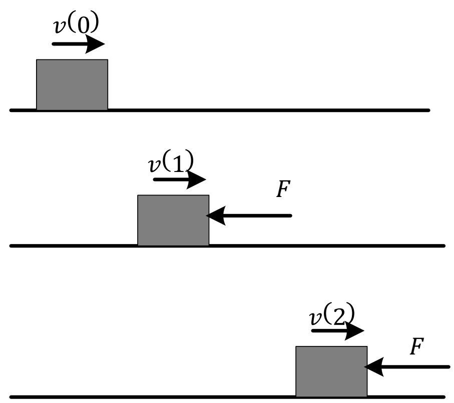
Physics 1 Mechanics - Impulse

In the previous lesson we learned that we could express Newton's 2nd law using momentum as follows.

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt}$$

where $\boldsymbol{p} = m\boldsymbol{v}$.

This equation relates the net force on an object to the change in momentum *per unit time*. Let's examine closer with an example of a body moving along the x-axis with a speed of $v(t)$.

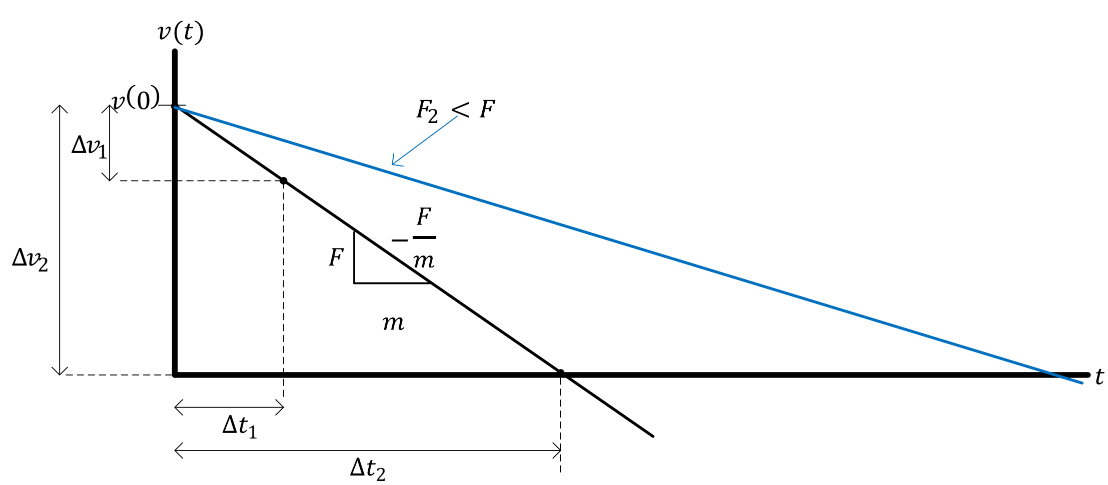


If an opposing constant force is applied to this object, it will act to slow it down. More specifically, assuming the mass is constant, the speed will change *at a constant rate* equal to the amount of force applied divided by the mass.

$$\frac{mdv}{dt} = F$$

$$\frac{dv}{dt} = \frac{F}{m}$$

This is shown below with a speed that starts at $v(0)$ and decreases linearly at a rate of $\frac{dv}{dt} = \frac{F}{m}$.



Note from the figure that the longer the force is applied to more the speed will change. For example, to bring the object to rest we would need to apply the force over a time of Δt_2. Conversely, as shown by the blue line, a smaller force can also bring the object to rest if the force is applied over a longer period. The quantity of force times the amount of time the force is applied is what we refer to as the impulse, J.

$$J = F\Delta t$$

The idea can be made more general using a non-constant applied force. To do this we again start with Newton's 2nd law, multiply through by dt and integrate both sides from t_1 to t_2.

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{p}}{dt}$$

$$\boldsymbol{F}_{net}(t)dt = d\boldsymbol{p}$$

$$\int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt = \int_{t_1}^{t_2} 1\, d\boldsymbol{p}$$

$$\int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt = \boldsymbol{p}(t_1) - \boldsymbol{p}(t_2)$$

$$\int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt = \Delta\boldsymbol{p}$$

The left-hand side is a more general expression for the impulse. It also gives us an additional way to define the impulse, i.e., using the right-hand side as the change in the momentum.

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt = \Delta\boldsymbol{p}$$

Note that the integral expression lines up with our simple example if we use a constant force.

$$\boldsymbol{F}\int_{t_1}^{t_2} 1dt = \boldsymbol{F}\Delta t = \Delta\boldsymbol{p}$$

Finally, if you are familiar with calculus you know that the average value of a function over a certain time interval can be computed as follows.

$$f_{avg} = \frac{1}{\Delta t}\int_{t_1}^{t_2} f(t)dt \rightarrow f_{avg}\Delta t = \int_{t_1}^{t_2} f(t)dt$$

This allows us to mathematically express the impulse in several different forms as shown below.

