Physics 1 Mechanics - Kinematics

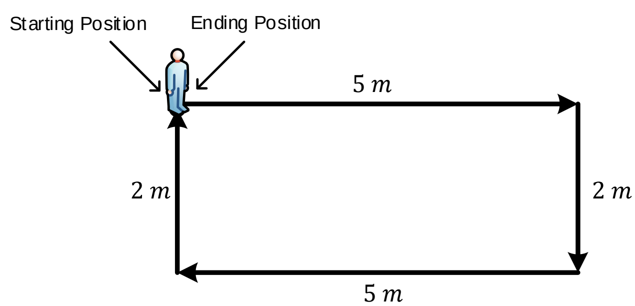
Kinematics is the study of the motion of bodies. As such it deals with the distance/displacement, speed/velocity, and the acceleration of bodies. Although we are familiar with the overall notion of these quantities, let's take a closer look to be sure…

Distance and Displacement:

These quantities deal with the position of a moving body. Although at first glance these words seem to describe the same quantity, they have a very important difference.

The figure below shows an example of a person walking in a rectangular path, with the same starting and ending position. In this case, we have drastically different results for distance and displacement, as shown below.

$$Distance = 5 + 2 + 5 + 2 = \mathbf{14\ m}$$
$$Displacement = Position\ 2 - Position\ 1 = \mathbf{0\ m}$$



Speed and Velocity:

These quantities deal with the rate of change in the position of a moving body over time. Like distance and displacement described above, speed and velocity have a very important difference.

Speed: A scalar quantity that describes the *rate of change of your **distance*** per unit time.

$$S_{av} = \frac{Distance}{\Delta t}$$

where S_{av} is the average speed over the time interval, $\Delta t = t_f - t_i$.

Velocity: A vector quantity that describes the *rate of change of your **displacement*** per unit time. In one dimension the average velocity can be described as follows:

$$\boldsymbol{v}_{av} = \frac{\Delta \boldsymbol{x}}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

where $\boldsymbol{v}_{av}$ is the average velocity over the time interval, Δt.

The figure below shows an example of a person traveling from point A to B and back to point A. Let's assume the distance between point A and B is 1 mile and that the journey was completed in a time interval, $\Delta t = t_f - t_i$, of 1 hour. With this we have:

$$S_{av} = \frac{Distance}{\Delta t} \qquad \boldsymbol{v}_{av} = \frac{\Delta x}{\Delta t}$$

$$S_{av} = \frac{2}{1} = 2\ mph \qquad \boldsymbol{v}_{av} = \frac{0}{1} = 0\ mph$$

Acceleration:

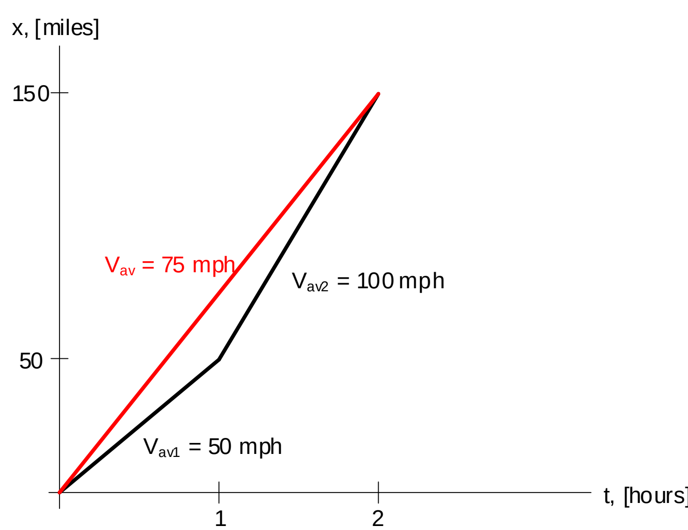
Acceleration is generally treated as a vector quantity and is defined as the rate of change of the velocity over time.

$$\boldsymbol{a}_{av} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

where $\boldsymbol{a}_{av}$ is the average acceleration over the time interval, Δt.

