Differential Equations – Exponential Growth and Decay

As we learned in the last section differential equations are one of the fundamental tools used by scientists and engineers to model all types of physical systems using mathematics. In this section we will use differential equations to model two types of physical systems. The first is a system in which the rate of change of some quantity is proportional to the amount present. The second is a system in which the rate of change of some quantity is proportional to the amount present minus a fixed constant. Both of these systems can be described with first-order separable differential equations, which as we now know can be solved using the *Separation of Variables* technique.

Rate of Change of Quantity Proportional to the Amount Present:

For certain physical systems it can be observed that the rate of change of a certain quantity is proportional to the amount present at any time, t. This scenario can be modeled with the following differential equation.

$$\frac{d}{dt}y(t) = ky(t)$$

Where, $y(t)$ is the quantity at time t, and k is the proportionality constant.

Since the above differential equation is separable, we can find the general solution as follows:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}dy = kdt$$

$$\int \frac{1}{y}dy = k\int dt$$

$$ln|y| = kt + C$$

$$y = \pm e^{c}e^{kt}$$

$$y(t) = Ce^{kt}$$

The particular solution is found by knowing the quantity at some time, t_0. If we let $t_0 = 0$, the particular solution can be written as

$$y(t) = y(0)e^{kt}$$

As you can see the solution is an exponential function that either grows or decays based on the sign of the proportionality constant, k, which is generally referred to as the time constant.

Modeling Exponential Growth and Decay
The differential equation that models a system in which the rate of change of a certain quantity, $y(t)$, is proportional to the quantity present is as follows: $$\frac{dy}{dt} = ky$$ Where k is referred to as the proportionality, or time, constant. Assuming we know the quantity at $t = 0$, the particular solution is given as $$y(t) = y(0)e^{kt}$$ • If $k < 0$ the solution is an exponential decaying function. • If $k > 0$ the solution is an exponential growth function.

Let's look at some systems that can be modeled using the above differential equation.

Example 1 - Population Growth : In the laboratory it is observed that the rate of growth for a certain bacterium is proportional to the amount present. The growth constant is measured experimentally to be $k = 0.41\ bacteria/hour$. Answer the following questions related to this scenario assuming that there are 1000 bacteria present at time $t = 0$.

1. Express the differential equation that models this scenario.
2. Solve the differential equation to find the expression for the bacteria population, $P(t)$.
3. How large is the population after 5 hours?
4. When will the population reach 10,000?

Solution:

1. The differential equation that models the above scenario is as follows:

$$\frac{dP(t)}{dt} = 0.41P(t)$$

Where $P(t)$ is the number of bacteria present at time t.

2. From above we know that the general solution to the differential equation is given as

$$P(t) = P(0)e^{0.41t}$$

Furthermore, we are told that the population at time $t = 0$ is 1000. Therefore, the particular solution is

$$P(t) = 1000e^{0.41t}$$

3. We can find the population at $t = 5$ with direct substitution as follows:

$$P(5) = 1000e^{0.41 \cdot 5}$$
$$P(5) \cong 7778$$

4. To find the time when the population reaches 10,000, we need to solve for t as follows:

$$10000 = 1000e^{0.41 \cdot t}$$
$$10 = e^{0.41 \cdot t}$$
$$ln(10) = ln(e^{0.41 \cdot t})$$
$$ln(10) = 0.41t$$
$$t = \frac{ln(10)}{0.41} \cong 5.62\ hours$$

Example 2 – "Population" Decay : Doctors have shown certain drugs leave a person's bloodstream at a rate that is proportional to the amount present. In an experiment a patient is injected with $450\ mg$ of a substance. Seven hours later it is found that $50\ mg$ of the substance remains. Assuming the proportional model is correct for the particular substance

1. Express the differential equation that models this scenario.
2. Find the time constant, k.
3. Find the time when $200\ mg$ of the substance remained in the bloodstream.

Solution:

1. With the amount of the substance represented as $A(t)$, we can write the differential equation as

$$\frac{dA}{dt} = kA$$

2. To find the time constant, k, we first write the expression for the general solution as

$$A(t) = 450e^{kt}$$

Where, $A(0) = 450$.

