Differential Equations – The Logistic Equation

When studying population growth, one may first think of the exponential growth model, where the growth rate is directly proportional to the present population. From the previous section, we have

$$\frac{dP}{dt} = kP$$

Where, k is the growth constant.

As we have learned, the solution to this equation is an ever-increasing exponential function, which we should quickly realize is unrealistic over large time spans. Intuition would seem to suggest that as a population grows resources become limited resulting in a slowing of the growth rate. Around 1840 P.F. Verhulst proposed an alternate model for population growth which attempts to take this fact into consideration. The model he used is based on the so-called logistic differential equation and is written as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$

Where, $A > 0$ is called the carrying capacity, which represents the largest population a particular environment can maintain.

To get an intuitive notion of how this new equation might account for the reduction in the growth rate as the population gets larger, we examine two extreme cases.

Case 1: $P \ll A$

When a population is small relative to the carrying capacity the ratio P/A becomes small and we can approximate the equation as follows

$$\frac{dP}{dt} \cong kP(1 - 0) = kP$$

Which matches the original exponential growth equation, as should be expected since the environment can more easily maintain a smaller population.

Case 2: $P \cong A$

When a population grows to the point where it approaches the carrying capacity, the ratio, P/A, becomes close to 1 and we can approximate the equation as follows

$$\frac{dP}{dt} \cong kP(1 - 1) = 0$$

Which indicates that the environment, with its limited resources, cannot maintain a population larger than A, so the growth rate approaches 0.

Next, let's try to solve this differential equation. The equation is first-order and separable, therefore we can again use the separation of variables technique.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$

$$\frac{dP}{dt} = kP\left(\frac{A - P}{A}\right)$$

$$\frac{A}{P(A - P)}dP = kdt$$

$$\int \frac{A}{P(A - P)}dP = k\int dt$$

For the left side integral, we use partial fraction expansion on the integrand.

$$\frac{A}{P(A - P)} = \frac{C_1}{P} + \frac{C_2}{A - P} \rightarrow A = C_1(A - P) + C_2P$$

We then solve for C_1 and C_2 as follows.

$Let\ P = 0$

$$A = C_1(A - 0) + C_2 \cdot 0$$

$$A = C_1A \rightarrow C_1 = 1$$

$Let\ P = A$

$$A = C_1(A - A) + C_2A$$

$$A = C_2A \rightarrow C_2 = 1$$

The integral equation can now be solved using the results from the partial fraction expansion.

$$\int \left(\frac{1}{P} + \frac{1}{A - P}\right)dP = k\int dt$$

$$ln|P| - ln|A - P| = kt + C$$

$$ln\left|\frac{P}{A - P}\right| = kt + C$$

$$\frac{P}{A - P} = Be^{kt}$$

$$P = ABe^{kt} - PBe^{kt}$$

$$P(1 + Be^{kt}) = ABe^{kt}$$

$$P = \frac{ABe^{kt}}{(1 + Be^{kt})}$$

To express the solution in a more familiar form we divide through by Be^{kt}.

$$P = \frac{A}{(1 + Be^{-kt})}$$

Finally, we can solve for B assuming an initial population of $P(0) = P_0$.

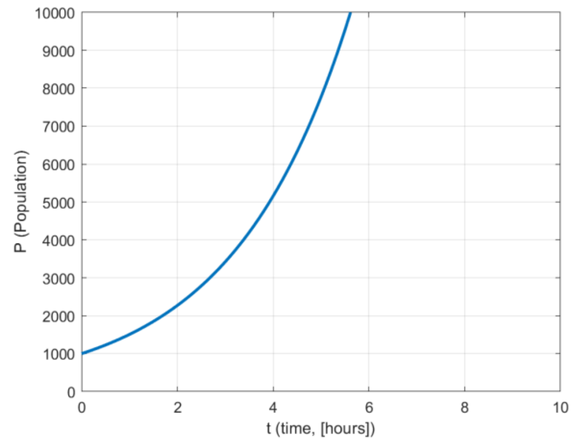
$$P_0 = \frac{A}{(1+B)}$$
$$P_0 + P_0 B = A$$
$$B = \frac{A - P_0}{P_0}$$

We summarize the above formally below.