In most cases, we are interested in the vector quantities related to a moving body, and hence we will focus on displacement, velocity, and acceleration. Our definitions above were for the *average* velocity and acceleration. However, we are most interested in the *instantaneous* quantities. Let's take a closer look at what is meant by the instantaneous velocity. To keep the discussion simpler, we will restrict the following discussion to one dimensional motion. The results can easily be extended to two and three dimensions, which we will summarize at the end.

The first example is of a person traveling 50 miles in the first hour and then continuing for another 100 miles in the second hour.



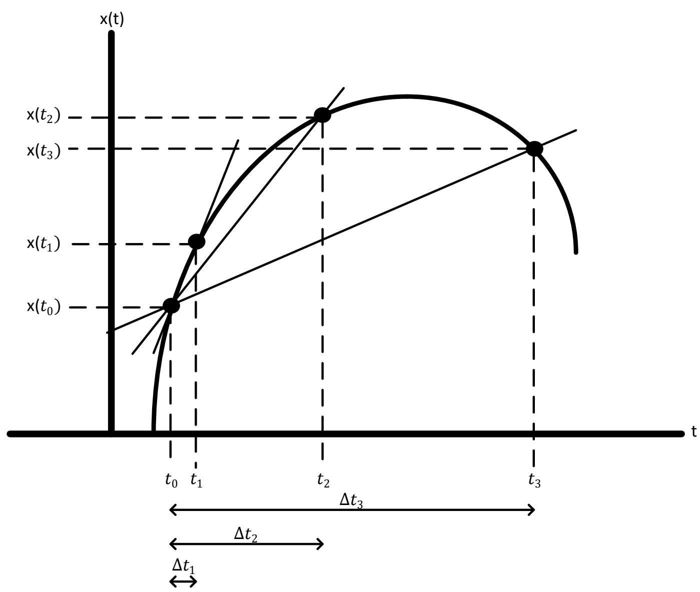
As the figure shows, unless the velocity is constant over the entire trip, the average velocity depends upon the interval over which we measure.

Averaged over 1st hour	Averaged over 2nd hour	Averaged over entire trip
$\boldsymbol{v}_{av1} = \frac{50}{1} = 50\ mph$	$\boldsymbol{v}_{av2} = \frac{100}{1} = 100\ mph$	$\boldsymbol{v}_{av} = \frac{150}{2} = 75\ mph$

The example above shows an abrupt change in velocity from the first hour to the second hour. Realistically, the velocity of a moving body changes gradually over time, as illustrated in the figure below. In this case we are interested in finding the so called instantaneous velocity. The instantaneous velocity is defined as velocity taken over an infinitesimally small time interval.

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \left(\frac{\Delta \boldsymbol{x}}{\Delta t} \right)$$

Which is read as: "The ratio of the change in position over the change in time as the change in time *approaches* zero."



The figure above is a position versus time graph for a body undergoing non-constant velocity. The figure is an illustration of how decreasing the Δt gives a better and better approximation of the instantaneous velocity at t_0. Obviously when Δt is equal to zero this ratio is undefined. However, as mentioned, Δt will *approach*, but not equal, zero. The mathematics that was developed to solve these types of problems is called calculus, and this limit is defined at the derivative of position, $x(t)$, with respect time. Note that the velocity becomes a function of time since its value changes for each t.

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t} \right) = \frac{dx(t)}{dt}$$

It is beyond the scope to review calculus, but if interested you can refer here for a full review of the subject.

We are now ready to present the fundamental equations of motion that can be used to solve various kinematic problems. What follows is a calculus based derivation of these equations. We are looking for equations that express relationships between the position, $x(t)$, the velocity, $v(t)$, and the acceleration, $a(t)$. In our derivations, we will take the special case where the acceleration is constant and not a function of time, i.e., $a(t) = a$. Hence the equations devlpoed are valid only for constant acceleration motion. If you are not familiar with calculus, feel free skip ahead to the results at the end. When solving various kinematics problems, we will simply use the equations that are derived below, and therefore skipping the derivations will not limit your ability to solve the problems.