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt = \boldsymbol{F}_{net,avg}\Delta t = \Delta\boldsymbol{p}$$

Let's do some examples to see how the impulse can be used to gain insight into various physical phenomena.

Example 1: (Stop a Moving Object)

Let's use the example in the introduction with the body having a mass, $m = 50\ kg$, moving at a constant velocity of $v = 10\ m/s$ along a frictionless surface.

a.) What is the magnitude of the force required to bring the body to rest in $10\ s$?
b.) If only half of this force is available, how much time would it then take to bring the body to rest?

Solution 1a: We can start by computing the impulse as the change in momentum for the body.

$$\begin{aligned} \boldsymbol{J} &= \Delta \boldsymbol{p} \\ &= m\boldsymbol{v_f} - m\boldsymbol{v_i} \\ &= 0 - 50 \cdot 10 \\ &= -500\ N \cdot s \end{aligned}$$

As we now know, the impulse can also be computed by multiplying the constant force by the time over which it is applied.

$$\begin{aligned} \boldsymbol{F}\Delta t &= \boldsymbol{J} \\ \boldsymbol{F} &= \frac{\boldsymbol{J}}{\Delta t} \end{aligned}$$

Substituting the impulse from above we can then solve for the force.

$$\begin{aligned} \boldsymbol{F} &= \frac{-500}{10} \\ \boldsymbol{F} &= -50\ N \end{aligned}$$

1b: Reducing the force by a factor of two, i.e., $\boldsymbol{F} = -25\ N$, will result in a doubling of the time required for the same change in momentum, i.e., to bring the body to rest.

$$\begin{aligned} \boldsymbol{F}\Delta t &= \boldsymbol{J} \\ \Delta t &= \frac{\boldsymbol{J}}{\boldsymbol{F}} \\ \Delta t &= \frac{-500}{-25} \\ \Delta t &= 20\ s \end{aligned}$$

Example 2: (Broken Leg)

Two people, both weighing $70\ kg$, jump out of a window that is $3\ m$ from the ground. Person A lands stiff-legged where we can assume the body moves only $1\ cm$ upon impact, while person B bends their legs so that the body moves $50\ cm$ upon impact. Find the average force exerted on the legs of each person by the ground.

Solution 2: We can start by finding the velocity, which is the same for each person, at the start of the impact using one of our formulas from kinematics.

$$v_f{}^2 = v_i{}^2 + 2g\Delta y$$
$$v_f = -\sqrt{0 + 2(-9.8)(-3)} = -7.67\ m/s$$

Additionally, an equation for the average force applied over the duration of the impact can be written using one of the equations for impulse as follows.

$$\boldsymbol{F}_{avg}\Delta t = \boldsymbol{J}$$
$$\boldsymbol{F}_{avg} = \frac{\boldsymbol{J}}{\Delta t}$$

Next, we can find the impulse using the change in momentum relationship.

$$\boldsymbol{J} = \Delta \boldsymbol{p} = (m\boldsymbol{v_f} - m\boldsymbol{v_i}) = -m\boldsymbol{v_i}$$

Where, $\boldsymbol{v_i}$ is the final velocity of $-7.67\ m/s$ from above.

Substituting we can rewrite the average force equation as follows.

$$\boldsymbol{F}_{avg} = \frac{m\boldsymbol{v_i}}{\Delta t}$$

The average force is therefore inversely proportional to the time duration of the impact. If we assume a constant rate of deceleration, we can again use one of our kinematic equations to find the time duration of the impact based on the initial speed at impact and the change in distance through the impact.

$$\Delta x = \frac{(v_i + v_f)}{2}\Delta t$$
$$\Delta t = \frac{2\Delta x}{v_i}$$

Substituting again, we see that the average force is equivalently inversely proportional to the distance over which the impact is spread.

$$\boldsymbol{F}_{avg} = \frac{m v_i{}^2}{2\Delta x}$$

Finally, the magnitude of the average force felt on the legs of each person is computed below.

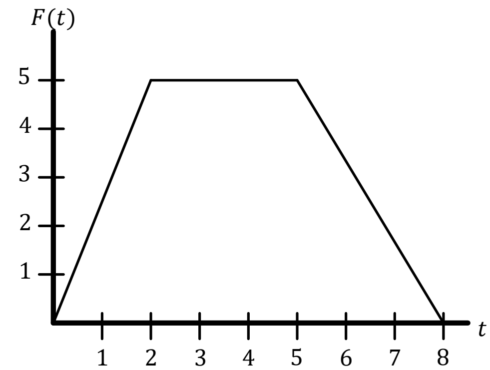
$$|\boldsymbol{F}_{avg,A}| = \frac{70 \cdot (\mathbf{7.67})^2}{2 \cdot 0.01}$$
$$|\boldsymbol{F}_{avg,A}| \cong 205{,}900\ N$$

$$|\boldsymbol{F}_{avg,B}| = \frac{70 \cdot (\mathbf{7.67})^2}{2 \cdot 0.1}$$
$$|\boldsymbol{F}_{avg,B}| \cong 4118\ N$$

As you can see, although the impulse, i.e., change in momentum, is the same in both cases, the average force experienced by each person is dramatically different. Person B, who extends the impact over a longer time, (and distance), will experience a much smaller average force than person A, who will likely experience a broken leg.

