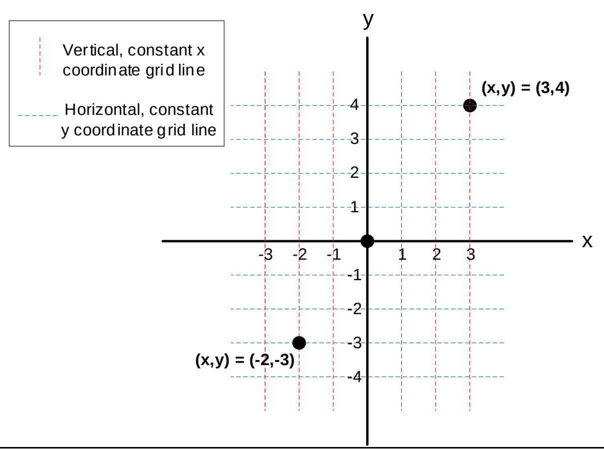
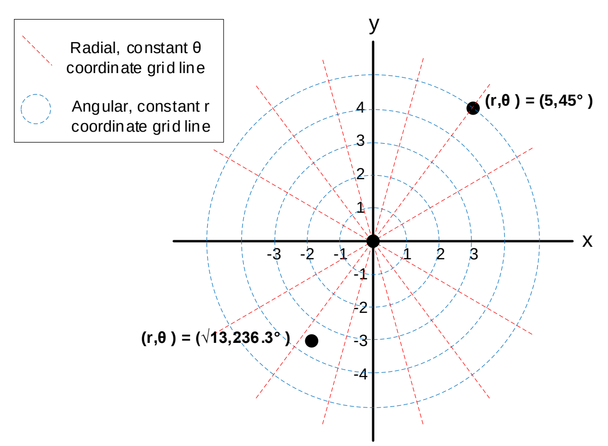
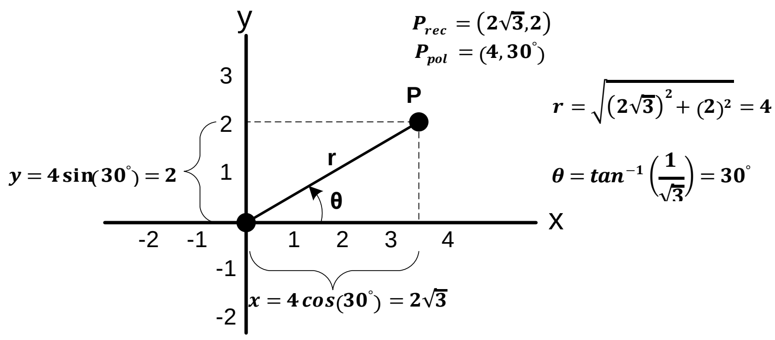
Polar Coordinates – Introduction

Up to now we have used the *rectangular coordinate system* to represent points in a plane. In this case, each point in a plane is represented by two real numbers, (x, y), giving the horizontal and vertical distance respectively from a specified origin. However, it is sometimes more convenient to use an alternate coordinate system to specify points in a plane. One such system is called the *polar coordinate system*. Each point is still specified by two real numbers, however, the numbers, (r, θ), represent the distance from the origin and the angular value between the positive x-axis and the line that connects the origin to the point. The coordinate, r, is referred to as the radial coordinate, and θ is known as the angular coordinate. These two systems are shown in the figure below.

Rectangular Coordinate System	**Polar Coordinate System**
	

Converting between the two coordinate systems is possible using basic trigonometry as shown in the figure below. Note the angle is always measured from the positive x-axis, therefore care must be taken when using the inverse tangent function outside of the first quadrant.

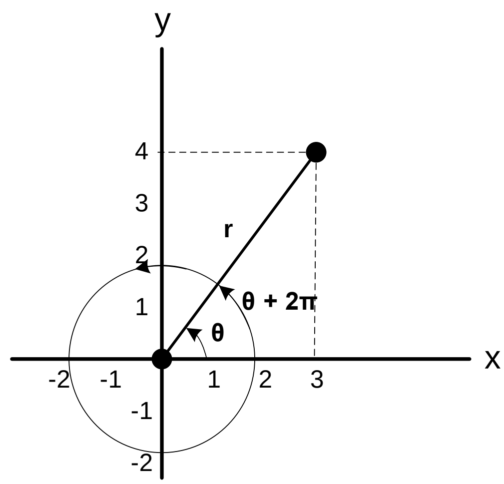


The table below summarizes the conversions.