We start by defining the instantaneous acceleration in the same way as we defined the instantaneous velocity above, i.e., as the instantaneous rate of change of the velocity.

$$\frac{dv(t)}{dt} = a$$

If we now integrate both sides of the equation, we get a relationship between the velocity and acceleration as shown.

$$v(t) = \int a\, dt$$
$$v(t) = at + v(0)$$

Where we assume knowledge of $v(0)$ as our initial condition for the indefinite integral.

This process can then be repeated to find the other fundamental motion equation:

$$\frac{dx(t)}{dt} = v(t)$$
$$\frac{dx(t)}{dt} = at + v(0)$$
$$x(t) = \int \big(at + v(0)\big)dt$$
$$x(t) = \frac{1}{2}at^2 + v(0)t + x(0)$$

Where again we assume knowledge of $x(0)$ as our initial condition for the indefinite integral.

That's it we are done! We now have our two fundamental equations for constant acceleration motion.

1.) $x(t) = x(0) + v(0)t + \frac{1}{2}at^2$
2.) $v(t) = v(0) + at$

We are usually presented with 5 motion equations. However, the 3 additional equations come from simple algebraic manipulation of the two fundamental equations above. For example, solving equation 2 for t and substituting the result into equation 1 we get a "new" equation as shown below.

$$x(t) = \frac{1}{2}a\left(\frac{v(t)-v(0)}{a}\right)^2 + v(0)\left(\frac{v(t)-v(0)}{a}\right) + x(0)$$

$$x(t) = \frac{1}{2}\cancel{a}\left(\frac{v^2(t)-2v(t)v(0)+v^2(0)}{a^{\cancel{2}}}\right) + v(0)\left(\frac{v(t)-v(0)}{a}\right) + x(0)$$

$$x(t) = \frac{v^2(t)}{2a} - \cancel{\frac{2v(t)v(0)}{2a}} + \frac{v^2(0)}{2a} + \cancel{\frac{v(t)v(0)}{a}} - \frac{v^2(0)}{a} + x(0)$$

$$x(t) = x(0) + \frac{v^2(t)}{2a} - \frac{v^2(0)}{2a}$$

$$x(t) - x(0) = \frac{1}{2a}\left(v^2(t) - v^2(0)\right)$$

which is usually written as:

$$v^2(t) = v^2(0) + 2a\big(x(t) - x(0)\big)$$

Note by initially solving for t from equation 2, we eliminated that variable in our new equation. We can repeat this process for other variables; however, I will leave this as an exercise to do on your own, and simply provide the results for the 5 equations below.

1.) $x(t) = x(0) + v(0)t + \frac{1}{2}at^2$

2.) $v(t) = v(0) + at$

3.) $v^2(t) = v^2(0) + 2a\big(x(t) - x(0)\big)$

4.) $x(t) = x(0) + \frac{1}{2}\big(v(t) + v(0)\big)t$

5.) $x(t) = x(0) + v(t)t - \frac{1}{2}at^2$

The other change that is usually made to these equations when they are presented is that we choose an initial and final time, t_0 and t_f respectively, and abbreviate the position and velocity variables as follows:

$$x(t_f) = x_f, \;\; x(t_0) = x_0$$
$$v(t_f) = v_f, \;\; v(t_0) = v_0$$

With this the variable t in the equation represents $t_f - t_i = \Delta t$.

With these changes we can write the equations as follows

$$\boxed{\begin{aligned} x_f &= x_0 + v_0\Delta t + \frac{1}{2}a\Delta t^2 \\ v_f &= v_0 + a\Delta t \\ {v_f}^2 &= {v_0}^2 + 2a(x_f - x_0) \\ x_f &= x_0 + \frac{1}{2}(v_f + v_0)\Delta t \\ x_f &= x_0 + v_f\Delta t - \frac{1}{2}a\Delta t^2 \end{aligned}}$$

The motions equations were derived in one dimension, however I mentioned that they can be easily extended for 2 or 3 dimensions. Recall that the position, velocity, and acceleration are vector quantities and therefore can have an x, y and z components. Focusing on the 2D case, and using what we learned in the previous lesson, we can write the position, velocity, and acceleration vectors in component form as shown.