Logistic Equation for Model Population Growth
A model for population growth which attempts to take into consideration the fact that as a population grows resources become limited, resulting in a slowing of the growth rate, is given by the following differential equation. $$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$ Where, $k > 0$ is the growth constant and $A > 0$ is called the carrying capacity.   The solution is given as $$P(t) = \frac{A}{(1 + Be^{-kt})}$$ Where, $$B = \frac{A - P(0)}{P(0)}, \ \ P(0) > 0$$

Example 1 - Population Growth : In the previous section we looked at an example of bacteria growth with a growth constant of $k = 0.41\ bacteria/hour$, and an initial population of $P(0) = 1000$. Using an exponential growth model, we found the following solution.

$$P(t) = 1000e^{0.41t}$$



As you can see from the figure, the population of bacteria grows without bound. Assuming the experiment provides the bacteria with a limited environmental capacity this unbound growth is obviously unrealistic. Instead, let's apply the logistic model using a carrying capacity of 8000.

In this case the differential equation is given as

$$\frac{dP}{dt} = 0.41P\left(1 - \frac{P}{8000}\right)$$

With a general solution of

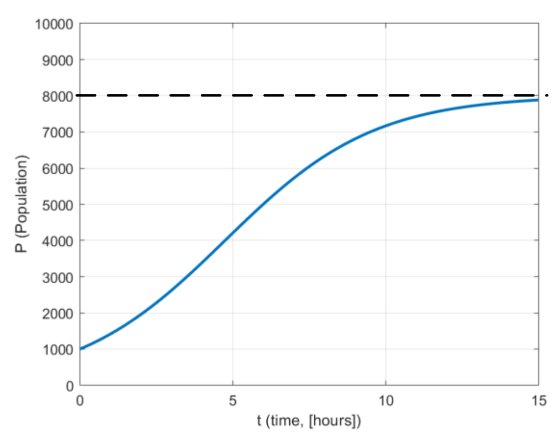
$$P(t) = \frac{8000}{(1 + Be^{-0.41t})}$$

Next, we can solve for B using the relationship derived above.

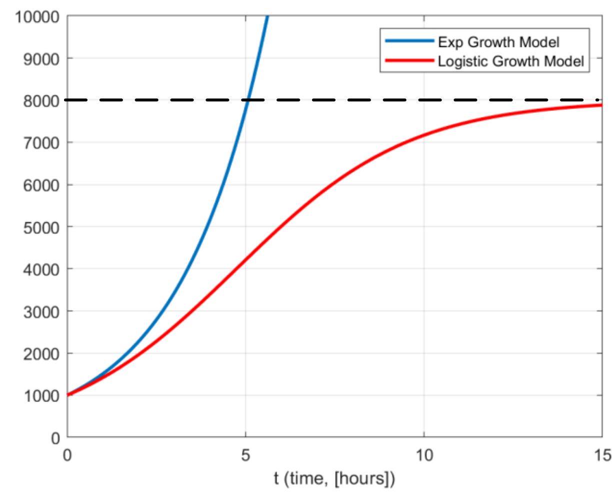
$$B = \frac{8000 - 1000}{1000} = 7$$

Finally, the equation and the resulting graph are shown below.

$$P(t) = \frac{8000}{(1 + 7e^{-0.41t})}$$



To compare the two models, we graph them together below.



The logistic model not only limits to population to 8000, but the overall growth rate is slower throughout. As you can see after 5 hours the exponential growth model has a population of nearly 8000, while the logistic model has a population of around 4000. The logistic model growth rate also decreases as the population increases, as expected, while the exponential model growth rate continues to increase.

Example 2 – Deer Population: A deer population grows logistically with a growth constant of $k = 0.4\ year^{-1}$ in a location where the carrying capacity is 1000 deer.

1. Find the expression for the deer population if the initial population is 100 deer.
2. How long does it take for the deer population to reach 500?