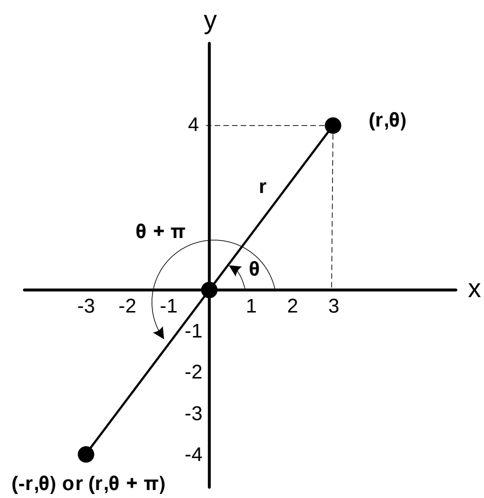
2D Coordinate Conversion Formulas	
Polar to Rectangular: $(r, \theta) \rightarrow (x, y)$	***Rectangular to Polar:*** $(x, y) \rightarrow (r, \theta)$
$x = r\cos(\theta)$ $y = r\sin(\theta)$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$

Before looking at some examples, we mention a few details about the polar coordinate system.

- Polar coordinates are not unique in the sense that the same point can be specified by different coordinate pairs. This is because adding 2π to the angular coordinate results in traveling one full revolution around the circle, and hence returning to the same point.
    - $(r, \theta) = (r, \theta + n2\pi)$, for any integer n.



- By convention we allow for negative radial coordinates. Therefore, by definition $(-r, \theta)$ is the reflection of (r, θ) through the origin. In other words, $(-r, \theta) = (r, \theta + \pi)$.



Lastly, we mention that we can avoid the ambiguity of polar coordinates from the above two points by restricting both r and θ. We commonly choose the following restrictions:

$$r > 0 \quad , \quad 0 \leq \theta < 2\pi$$

Now that we understand the basics of a polar coordinate system let's do dome examples.

Example 1: Convert the following rectangular coordinates to polar coordinates.

a. $(2,3)$ *b.* $(4,-7)$ *c.* $(-3,-8)$ *d.* $(-5,2)$

Solution: When converting from rectangular to polar coordinates care needs to be taken when using the inverse tangent function with regard to the quadrant the point lies.

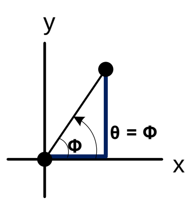
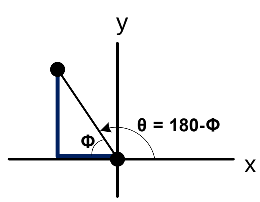
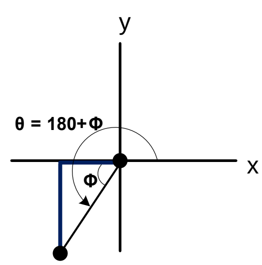
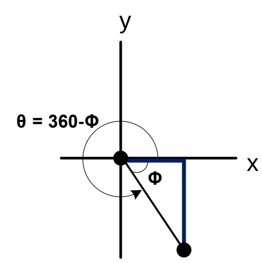
a. This point lies in the first quadrant; therefore, we can use the inverse tangent directly.

$$r = \sqrt{2^2 + 3^2} = \sqrt{13}$$
$$\theta = tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ = 0.983\ rad$$

b. This point lies in the fourth quadrant; therefore, we need to use care when computing the angular coordinate.

$$r = \sqrt{4^2 + (-7)^2} = \sqrt{65}$$

One way to compute the angle is to use the absolute value of the x and y coordinates to compute an angle, φ, and then compute θ based on the quadrant as illustrated below.

<table>
<tr><th>1st quadrant
$+x, +y$</th><th>2nd quadrant
$-x, +y$</th><th>3rd quadrant
$-x, -y$</th><th>4th quadrant
$+x, -y$</th></tr>
<tr><td>$\theta = \varphi$</td><td>$\theta = \pi - \varphi$</td><td>$\theta = \pi + \varphi$</td><td>$\theta = 2\pi - \varphi$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
</table>

$$\varphi = tan^{-1}\left(\frac{7}{4}\right) = 60.3^\circ \ \rightarrow\ \theta = 360^\circ - \varphi = 299.7^\circ$$

c. This point lies in the third quadrant.

$$r = \sqrt{(-3)^2 + (-8)^2} = \sqrt{73}$$
$$\varphi = tan^{-1}\left(\frac{8}{3}\right) = 69.4^\circ \ \rightarrow\ \theta = 180^\circ + \varphi = 249.4^\circ$$

d. This point lies in the second quadrant.

$$r = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$$
$$\varphi = tan^{-1}\left(\frac{2}{5}\right) = 21.8^\circ \ \rightarrow\ \theta = 180^\circ - \varphi = 158.2^\circ$$

Example 2: Convert the following polar coordinates to rectangular coordinates.