Example 3: (Integrating to Find the Impulse)

The net force acting on a body in the x-direction varies with time as shown in the figure below. Find the impulse of this force.



Solution 3: Recall the most general form of the impulse is given as an integral of the net force with respect to time as follows.

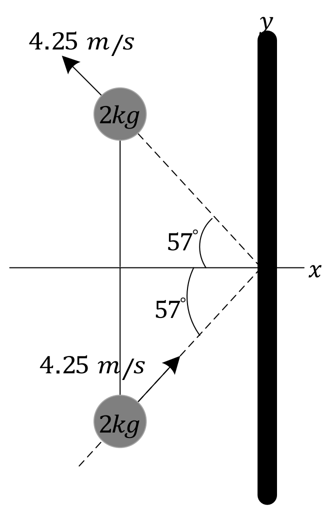
$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt$$

We can compute this integral graphically as the area under the force curve. The area can be computed graphically with two triangles and a rectangle as follows:

$$\boldsymbol{J} = \left(\frac{1}{2}2 \cdot 5\right) + (3 \cdot 5) + \left(\frac{1}{2}3 \cdot 5\right)$$
$$\boldsymbol{J} = 27.5\ N \cdot s$$

Example 4: (2D Velocity)

A 2 kg steel ball strikes a wall with a speed of $4.25\ m/s$ at an angle of 57° with the normal of the wall. It bounces off with the same speed and angle. If the ball is in contact with the wall for $0.25\ s$, what is the magnitude of the average force exerted on the ball by the wall?



Solution 4: When the ball makes impact with the wall it experiences an impulse that can be expressed as both the change in momentum of the ball as well as the average force felt by the ball multiplied by the impact time.

$$\boldsymbol{J} = \boldsymbol{F}_{avg}\Delta t = \Delta \boldsymbol{p}$$

Note that the impulse, $\boldsymbol{J}$, the net force, $\boldsymbol{F}_{avg}$, and the momentum, $\boldsymbol{p}$, are all vector quantities.

Using vector notation then, we can write the following vector equation.

$$\langle F_{avg,x}, F_{avg,y}\rangle = \frac{m}{\Delta t}\langle (v_{f,x} - v_{i,x}), (v_{f,y} - v_{i,y})\rangle$$

The velocity components can be taken directly from the figure.

$$v_i = \langle v\, cos(\theta)\,, v\, sin(\theta)\rangle$$
$$v_f = \langle -v\, cos(\theta)\,, v\, sin(\theta)\rangle$$

Substituting and plugging in the given values, we can find the average force.

$$\langle F_{avg,x}, F_{avg,y}\rangle = \frac{mv}{\Delta t}\langle(-\,cos(\theta) - cos(\theta)), (sin(\theta) - sin(\theta))\rangle$$
$$\langle F_{avg,x}, F_{avg,y}\rangle = -\frac{2mv}{\Delta t}\langle cos(\theta)\,, 0\rangle$$
$$\langle F_{avg,x}, F_{avg,y}\rangle = -\frac{2 \cdot 2 \cdot 4.25}{0.25}\langle cos(57^{\circ})\,, 0\rangle$$
$$\langle F_{avg,x}, F_{avg,y}\rangle = \langle -37, 0\rangle\ N$$

The magnitude of the average force is then

$$\left|\boldsymbol{F}_{avg}\right| = 37\ N$$

Final Summary for Impulse

Impulse

The impulse is defined as the integral of the net force over the time it acts on a body.

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}_{net}(t)dt$$

While it may be difficult to know the exact equation of the force upon an impact, we can also compute the impulse if we know the average force over that same time interval.

$$\boldsymbol{J} = \boldsymbol{F}_{net,avg}\Delta t$$

Where, the average value of a function is defined as follows.

$$\boldsymbol{f}_{avg} = \frac{1}{\Delta t}\int_{t_1}^{t_2} f(t)dt$$

And when the net force is a constant, we have

$$\boldsymbol{J} = \boldsymbol{F}\Delta t$$

Finally, the impulse is also equivalent to the change in momentum of the body on which it acts.

$$\boldsymbol{J} = \Delta \boldsymbol{p}$$

**This fact can be useful to find the average force if we can estimate the time of impact.*

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