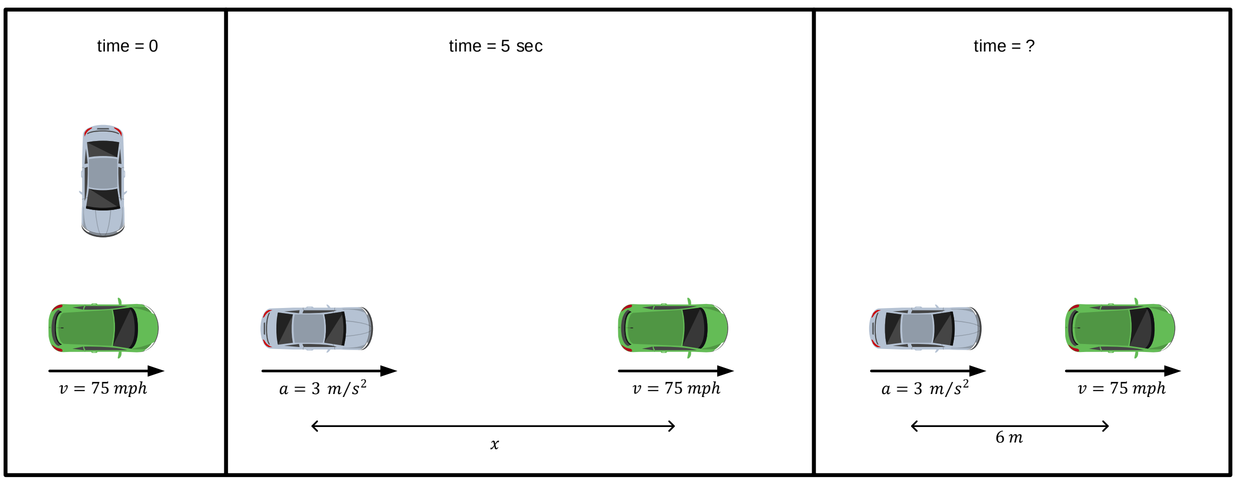
$$\begin{aligned} \boldsymbol{p}(t) &= \langle x(t), y(t) \rangle \\ \boldsymbol{v}(t) &= \langle v_x(t), v_y(t) \rangle \\ \boldsymbol{a}(t) &= \langle a(t), a_y(t) \rangle \end{aligned}$$

Next, recall from our lesson on vectors, we can treat each component of a vector separately. Therefore, we simply repeat our motion equations for each of the dimensions required to solve our problem. Let's practice using the equations with some examples.

Example 1: (1D Horizontal Motion)

A police car is watching for speeding cars on the side of the road when he spots a car passing him at a constant speed of 75 mph. After a 5 second delay the police begins to chase the speeder with an acceleration of 3.0 m/s^2.

How much time passes after the 5 second delay before the police car is 6 m behind the speeder? How fast is the police car going at this point?



Solution: Let's start by finding the distance, x, the speeder has gotten before the police car start to chase him. We first convert mph to m/s.

$$\begin{aligned} v_s &= 75 \, \frac{\cancel{mi}}{\cancel{hr}} \left(\frac{1 \, \cancel{hr}}{3600 \, sec} * \frac{1609.34 \, m}{1 \, \cancel{mi}} \right) \\ v_s &= 33.53 \, \, m/s \end{aligned}$$

Then using the first kinematic equation with $a = 0$, $x_{s,0} = 0$, $\Delta t = 5$, and $v_0 = v_s$, we have

$$x_{s,f} = x_0 + v_0 \Delta t + \frac{1}{2} a \Delta t^2$$
$$x_{s,f} = 0 + 33.53 * 5 + 0$$
$$x_{s,f} = 167.65\ m$$

We can use the first equation again to write the position of the police car and the speeder as functions of time, starting when the police car begins. Note the police car starts at $x_0 = 0$, whereas the speeder starts at $x_0 = x_{s,f}$ from above. For the police car we have

$$x_p(t) = \cancel{x_p(0)} + \cancel{v_p(0)t} + \frac{1}{2} a t^2$$
$$x_p(t) = \frac{1}{2} a_p t^2$$

Where $a_p = 3\ m/s^2$.