a. $\left(3, \frac{\pi}{6}\right)$ *b.* $\left(6, \frac{3\pi}{4}\right)$ *c.* $\left(0, \frac{\pi}{5}\right)$ *d.* $\left(5, -\frac{\pi}{2}\right)$

Solution: When converting from polar to rectangular coordinates the signs of the x and y coordinates are handled automatically from the sine and cosine function.

a. The angle, $\frac{\pi}{6}$, which is equivalent to 30°, represents a point in the first quadrant where x and y are both positive.

$$x = 3\,cos\left(\frac{\pi}{6}\right) = 3\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \qquad y = 3\,sin\left(\frac{\pi}{6}\right) = 3\frac{1}{2} = \frac{3}{2}$$

b. The angle, $\frac{3\pi}{4}$, which is equivalent to 135°, represents a point in the second quadrant where x is negative and y is positive.

$$x = 6\,cos\left(\frac{3\pi}{4}\right) = 6\frac{-\sqrt{2}}{2} = -3\sqrt{2} \qquad y = 6\,sin\left(\frac{3\pi}{4}\right) = 6\frac{\sqrt{2}}{2} = 3\sqrt{2}$$

c. The angle, $\frac{\pi}{5}$, which is equivalent to 36°, represents a point in the first quadrant, however, since the length of the line to this point is zero it lies at the origin.

$$x = 0\,cos\left(\frac{\pi}{5}\right) = 0 \qquad y = 0\,sin\left(\frac{\pi}{5}\right) = 0$$

d. The angle, $-\frac{\pi}{2}$, which is equivalent to -90°, represents a point in that is located below the x-axis and directly on the y-axis. Note, if we were to restrict the angles from 0 to 2π this point is equivalent to $(5, 3\pi/2)$

$$x = 5\,cos\left(-\frac{\pi}{2}\right) = 5 \cdot 0 = 0 \qquad y = 5\,sin\left(-\frac{\pi}{2}\right) = 5(-1) = -5$$

Example 3: Find two polar representations of $(-1,1)$. One with $r > 0$ and one with $r < 0$.

Solution:

For $r > 0$ we can use the conventional conversion from above.

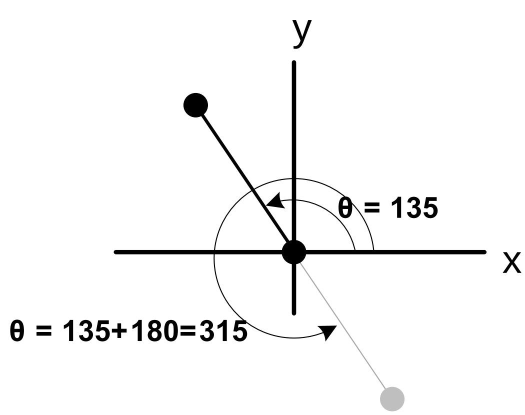
$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

For the angle since the point is in quadrant two, we proceed as follows.

$$\varphi = tan^{-1}\left(\frac{1}{1}\right) = 45^{\circ} \ \rightarrow \theta = 180^{\circ} - \varphi = 135^{\circ}$$

Next, for $r < 0$, i.e., $r = -\sqrt{2}$, the point is reflected through the origin. To counteract this reflection, we can add π to the original angle. Therefore, we have

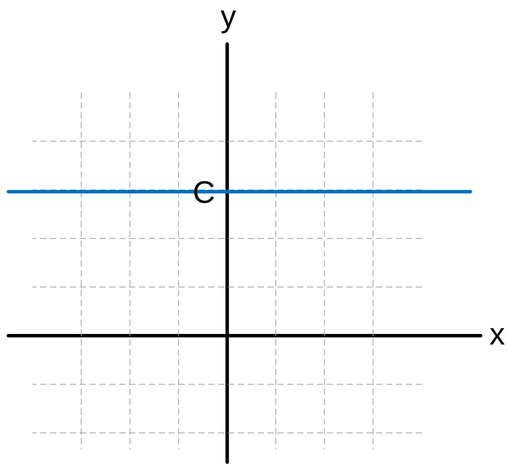
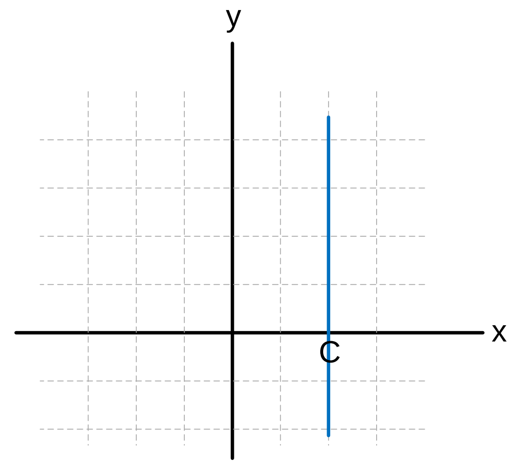
$$\theta = 135^{\circ} + 180^{\circ} = 315^{\circ}$$