Solution:

1. In this case the differential equation is given as

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{1000}\right)$$

The general solution is given as

$$P(t) = \frac{1000}{(1 + Be^{-0.4t})}$$

Where,

$$B = \frac{A - P(0)}{P(0)} = \frac{1000 - 100}{100} = 9$$

Therefore, we have

$$P(t) = \frac{1000}{(1 + 9e^{-0.4t})}$$

2. The time for the population to reach 500 is found as follows:

$$500 = \frac{1000}{(1 + 9e^{-0.4t})}$$

$$(1 + 9e^{-0.4t}) = \frac{1000}{500}$$

$$e^{-0.4t} = \frac{1}{9}$$

$$t = \frac{ln(1/9)}{-0.4}$$

$$t = \frac{ln(1/9)}{-0.4} \cong 5.5\ years$$

Example 3 – U.S. Population : In 1751 Benjamin Franklin predicted that the U.S. population would increase with a growth constant of $k = 0.028\ year^{-1}$. According to the census, the population in 1800 was 5 million and 76 million in 1900.

1. Assuming exponential growth and an initial population of 5 million in 1800, predict the population in 1900.
2. Assuming logistic growth find the carrying capacity for the U.S. population.

Solution:

1. For exponential growth we have

$$P(t) = P(0)e^{kt} = 5E^6e^{0.028t}$$

To find the predicted population in 1900 we use $t = 100$.

$$P(100) = 5E^6e^{0.028\cdot 100} \cong\ 82.2\ million$$

2. Using the exponential growth model, we predicted the population to be about 82.2 million in 1900. The census however showed that the population was 76 million. This indicates that the exponential growth model doesn't fit well. Instead we may assume a logistic growth model and find the carrying capacity based on the data provided.

The general solution for the logistic model is:

$$P(t) = \frac{A}{(1 + Be^{-0.028t})}, with\ \ B = \frac{A - P(0)}{P(0)}$$

Substituting B into the general solution and simplifying we can find an expression for A.

$$P(t) = \frac{A}{\left(1 + \left(\frac{A - P(0)}{P(0)}\right) e^{-0.028t}\right)}$$

$$P(t) = \frac{AP(0)}{P(0) + Ae^{-0.028t} - P(0)e^{-0.028t}}$$

$$P(t)P(0) + P(t)Ae^{-0.028t} - P(t)P(0)e^{-0.028t} = AP(0)$$

$$A\big(P(t)e^{-0.028t} - P(0)\big) = P(t)P(0)e^{-0.028t} - P(t)P(0)$$

$$A = \frac{P(t)P(0)e^{-0.028t} - P(t)P(0)}{P(t)e^{-0.028t} - P(0)}$$

Now, using $t = 100$, $P(t) = 76E^6$, and $P(0) = 5E^6$ we can solve for the carrying capacity.

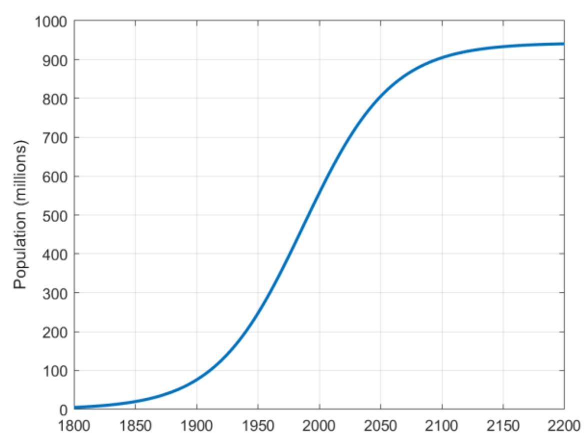
$$A = \frac{76E^6 5E^6 e^{-0.028\cdot100} - 76E^6 5E^6}{76E^6 e^{-0.028\cdot100} - 5E^6}$$

$$= \frac{380E^6(e^{-2.8} - 1)}{76e^{-2.8} - 5} \cong 943\ million$$

Next, we can solve for B and write the final equation. A graph is also shown for illustration.

$$B = \frac{943 - 5}{5} = 187.6\ million$$

$$P(t) = \frac{943}{(1 + 187.6e^{-0.028t})}\ million$$



Final Summary for Differential Equations – The Logistic Equation

The Logistic Equation for Population Growth
Around 1840 P.F. Verhulst proposed an alternate model for population growth which is based on a logistic differential equation and is written as $$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$ Where, $k > 0$ is the growth constant and $A > 0$ is called the carrying capacity. The solution is given as $$P(t) = \frac{A}{(1 + Be^{-kt})}$$ Where, $$B = \frac{A - P(0)}{P(0)}, \;\; P(0) > 0$$

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