Differential Equations Introduction – Separation of Variables

Differential equations are one of the fundamental tools used by scientists and engineers to model all types of physical systems using mathematics. Recall, algebraic equations are used to express how one or more dependent variables vary with respect to one or more independent variables. For example, $y = 3x^2 + 5x + 7$, is a 2nd order polynomial equation that relates the independent variable, x, to the dependent variable, y. Differential equations, on the other hand, express how *the derivatives* of one or more dependent variables vary with respect to one or more independent variables, e.g. $\frac{dy}{dx} = 4x^2 + 3x$. Generally speaking, "solving" an algebraic equation involves finding all of the y value that satisfy the equation. On the other hand, "solving" a differential equation involves finding all of the function(s), e.g., $y = f(x)$, that satisfy the equation. Differential equations are generally difficult to solve. In fact, many differential equations do not have solutions that can be expressed with explicit formulas. Certain classes of differential equations, however, can be solved explicitly using different methods that have been developed over time. In this section we introduce some of the basic definitions and terminology used with differential equations. We will also learn to solve differential equations of a certain kind using a method called the *Separation of Variables*.

Differential Equation Introduction

A definition for a differential equation may be formally stated as below.

Differential Equation
An equation that relates derivatives of one or more dependent variables to one or more independent variables.

Differential equations are classified according to *type*, *order*, and *linearity*.

Classification by Type:

A differential equation is called an *ordinary differential equation,* (ODE)*,* if it has only one independent variable. Differential equations that contain more than one independent variable are called *partial differential equations,* (PDE).

Ordinary Differential Equation Example	***Partial Differential Equation Example***
$3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 6y = 0$	$3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} + 6u = 0$

Classification by Order:

The highest order derivative in a differential equation determines the *order* of the differential equation. Examples are shown below.

Differential Equation Order	
First-Order	$x^2 \frac{dy}{dx} + 6y = 40$
Second-Order	$3\frac{d^2y}{dx^2} - sin(x)\frac{dy}{dx} = 0$
Third-Order	$3\frac{d^3y}{dx^3} = x\frac{dy}{dx} + cos(x^4)$

Linear vs. Non-Linear:

A differential equation is called linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

The above equation is characterized by the following two properties:

1. The dependent variable, y, is only to the first degree.
   - E.g., $5x^2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \boldsymbol{y^2} = e^x$ is non-linear because of the y^2 term.
2. Each coefficient, a_n, depends only on the independent variable, x.
   - E.g., $5x^2\frac{d^2y}{dx^2} + \boldsymbol{3y}\frac{dy}{dx} + y = e^x$ is non-linear because of the coefficient $3y$.

In this section we will restrict ourselves to first-order ODEs. More specifically, we will learn to solve differential equations that are said to be *separable* using a method called the *Separation of Variables*.

Separation of Variables:

A first-order differential equation is called separable if the first-order derivative can be expressed as the ratio of two functions; one a function of x and the other a function of y.

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

For example, $\frac{dy}{dx} = sin(x)\,y^2$ is separable since we can rewrite the equation as $\frac{dy}{dx} = \frac{sin(x)}{1/y^2}$. However, $\frac{dy}{dx} = (x + y)$ is not separable.

First-order separable differential equations are solved using the method of *Separation of Variables*, which is described below.

Given a first order separable differential equation:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

We proceed as follows:

1. Move the terms involving y and dy to one side and the terms involving x and dx to the other.
$$\frac{1}{g(y)}dy = f(x)dx$$
2. Integrate both sides of the equation.
$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

Assuming these integrals can be evaluated, we can then try to solve for y as a function of x.

We illustrate the method using the example below.

Example 1: Solve the following differential equation.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solution: By inspection we can see the equation is a first-order separable equation and therefore we can use the separation of variables technique to solve.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$ydy = xdx$$

$$\int ydy = -\int xdx$$

Next, we evaluate the integrals as shown below.