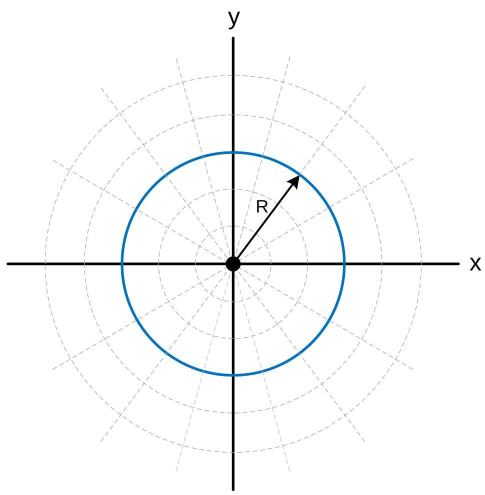
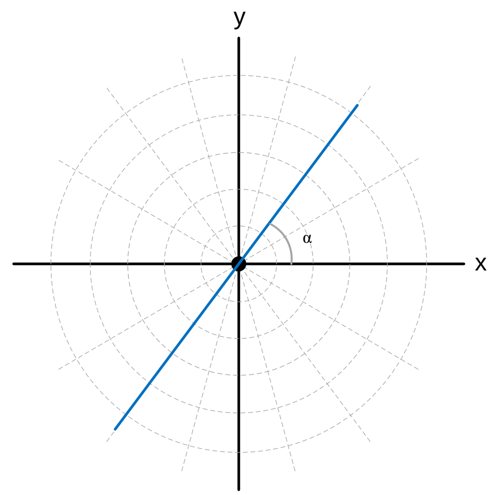
Polar Equations

A curve described by an equation involving polar coordinates is called a polar equation. When trying to sketch the graph of a polar equation it's important to keep in mind the fact that the grid lines are quite different than they are in the rectangular coordinate system. To make this point clearer let's begin by describing two trivial equations in each system.

In rectangular coordinates these equations describe a horizontal and vertical line, respectively.

Horizontal Line, $y = C$	**Vertical Line,** $x = C$
	
y is always equal to C for any x value.	x is always equal to C for any y value.

In polar coordinates these equations describe a circle and a line through the origin respectively.

Circle, $r = R$	**Line through the origin,** $\theta = \alpha$
	
r is always equal to R for any θ value.	θ is always equal to α for any r value.

With these ideas in mind let's look at some other common polar graphs.

Lines not passing through the origin

In rectangular coordinates if we are given a point, e.g., (x_0, y_0), and a slope, e.g., m, we can write the equation of the line as follows.

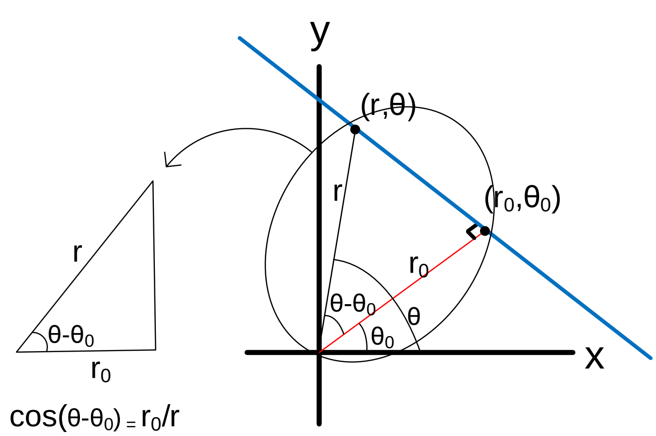
$$y(x) = m(x - x_0) + y_0$$

A polar equation for this line can be created by substituting $y = r\,sin(\theta)$ and $x = r\,cos(\theta)$ as follows.

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ r\,sin(\theta) &= m(r\,cos(\theta) - x_0) + y_0 \\ r\,sin(\theta) - mr\,cos(\theta) &= -mx_0 + y_0 \\ r(sin(\theta) - m\,cos(\theta)) &= y_0 - mx_0 \\ r(\theta) &= \frac{y_0 - mx_0}{sin(\theta) - m\,cos(\theta)} \end{aligned}$$

As you can see this equation requires knowledge of a rectangular coordinate, (x_0, y_0), from the line and its slope, m. However, we would also like to find a general form for the equation of a line using polar coordinates from the start. We do this below.

We start with a point given in polar coordinates as (r_0, θ_0). We then draw a line through this point such that it is perpendicular to another line which is drawn from the origin to the point. This is shown in the figure below by the blue and red line, respectively.