Similarly, for the speeder we have

$$x_s(t) = x_s(0) + v_s(0)t + \cancel{\frac{1}{2} a t^2}$$
$$x_s(t) = x_{s,f} + v_s t$$

Where $x_{s,f} = 167.65\ m$, and $v_s = 33.53\ m/s$.

With an equation for the position of both cars we can write the following relationship to find the time when the position of the speeder is exactly 6 meters ahead of the police car.

$$x_s(t) = \left(x_p(t) + 6\right)$$

Substituting from above we find

$$\left(x_{s,f} + v_s t\right) = \left(\frac{1}{2} a t^2 + 6\right)$$
$$167.65 + 33.53t = 1.5t^2 + 6$$
$$1.5t^2 - 33.53t - 161.65 = 0$$

Using the quadratic formula to solve for t we find:

$$t_1 = 26.4$$
$$t_2 = -4.1$$

We disregard the negative time value and find that the police car is 6 meters behind the speeder 26.4 second after he starts chasing.

Finally, to find how fast the police car is traveling at this time instant we use the second motion equation.

$$v_{p,f} = \cancel{v_0} + a_p \Delta t$$
$$v_{p,f} = (3 * 26.4)\left(\frac{3600\ \cancel{sec}}{1\ hr} * \frac{1\ mi}{1609.34\ \cancel{m}}\right)$$
$$v_{p,f} = 177.2\ mph$$

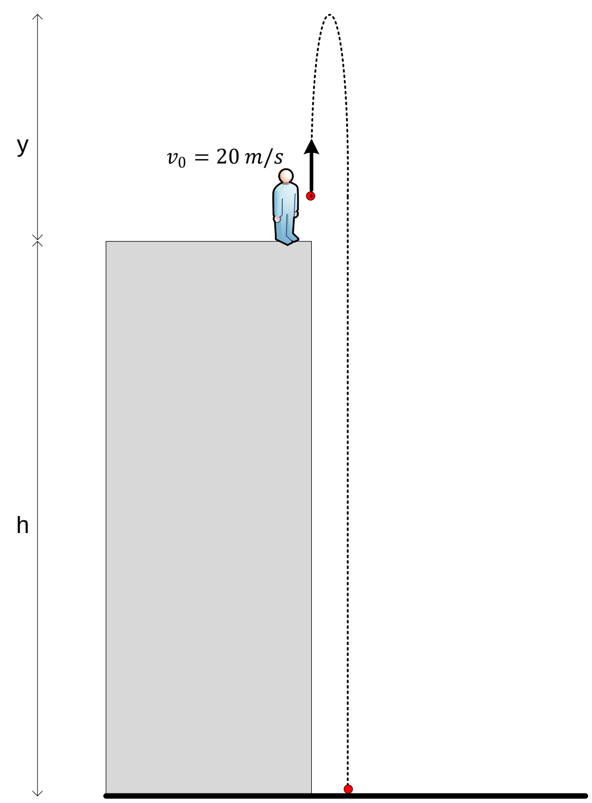
Note: This is an unrealistic high speed which leads us to believe that the police car will not maintain this acceleration for entire chase. The hope is that the speeder will hear the police car sirens and eventually slow down.

Example 2: (1D Vertical Motion - Free Fall)

A person standing on top of a $100\ m$ tall building throws a ball upward with an initial velocity of $20\ m/s$.

a.) How high does the ball reach above the building before starting to fall again?

b.) How long does it take for the ball to reach the ground and what is its speed when it lands?



