}$$

Conservation of Linear Momentum

The total momentum of an isolated system, (one in which there are no external forces), remains constant.

$$\sum_{k=1}^{N} \boldsymbol{p}_{\boldsymbol{k},\boldsymbol{i}} = \sum_{k=1}^{N} \boldsymbol{p}_{\boldsymbol{k},\boldsymbol{f}}$$

For two colliding objects, we can write:

$$m_1\boldsymbol{v}_{\boldsymbol{1}}^{\boldsymbol{i}} + m_2\boldsymbol{v}_{\boldsymbol{2}}^{\boldsymbol{i}} = m_1\boldsymbol{v}_{\boldsymbol{1}}^{\boldsymbol{f}} + m_2\boldsymbol{v}_{\boldsymbol{2}}^{\boldsymbol{f}}$$

Collision Classification

Elastic Collision:

One in which **kinetic energy is conserved**:

$$\sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,i}}^2 = \sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,f}}^2$$

For two colliding objects, we can write:

$$\frac{1}{2} m_1 {v_{1,i}}^2 + \frac{1}{2} m_2 {v_{2,i}}^2 = \frac{1}{2} m_1 {v_{1,f}}^2 + \frac{1}{2} m_2 {v_{2,f}}^2$$

Inelastic Collision:

One in which **kinetic energy is NOT conserved**:

$$\sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,i}}^2 = \sum_{k=1}^{N} \frac{1}{2} m_k {v_{k,f}}^2 + thermal\ and\ other\ form\ of\ energy$$

For two colliding objects, we can write:

$$\frac{1}{2} m_1 {v_{1,i}}^2 + \frac{1}{2} m_2 {v_{2,i}}^2 = \frac{1}{2} m_1 {v_{1,f}}^2 + \frac{1}{2} m_2 {v_{2,f}}^2 + thermal\ and\ other\ form\ of\ energy$$

Note: The maximum amount of kinetic energy is lost when the colliding object stick together and this is called a ***completely inelastic collision***.

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