Next, we choose a variable point, (r, θ), on the line and form a right triangle. The equation of the line is then derived using the trigonometric relationship between the adjacent side and the hypotenuse as shown in the figure and below.

$$\begin{aligned} cos(\theta - \theta_0) &= \frac{r_0}{r} \\ r(\theta) &= \frac{r_0}{cos(\theta - \theta_0)} \\ r(\theta) &= r_0\,sec(\theta - \theta_0) \end{aligned}$$

Circle not centered at origin

We can start by showing that the following polar equation describes a circle of a radius of a centered at $(a, 0)$.

$$r(\theta) = 2a\,cos(\theta)$$

Since we are already familiar with the equation of a circle in rectangular coordinates, let's show the above polar equation describes the circle stated by converting it to rectangular coordinates. We start by multiplying through by r and then using the conversion formulas we learned above.

$$r^2 = 2ar\,cos(\theta)$$
$$x^2 + y^2 = 2ax$$
$$x^2 - 2ax + y^2 = 0$$

Next, we complete the squares for the x terms.

$$(x - a)^2 - a^2 + y^2 = 0$$
$$(x - a)^2 + y^2 = a^2$$

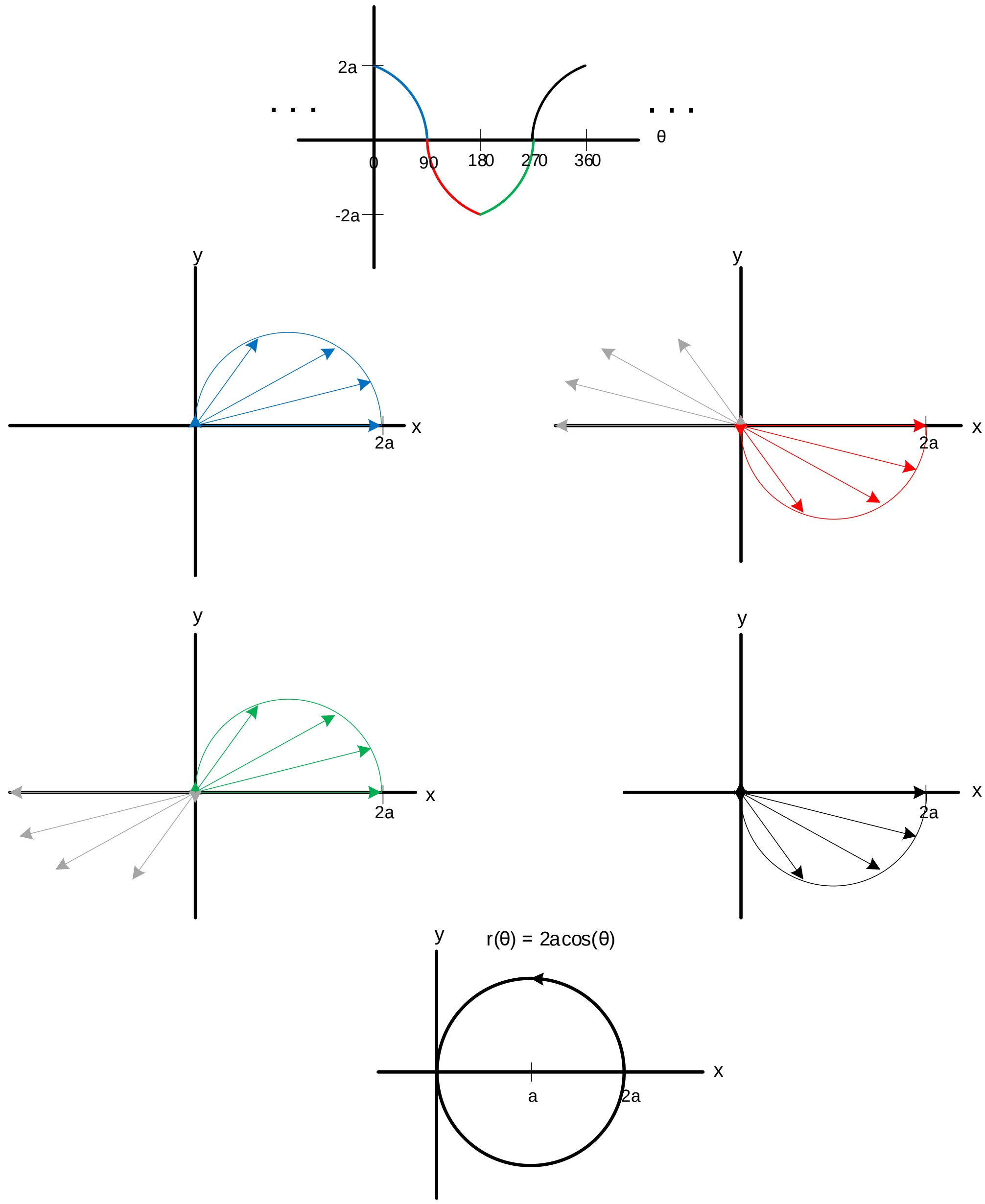
Which, we should notice describes a circle of radius a, centered at $(a, 0)$ as was claimed above.