Next, we use the fact that 50 mg was found in the bloodstream after 7 hours to solve for k.

$$A(7) = 450e^{k\cdot 7}$$
$$50 = 450e^{k\cdot 7}$$
$$\frac{1}{9} = e^{k\cdot 7}$$
$$ln\left(\frac{1}{9}\right) = k \cdot 7$$
$$k = \frac{-\,ln(9)}{7} \cong -0.31$$

As expected, k is negative since the substance decays over time.

3. We can now find the time when the amount of substance remaining was 200 mg.

$$200 = 450e^{-0.31\cdot t}$$
$$\frac{4}{9} = e^{-0.31\cdot t}$$
$$ln\left(\frac{4}{9}\right) = -0.31 \cdot t$$
$$t = \frac{-\,ln\left(\frac{4}{9}\right)}{-0.31} \cong 2.62\ hrs$$

Example 3 – Carbon Dating : An American chemist named William Libby developed a technique to estimate the age of fossils in the late 1950s. He was awarded the Nobel Prize in 1960 for this work. The technique relies on the fact that the ratio of the amount of radioactive carbon, C^{14}, to nonradioactive carbon, C^{12}, appears to be constant in all living organisms. However, when the organism dies C^{14} begins to decay with a half-life of 5600 years, i.e. half of the present amount decays every 5600 years. Using this information estimate the age of a fossil that is found to contain $1/1000^{th}$ the original amount of C^{14}.

Solution: Assuming the C^{14} decay is proportional to the amount present we can use the exponential decay model, which is described by the following differential equation.

$$\frac{dA}{dt} = kA$$

Where, A represents the amount of C^{14} present. The general solution, as we know, is given as

$$A(t) = A(0)e^{kt}$$

We can find k by using the half life of 5600 years as follows:

$$\frac{\cancel{A(0)}}{2} = \cancel{A(0)}e^{k\cdot 5600}$$
$$ln\left(\frac{1}{2}\right) = k \cdot 5600$$
$$k = \frac{-\,ln(2)}{5600} \cong -0.00012378$$

We can now estimate the age of the fossil using the fact that it is currently measured to contain $1/1000^{th}$ the original amount of C^{14}.

$$\frac{\cancel{A(0)}}{1000} = \cancel{A(0)}e^{-0.00012378t}$$
$$ln\left(\frac{1}{1000}\right) = -0.00012378t$$
$$t = \frac{-\,ln(1000)}{-0.00012378} \cong 55{,}808\ years$$

Example 4 – Compound Interest: Another situation for which we can apply an exponential growth model is related to banking. Suppose we deposited an initial amount of money, $P(0)$, into an interest-bearing account. If the interest is compounded continuously then the rate of change of the balance in the account is proportional to the present amount, with the interest rate, r, being the proportionality constant.

$$\frac{dP}{dt} = rP$$

As we well know by now, the general solution to this equation is given as

$$P(t) = P(0)e^{rt}$$

Where, in this case t is in years

Assuming we initially deposited $\$100{,}000$ into an account that was offering a 10% interest rate compounded continuously

1. Find the amount in the account after 20 years.
2. Find the number of years for the balance to reach $1,000,000.

Solution:

1. We can find the balance after 20 years with direct substitution.

$$P(20) = 100{,}000e^{0.1\cdot 20} = \$738{,}905.61$$

2. The number of years to reach 1 million dollars is computed as follows:

$$1{,}000{,}000 = 100{,}000\mathrm{e}^{0.1\cdot \mathrm{t}}$$
$$\ln(10) = 0.1\cdot \mathrm{t}$$
$$\mathrm{t} = \frac{\ln(10)}{0.1} \cong 23 \text{ years}$$

Rate of Change of Quantity Proportional to the Amount Present Minus a Fixed Value:

The following differential equation represents a scenario when the rate of change of a quantity is proportional to the amount present.

$$\frac{dy}{dt} = ky$$

A closely related scenario is one in which the rate of change of a quantity is proportional to the amount present minus a fixed value. The differential equation for this scenario is as follows:

$$\frac{dy}{dt} = k(y - b)$$

Which can also be solved using the separation of variables technique.

$$\frac{1}{(y-b)}dy = kdt$$
$$\int \frac{1}{(y-b)}dy = \int kdt$$
$$ln|y - b| = kt$$
$$y - b = Ce^{kt}$$
$$y(t) = Ce^{kt} + b$$

Let's look at some systems where this differential equation applies.