$$\frac{1}{2}y^2 + C_1 = -\frac{1}{2}x^2 + C_2$$

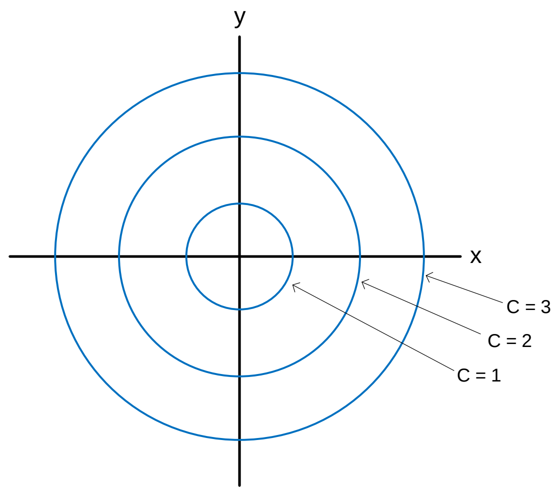
$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = C_2 - C_1$$

$$y^2 + x^2 = 2(C_2 - C_1)$$

Finally, since C_1 and C_2 are both completely arbitrary we can replace $2(C_2 - C_1)$ with a new constant C^2, giving us the equation of a circle with radius C.

$$x^2 + y^2 = C^2$$

Therefore, we find that there are an infinite number of solutions to the given differential equation. Each solution is a circle with radius C.



The solutions we found are called the *general solutions* of the differential equation. In order to select a single solution from these general solutions we will need additional information. As you can see from above each solution passes through a unique set of points in the x-y plane. Therefore, if we had knowledge of a single point in the solution, e.g. $y(x_0) = y_0$, we can use this information to find what is called a *particular solution* to the differential equation. We refer to this information as an *initial condition*, (even though it not necessary to have $x_0 = 0$). When we are given a differential equation with initial conditions, we refer to this as an *initial value problem*. For example, the previous example could have been given as an initial value problem as follows:

Solve the following differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Subject to the initial condition

$$y(1) = 2$$

To find the particular solution we start with the general solution shown below.

$$x^2 + y^2 = C^2$$

Next, we use the initial condition to solve for the unknown constant.

$$1^2 + 2^2 = C^2$$
$$\sqrt{5} = C$$

Finally, we can write the particular solution as follows.

$$x^2 + y^2 = 5$$

Let's finish this section with a few more examples of first order separable differential equations with initial conditions.

Example 2: Solve the following first order differential equation with the given initial conditions.

$$\frac{dy}{dx} + 2y = 0 \qquad\qquad y(ln(5)) = 3$$

Solution:

$$\frac{dy}{dx} = -2y$$
$$\frac{1}{y}dy = -2dx$$
$$\int \frac{1}{y}dy = -2\int dx$$
$$ln|y| = -2x + C$$
$$|y| = e^{-2x+C}$$
$$|y| = e^C e^{-2x}$$
$$y = \pm e^C e^{-2x}$$

Since C is an arbitrary number then e^C represents an arbitrary *positive* number. Therefore, we can replace the entire $\pm e^C$ with a new arbitrary nonzero, i.e., it can be both positive or negative, constant, K. The general solution is then given as shown.

$$y = Ke^{-2x}$$

To find the particular solution we use the initial condition from above and solve for K.

$$3 = Ke^{-2\,ln(5)}$$
$$3 = K\left(e^{ln(5)}\right)^{-2}$$
$$3 = K5^{-2}$$
$$75 = K$$

Finally, we can write the particular solution.

$$y = 75e^{-2x}$$

Example 3: Solve the following first order differential equation with the given initial conditions.

$$yy' = xe^{-y^2} \qquad\qquad y(0) = -2$$

Solution:

$$y\frac{dy}{dx} = xe^{-y^2}$$
$$ye^{y^2}dy = xdx$$
$$\int ye^{y^2}dy = \int xdx$$

To evaluate the y integral we use the following substitution.

$$u = y^2 \qquad\qquad \begin{aligned} du &= 2ydy \\ \frac{1}{2}du &= ydy \end{aligned}$$

$$\frac{1}{2}\int e^u du = \int xdx$$
$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2 + C$$
$$e^{y^2} = x^2 + C$$
$$y^2 = ln(x^2 + C)$$

In this case, we can leave the general solution as an implicit equation and use the initial condition from above to solve for C.

$$(-2)^2 = ln(C)$$
$$C = e^4$$

Therefore, the particular solution is given as follows.

$$y^2 = ln(x^2 + e^4)$$

Example 4: Solve the following first order differential equation with the given initial conditions.