Solution: Note the motion in this example is still one dimensional but is in the vertical direction. We can use the same motions equations, replacing x with y.

Part a: The key here is to realize that when the ball reaches its maximum height the velocity is zero. With that we can use the third motion equation to solve for the height when the velocity of the ball is zero. To solve directly for the height above the building we will let the top of the building be zero, $y_0 = 0$.

$$\begin{aligned} {v_f}^2 &= {v_0}^2 + 2a(y_f - y_0) \\ 0 &= 400 - 2g(y_f - 0) \\ 2gy_f &= 400 \\ y_f &= \frac{400}{2 * 9.8} \\ y_f &= 20.4\ m \end{aligned}$$

The maximum height above the building the ball will reach before beginning to fall is $20.4\ m$.

Part b: We can use the first motion equation to find the time it takes for the ball to reach the ground. In this case, we'll write the first equation as a function so that is represents the position of the ball over time.

$$\begin{aligned} y(t) &= y(0) + v_y(0)t + \frac{1}{2}at^2 \\ y(t) &= 100 + 20t - 4.9t^2 \end{aligned}$$

This equation gives the position of the ball as a function of t. We want to solve for the time when the height of the ball is zero, therefore we let $y(t) = 0$

$$0 = 100 + 20t - 4.9t^2$$

Using the quadratic formula to solve for t, and ignoring the negative answer we find $t = 7$.

Finally, to find the speed of the ball when it hits the ground, we use the second motion equation with $t = 7$.

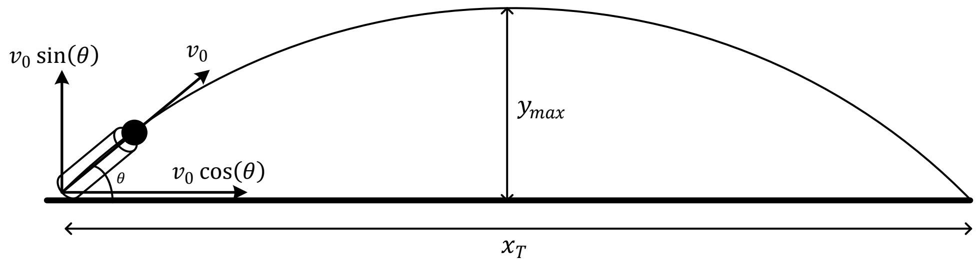
$$\begin{aligned} v_f &= v_0 + at \\ v_f &= 20 - 9.8 * 7 \\ v_f &= -48.6 \frac{m}{s} \end{aligned}$$

Which is negative as expected since the ball is falling.

Example 3: (2D Motion (Projectile Motion))

A cannon can be fired from the ground with an initial speed of $100\ m/s$. The target on the ground is at a distance of $x_T = 500\ m$.

a.) What angle, θ, should the cannon be aimed to hit the ground target?
b.) What is the maximum height the cannon reaches using the angle from part a?



Solution 3: This example uses 2 dimensional motion. As mentioned, the motion equations apply to the x and y components independently. Note how the velocity vector has been decomposed into an x and y component as explained in our lesson on vectors.

Part a: In this case, we'll start by writing the first motion equation as functions of time in both the x and y directions.

Horizontal Direction	***Vertical Direction***
$x(t) = \cancel{x(0)} + v_x(0)t + \cancel{\frac{1}{2}a_x t^2}$ $x(t) = v_0 \cos(\theta)\, t$	$y(t) = \cancel{y(0)} + v_y(0)t + \frac{1}{2}a_y t^2$ $y(t) = v_0 \sin(\theta)\, t - 4.9t^2$
Where we use the fact that $x(0)$ and a_x are both zero	Where we use the fact that $y(0) = 0$ and $a_y = g = -9.8$.