Example 5 – Newton's Law of Cooling: Consider a hot object that is placed in a room where the ambient temperature is T_0. Newton assumed that the rate of change of the temperature of the object would be proportional to the difference in the current temperature of the object and the ambient temperature of the room. This can be expressed as

$$\frac{dT}{dt} = k(T - T_0)$$

Where, k is referred to as the cooling constant.

Assume we initial place a hot bar with a cooling constant, k, of $-2.1\ min^{-1}$, into a large tank of water which is held at $T_0 = 10^{\circ}C$.

1. Write the differential equation that models this scenario and solve for the general solution.
2. What is the bar's temperature after $1\ min$ if its initial temperature was $180^{\circ}C$?
3. What was the bar's initial temperature if it cooled to $80^{\circ}C$ in $30\ sec$?

Solution:

1. The differential equation to model the above scenario is as follows:

$$\frac{dT}{dt} = -2.1(T - 10)$$

Which has a general solution of

$$T(t) = Ce^{-2.1t} + 10$$

2. To find the bar's temperature after $1\ min$ we must first solve for the unknown constant, C.

Since the initial temperature of the bar was $180^{\circ}C$ we proceed as follows:

$$180 = Ce^{-2.1\cdot 0} + 10$$
$$180 = C + 10$$
$$C = 170$$

Therefore, the particular solution is

$$T(t) = 170e^{-2.1t} + 10$$

We can now solve for the temperature after 1 min using direct substitution.

$$T(1) = 170e^{-2.1\cdot 1} + 10 \cong 30.8^{\circ}C$$

3. We start with the general solution and find a new constant based on the condition given, i.e. $80^{\circ}C$ after 30 sec.

$$80 = Ce^{-2.1\cdot 0.5} + 10$$
$$C = \frac{70}{e^{-1.05}}$$
$$C = 200$$

The particular solution in this case is then

$$T(t) = 200e^{-2.1t} + 10$$

The initial temperature is then found as follows:

$$T(0) = 200e^{-2.1\cdot 0} + 10 = 210^{\circ}C$$

Example 6 – Annuity: In example 4 we described a scenario where an initial amount of money, $P(0)$, was placed into an interest-bearing account that was compounded continuously. As long as no money was ever removed from the account the differential equation describing this scenario is

$$\frac{dP}{dt} = rP$$

Which has a solution given by

$$P(t) = P(0)e^{rt}$$

An annuity is an investment in which an initial amount of money is placed into the same type of interest-bearing account, however, money is also withdrawn at regular intervals. If the withdrawals occur over a short enough interval, we can assume that the money is withdrawn at a rate of N dollars per year. In this case the differential equation contains an extra term to account for this withdrawal.

$$\underbrace{\frac{dP}{dt}}_{rate\ of\ change\ in\ account} = \underbrace{rP}_{interest\ rate} - \underbrace{N}_{withdrawal\ rate}$$

If we factor out an r the equation has the same form as the differential equation introduced in this section.

$$\frac{dP}{dt} = r\left(P - \frac{N}{r}\right)$$

The general solution is then given as

$$P(t) = Ce^{rt} + \frac{N}{r}$$

An annuity is offering an interest rate, r, of 7% and you wish to make withdrawals at a rate of $\$500$ per year.

1. Find when the annuity will run out of money if the initial deposit is $\$5000$.
2. Find when the annuity will run out of money if the initial deposit is $\$9{,}000$.

Solution:

1. We first need to find the unknown constant, C, using the initial deposit, i.e. $P(0) = 5000$.

$$P(0) = Ce^{r\cdot 0} + \frac{N}{r}$$
$$5000 = Ce^{0.07\cdot 0} + \frac{500}{0.07}$$
$$C = 5000 - \frac{500}{0.07} \cong -2143$$

The particular solution can then be written as

$$P(t) = -2143e^{0.07t} + 7143$$

The time when the balance will be zero is then found as

$$0 = 7143 - 2143e^{0.07t}$$
$$ln\left(\frac{7143}{2143}\right) = 0.07t$$
$$t = ln\left(\frac{7143}{2143}\right)/0.07 \cong 17\ years$$

2. We need to again find the unknown constant, C, using the initial deposit, i.e., $P(0) = 9000$.

$$9000 = Ce^{0.07\cdot 0} + \frac{500}{0.07}$$
$$C = 9000 - \frac{500}{0.07} \cong 1857$$

The particular solution can then be written as

$$P(t) = 1857e^{0.07t} + 7143$$

In this case, since the constant, C, is positive the account will continue to grow forever. The account will never run out of money!