$$y' = (x-1)(y-2) \qquad\qquad y(2) = 4$$

Solution:

$$\frac{dy}{dx} = (x-1)(y-2)$$

$$\frac{1}{(y-2)}dy = (x-1)dx$$

$$\int \frac{1}{(y-2)}dy = \int (x-1)dx$$

$$ln|y-2| = \frac{1}{2}x^2 - x + C$$

$$|y-2| = e^C e^{\left(\frac{1}{2}x^2 - x\right)}$$

$$y - 2 = Ke^{\left(\frac{1}{2}x^2 - x\right)}$$

$$y = Ke^{\left(\frac{1}{2}x^2 - x\right)} + 2$$

Where we employed the same technique from example 2 to remove the absolute value and replace $\pm e^C$ with a new constant, K. Next, we use the initial conditions to solve for K and write the particular solution.

$$4 = Ke^0 + 2$$

$$2 = K$$

$$y = 2e^{\left(\frac{1}{2}x^2 - x\right)} + 2$$

Example 5: Solve the following first order differential equation with the given initial conditions.

$$(1-t)\frac{dy}{dt} - y = 0 \qquad\qquad y(2) = -4$$

Solution:

$$(1-t)\frac{dy}{dt} = y$$

$$\frac{1}{y}dy = \frac{1}{(1-t)}dt$$

$$\int \frac{1}{y}dy = \int \frac{1}{(1-t)}dt$$

$$ln|y| = -\,ln|1-t| + C$$

$$|y| = e^C e^{(-\,ln|1-t|)}$$

$$|y| = e^C e^{(ln|1-t|)^{-1}}$$

$$y = \frac{\pm e^C}{|1-t|}$$

$$y = \frac{K}{|1-t|}$$

We then find K and write the particular solution.

$$-4 = \frac{K}{|1-2|}$$

$$-4 = K$$

$$y = \frac{-4}{|1-t|}$$

Example 6: Solve the following first order differential equation with the given initial conditions.

$$t^2\frac{dy}{dt} - t = 1 + y + ty \qquad\qquad y(1) = 0$$

Solution: We need to perform some algebraic rearranging before being able to use the separation of variables technique.

$$t^2\frac{dy}{dt} - t = 1 + y + ty$$

$$t^2\frac{dy}{dt} = 1 + y + ty + t$$

$$t^2\frac{dy}{dt} = (1 + y)(1 + t)$$

$$\frac{1}{(1+y)}dy = \frac{(1+t)}{t^2}dt$$

$$\int \frac{1}{(1+y)}dy = \int t^{-2} + \frac{1}{t}dt$$

$$ln|1 + y| = -\frac{1}{t} + ln|t| + C$$

$$1 + y = \pm e^C|t|e^{-\frac{1}{t}}$$

$$y = K|t|e^{-\frac{1}{t}} - 1$$

Next, we can solve for K as shown.

$$0 = K|1|e^{-1} - 1$$

$$1 = Ke^{-1}$$

$$e^1 = K$$

Which gives is the following particular solution.

$$y = e|t|e^{-\frac{1}{t}} - 1$$

Final Summary for Differential Equation Introduction – Separation of Variables

Differential Equation Definition

An equation that relates derivatives of one or more dependent variables to one or more independent variables.

Differential Equation Classifications

Type:

1. *Ordinary differential equation,* (ODE): Has only one independent variable.
2. *Partial differential equation,* (PDE): Has more than one independent variable.

Ordinary Differential Equation Example	***Partial Differential Equation Example***
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Order:

Order is determined by the highest order derivative in a differential equation.

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The above equation is characterized by the following two properties:

1. The dependent variable, y, is only to the first degree.
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Separation of Variables

A first-order differential equation is called *separable* if the first-order derivative can be expressed as the ratio of two functions; one a function of x and the other a function of y.

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

First-order separable differential equations are solved using the method of the *Separation of Variables* as follows:

1. Move the terms involving y and dy to one side and the terms involving x and dx to the other.
$$\frac{1}{g(y)} dy = f(x)dx$$
2. Integrate both sides of the equation.
$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

Assuming these integrals can be evaluated, we can then try to solve for y as a function of x.

$$y = f(x) + C$$

The set of solutions defined by the above equation is known as the general solution to the differential equations.

Initial Value Problem

A differential equation that is subject to an initial condition is called an *initial value problem*. An example is given as follows:

Solve the following differential equation

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Subject to the initial condition

$$y(x_o) = y_o$$

After finding the general solution, the initial condition can be used to solve for the unknown constant to obtain the so-called *particular solution*.

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