Although this is a valid method for graphing a polar equation, it is also useful to learn how to graph polar equations directly. Next, we illustrate a general procedure that can be used to sketch the graphs of polar equations using the same circle from above.

We start by sketching the polar equation on rectangular axes. In this case the cosine function is well known and is sketched below for one period. Next, we map the rectangular coordinates to the polar plane one quadrant at a time. The table below list values of θ and r for the first quadrant.

θ	$r = 2a\,cos(\theta)$
0°	$2a(1)$
30°	$2a(\sqrt{3}/2)$
45°	$2a(\sqrt{2}/2)$
60°	$2a(1/2)$
90°	0

If we imagine an arrow pointing to the coordinates, we notice that the length of the arrow starts at $r = 2a$ and decreases to $r = 0$ as the theta varies from $0^\circ$to 90°. Connecting these points with a smooth curve gives the half circle shown in blue. We do a similar thing for the second quadrant, however in this case, since the r value is negative, the arrows are reflected through the origin resulting in the bottom half of the circle, shown in red. Note that the circle is complete then after only the first tow quadrants. However, for illustrative purposes we will continue the procedure with the next two quadrants in a similar fashion. As you can see the same circle is retraced. The final curve, shown at the bottom of the figure, is a circle of radius a centered on $(a, 0)$ as expected.



We can similarly show that a circle of radius a centered at $(0, b)$ is given as

$$r(\theta) = 2b\,sin(\theta)$$

Finally, combining the two equations we can derive the polar equation of a circle with radius $\sqrt{a^2 + b^2}$, centered at (a, b).

$$r(\theta) = 2a\,cos(\theta) + 2b\,sin(\theta)$$

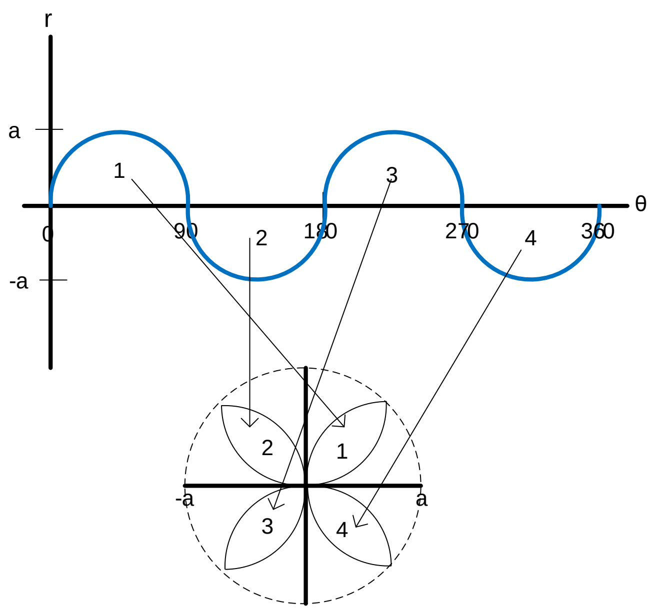
Rose Curves

Another interesting class of polar curves are called rose curves, and are defined by the following equations

$$r(\theta) = a\,sin(n\theta) \qquad \textit{or} \qquad r(\theta) = a\,cos(n\theta)$$

Let's take a look at the sine rose curve. When $n = 1$ this is simply the equation of a circle with radius $a/2$ centered at $(0, a/2)$ as we have seen in the previous section. We can say this is the graph of a rose with one pedal.

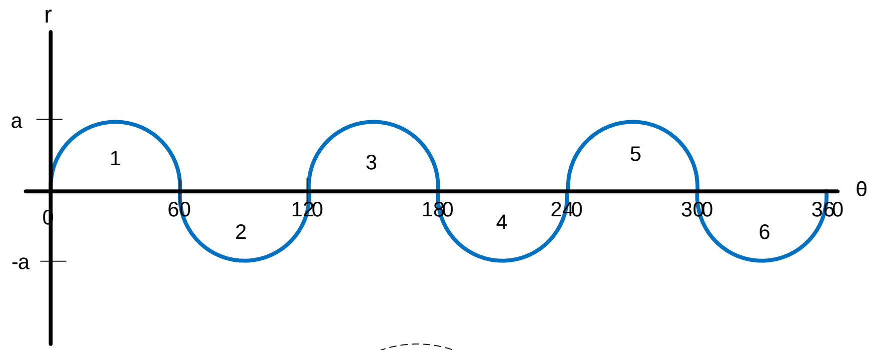
For $n = 2$ we can use the same technique we did with the circle. The procedure is captured in the figure below.