Before summarizing, let's look a little closer at the annuity. The general solution was given as

$$P(t) = Ce^{rt} + \frac{N}{r}$$

Where, $P(t)$ is the amount in the account at time, t. The yearly interest rate, compounded continuously, is given as r, and the amount of money per year being withdrawn is N.

To find a general expression for the constant, C, we assume the initial amount is $P(0) = P_0$.

$$P(0) = Ce^{r \cdot 0} + \frac{N}{r}$$
$$C = P_0 - \frac{N}{r}$$

Therefore, we can write

$$P(t) = \left(P_0 - \frac{N}{r}\right) e^{rt} + \frac{N}{r}$$

With this we can answer some interesting question that one may pose for an annuity. Two examples are given below.

1. Given an interest rate, r, and an initial amount, P_0, what is the maximum value that can be withdrawn from the account each year while leaving the initial amount constant, i.e., "living off the interest"?

To answer this question, we set $P(t) = P_0$, and solve for N.

$$P_0 = \left(P_0 - \frac{N}{r}\right) e^{rt} + \frac{N}{r}$$
$$\left(P_0 - \frac{N}{r}\right) = \left(P_0 - \frac{N}{r}\right) e^{rt}$$

Since $e^{rt} > 0$, for all t, the equality can only occur when $\left(P_0 - \frac{N}{r}\right) = 0$, i.e.,

$$N = rP_0$$

For example, if you had account with $P_0 = \$1{,}000{,}000$ getting an interest rate of 5%, you could withdrawal $rP_0 = 0.05 * \$1{,}000{,}000 = \$50{,}000$ per year forever, "Live off the interest".

2. Given an interest rate, r, and an initial amount, P_0, how much can you withdrawal each year if you wanted the money to last for T years?

To answer this question, we set $P(T) = 0$ and solve for N.

$$0 = \left(P_0 - \frac{N}{r}\right)e^{rT} + \frac{N}{r}$$

$$0 = P_0 e^{rT} - \frac{N}{r}e^{rT} + \frac{N}{r}$$

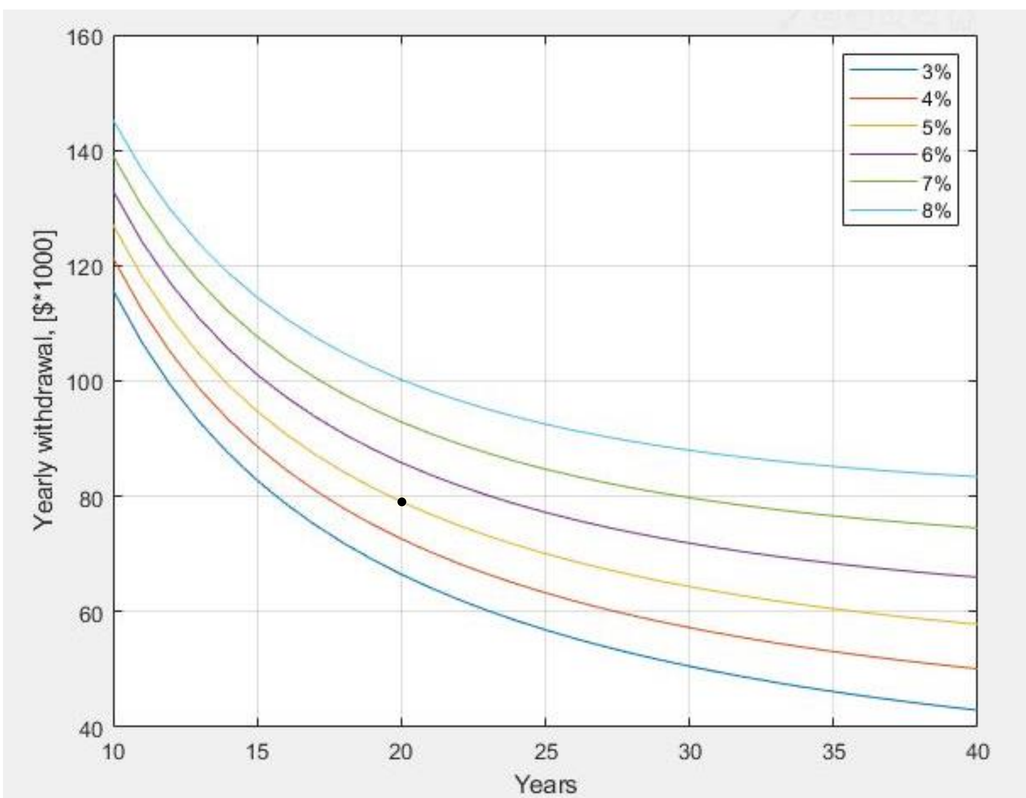
$$\frac{N}{r}(e^{rT} - 1) = P_0 e^{rT}$$