These equations independently describe the position of the ball in the horizontal and vertical directions as functions of time. For the cannon to hit its target we want $x(t) = x_T$ and $y(t) = 0$ at the same time, which we can refer to as t_T. The two equations then become

$$x(t_T) = v_0 \cos(\theta)\, t_T = x_T$$
$$y(t_T) = v_0 \sin(\theta)\, t_T - 4.9{t_T}^2 = 0$$

Solving the first equation for t_T and plugging the result into the second equation we have

$$v_0\, sin(\theta)\left(\frac{x_T}{v_0\, cos(\theta)}\right) - 4.9\left(\frac{x_T}{v_0\, cos(\theta)}\right)^2 = 0$$
$$\frac{sin(\theta)}{cos(\theta)} x_T = \frac{4.9}{{v_0}^2\, cos^2(\theta)} {x_T}^2$$
$$sin(\theta)\, cos(\theta) = \frac{4.9}{{v_0}^2} x_T$$

Next, using the trigonometry identity, $sin(\theta)\, cos(\theta) = \frac{1}{2} sin(2\theta)$, we can write

$$\frac{1}{2} sin(2\theta) = \frac{4.9x_T}{{v_0}^2}$$

$$sin(2\theta) = \frac{2 * 4.9x_T}{{v_0}^2}$$

$$\theta = \frac{sin^{-1}\left(\frac{2 * 4.9x_T}{{v_0}^2}\right)}{2}$$

$$\theta = \frac{sin^{-1}\left(\frac{2 * 4.9 * 500}{100^2}\right)}{2}$$

$$\theta = 14.67^{\circ}$$

Part b: We can find the maximum height the same way we did for example 2 since the y-component of the cannon is zero when it reaches this point.

$${v_{f,y}}^2 = {v_{0,y}}^2 + 2a_y(y_{max})$$

$$0 = (v_0\, sin(\theta))^2 - 2gy_{max}$$

$$y_{max} = \frac{(v_0\, sin(\theta))^2}{2g}$$

$$y_{max} = \frac{(100\, sin(14.67))^2}{2 * 9.8}$$

$$y_{max} = 32.7\ m$$

Additional Insight:

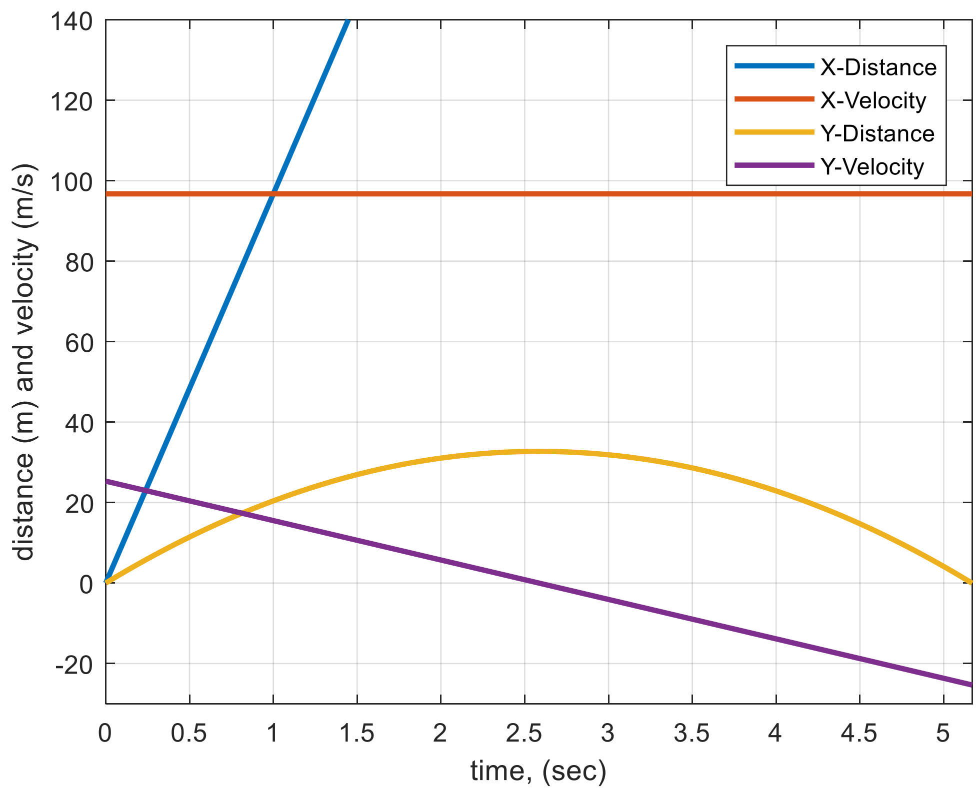
For addition insight, we will plot the velocity and position equations in the x and y directions for example 3 above.