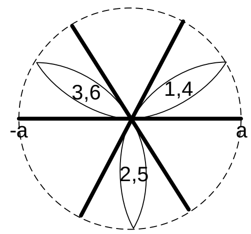


Since $n = 2$ the sine wave goes through two full cycles as θ varies from 0° to 360°. Furthermore, for each half cycle the length of r goes from 0 to a and then back to 0. Mapping this to the polar plane we get a loop, (pedal), in the section of the plane that is covered by the angle. These pedals are formed for each half cycle; however, we must keep in mind that for half cycles that are below the x-axis the pedal is reflected through the origin. With this in mind we divide the polar plane into 4 equal sections and sketch a loop in each section, reflecting through the origin when the half cycle is negative. The final graph for $n = 2$ contains 4 pedals as shown above.

Let's see what happens for $n = 3$.

In this case the sine wave goes through 6 half cycles as θ varies from 0° to 360°. Therefore, we divide the coordinate plane into six 60° sectors and sketch loops for each half cycle, reflecting through the origin when the half cycle is negative. As the figure below shows sections 2, 4, and 6 are reflecting so that they overlap 5, 1, and 3, respectively. Therefore, the graph for $n = 3$ contains 3 pedals.





The patterns displayed above continue for larger values of n and are summarized below.

$r(\theta) = a\,sin(n\theta)$	
n even	n odd
Graphs contain $2n$ pedals	Graph contains n pedals.
Graphing procedure: Divide the polar region into $2n$ sectors and create "pedals" in each sector	Graphing procedure: Divide the polar region into $2n$ sectors and create "pedals" in each sector remembering to reflect every other pedal through the origin.

The rules are identical for the cosine function, except the rose will be rotated to account for the phase offset between the sine and cosine functions.

Various other "standard" polar curves such as Cardioid, Limacon, Lemniscate, and others exist. Each could be sketched following a procedure similar to the one above and standard rules can be developed. Rather than memorizing the various curves we gain more insight by following the procedure above for each new curve.

Let's do a few more examples before summarizing this section.

Example 4: Find an equation in polar coordinates of the line with the given description.

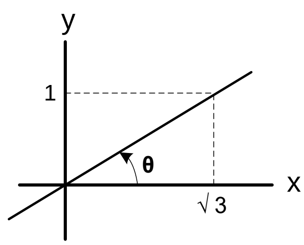
a. It passes through the origin and has a slope of $1/\sqrt{3}$.
b. It's tangent to the circle $r = 2\sqrt{10}$ at the point with rectangular coordinates $(-2, -6)$

Solution:

a. The general polar equation of a line that passes through the origin is

$$r = \theta_0$$

To find the value of θ_0 we can use the triangle shown below.

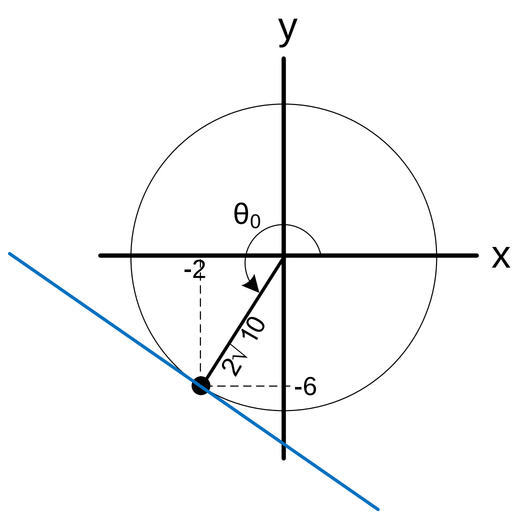


$$\theta_0 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$r = \frac{\pi}{6}$$

b. The standard equation of a line not through the origin is $r(\theta) = r_0\, sec(\theta - \theta_0)$. In this case $r_0 = 2\sqrt{10}$ and θ_0 is found with the help of the figure below as $\theta_0 = \pi + \tan^{-1}\left(\frac{-6}{-2}\right) \cong 4.39$. Therefore, the equation is

$$r(\theta) = 2\sqrt{10}\, sec(\theta - 4.39)$$



Example 5: Show that the following polar equation describes a circle that is centered at (a, b) and has a radius of $\sqrt{a^2 + b^2}$.

$$r(\theta) = 2a\,cos(\theta) + 2b\,sin(\theta)$$

Solution: To show this we can convert the equation to rectangular form by multiplying through by r, using the conversion formulas, and completing the squares.