$$N = \frac{(P_0 r e^{rT})}{(e^{rT} - 1)}$$

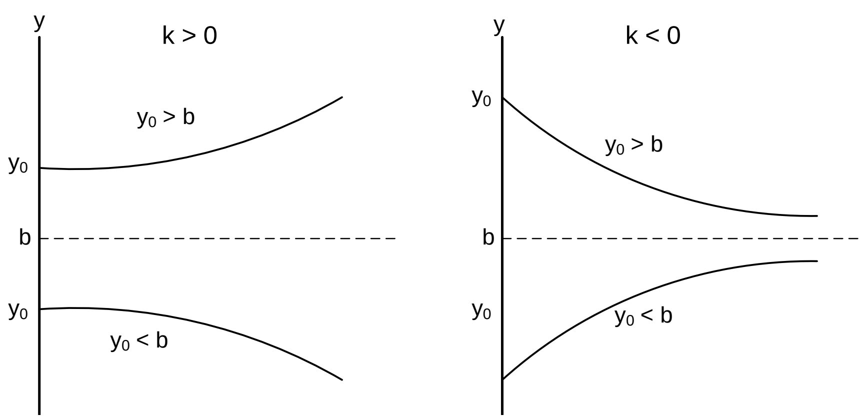
For example, if you had account with $P_0 = \$1{,}000{,}000$ getting an interest rate of 5%, and you wanted your money to last for 20 years, you could withdrawal

$$N = \frac{(\$1{,}000{,}000 \cdot 0.05 \cdot e^{0.05 \cdot 20})}{(e^{0.05 \cdot 20} - 1)} \cong \$79{,}000\ per\ year$$

For illustration, the figure below shows the amount you can withdraw as a function of the number of years for various interest rates. The example value above is highlighted.



Final Summary for Differential Equations – Exponential Growth and Decay

Exponential Growth/Decay – Rate ∝ Amount Present
The differential equation that models a system in which the rate of change of a certain quantity, $y(t)$, is proportional to the quantity present is as follows: $$\frac{dy}{dt} = ky$$ Where k is referred to as the proportionality, or time, constant. The general solution to this differential equation is given as $$y(t) = Ce^{kt}$$ Assuming we know the quantity at $t = 0$, the particular solution is given as $$y(t) = y(0)e^{kt}$$ • If $k < 0$ the solution is an exponential decaying function. • If $k > 0$ the solution is an exponential growth function.
Exponential Growth/Decay – Rate ∝ (Amount Present – Fixed Value)
The differential equation that models a system in which the rate of change of a certain quantity, $y(t)$, is proportional to the quantity present minus a fixed value, b, is as follows: $$\frac{dy}{dt} = k(y - b)$$ Where k is referred to as the proportionality constant and b is a fixed value. The general solution to this differential equation is given as $$y(t) = Ce^{kt} + b$$ Assuming we know the quantity at $t = 0$, the particular solution is given as $$y(t) = (y(0) - b)e^{kt} + b$$ We can identify the behavior of 4 cases based on the sign of k and $(y(0) - b)$.  Note: this behavior would be identical to the first type of differential equation if let $b = 0$.

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