<u>x direction equations</u>

$$x(t) = v_0 \cos(\theta)\, t$$
$$v_x(t) = v_0 \cos(\theta)$$

<u>y direction equations</u>

$$y(t) = v_0 \sin(\theta)\, t - \frac{1}{2} t^2$$
$$v_y(t) = v_0 \sin(\theta) - gt$$



Key Observations:

- The x distance is linear with time and reaches 500 m (not shown) when the y-distance reaches the ground.
- The x velocity is constant since no forces are acting on the cannon after it is fired in the x direction.
- The y distance is in the shape of a parabola and reaches its maximum height exactly halfway between the starting and landing position.
- The y velocity is linear with time, but it starts out positive, is zero at the maximum height, and then is negative until the cannon lands. Note that when the cannon lands the magnitude of the y velocity is exactly the same as it is when it started.

Final Summary for Kinematics

1D Kinematic Definitions

- **Distance**: A scalar quantity that describes the overall length along a path traveled.

- **Displacement**: A vector quantity that describes the change in the objects *position*.

$$\Delta \boldsymbol{x} = x_2 - x_1$$

- **Average Speed**: A scalar quantity that describes the *rate of change of **distance*** per unit time, Δt.

$$S_{av} = \frac{D}{\Delta t}$$

- **Average Velocity**: A vector quantity that describes the *rate of change of **displacement*** per unit time, Δt.

$$\boldsymbol{v}_{av} = \frac{\Delta \boldsymbol{x}}{\Delta t}$$

- **Average Acceleration:** A vector quantity describes the *rate of change of **velocity*** per unit time, Δt.

$$\boldsymbol{a}_{av} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

- **Instantaneous Velocity**: A vector quantity that describes the *rate of change of **displacement*** per unit time, Δt, where Δt approaches 0.

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \left(\frac{\Delta \boldsymbol{x}}{\Delta t}\right) = \frac{d\boldsymbol{x}(t)}{dt}$$

- **Instantaneous Acceleration:** A vector quantity that describes the *rate of change of **velocity*** per unit time, Δt, where Δt approaches 0.

$$\boldsymbol{a}(t) = \lim_{\Delta t \to 0} \left(\frac{\Delta \boldsymbol{v}}{\Delta t}\right) = \frac{d\boldsymbol{v}(t)}{dt} = \frac{d^2\boldsymbol{x}(t)}{dt^2}$$

1D Motion Equations

For a Specific Time Delta	***As Functions for all Times***
$x_f = x_0 + v_0 \Delta t + \frac{1}{2} a \Delta t^2$	$x(t) = x(0) + v(0)t + \frac{1}{2} a t^2$
$v_f = v_0 + a\Delta t$	$v(t) = v(0) + at$
${v_f}^2 = {v_0}^2 + 2a(x_f - x_0)$	$v(t)^2 = v(0)^2 + 2a\big(x(t) - x(0)\big)$
$x_f = x_0 + \frac{1}{2}(v_f + v_0)\Delta t$	$x(t) = x(0) + \frac{1}{2}\big(v(t) + v(0)\big)t$
$x_f = x_0 + v_f \Delta t - \frac{1}{2} a \Delta t^2$	$x(t) = x(0) + v(t)t - \frac{1}{2} a t^2$

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