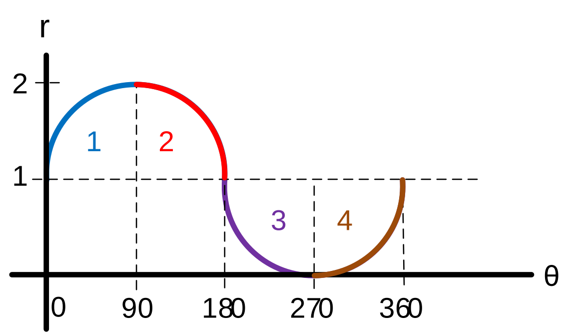
$$\begin{aligned} r^2 &= 2ar\,cos(\theta) + 2br\,sin(\theta) \\ x^2 + y^2 &= 2ax + 2by \\ (x^2 - 2ax) + (y^2 - 2by) &= 0 \\ ((x-a)^2 - a^2) + ((y-b)^2 - b^2) &= 0 \\ (x-a)^2 + (y-b)^2 &= (a^2 + b^2) \end{aligned}$$

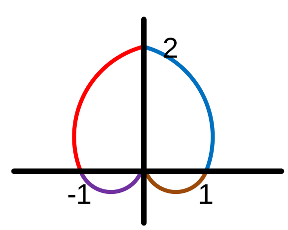
Example 6: Sketch the curve corresponding to $r = 1 + sin(\theta)$, referred to as a Cardioid.

Solution: We take the same approach we did for sketching the rose curves. The figure below shows the procedure.

The top figure is a graph of the curve placed on rectangular coordinates. The bottom figure is the polar plot generated by mapping one quadrant at a time as explained below.

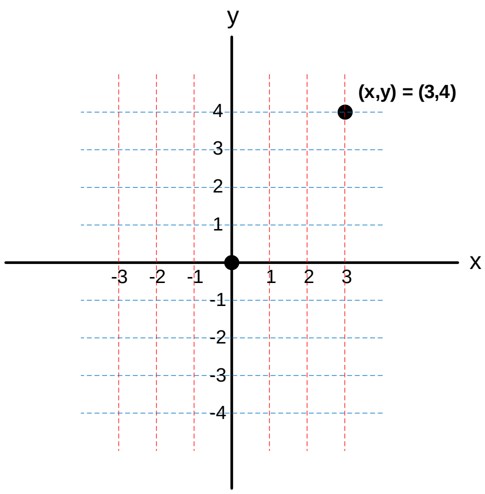
- First Quadrant
  - θ varies from 0° to 90° while r varies from 1 to 2.
- Second Quadrant
  - θ varies from 90° to 180° while r varies from 2 to 1.
- Third Quadrant
  - θ varies from 180° to 270° while r varies from 1 to 0.
- Fourth Quadrant
  - θ varies from 270° to 360° while r varies from 0 to 1.

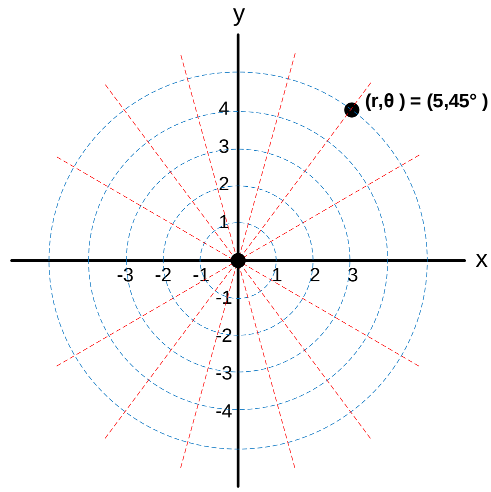




Final Summary for Polar Coordinates – Introduction

Polar Coordinates
A point P can be specified in both the rectangular coordinate system as (x, y) and a polar coordinate system as (r, θ). The two systems are shown below.





2D Coordinate Conversion Formulas

Polar to Rectangular: $(r, \theta) \rightarrow (x, y)$	***Rectangular to Polar:*** $(x, y) \rightarrow (r, \theta)$
$x = r\,cos(\theta)$ $y = r\,sin(\theta)$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ Note: The quadrant where the point lies must be considered when using $tan^{-1}\left(\frac{y}{x}\right)$.

Common Polar Curves

Description of Curve	**Polar Equation**
Circle of radius r_0 centered at the origin	$r = r_0$
Line through the origin with slope $\tan(\theta_0)$	$\theta = \theta_0$
Line on which the point (r_0, θ_0) is the point closest to the origin	$r(\theta) = r_0\,sec(\theta - \theta_0)$
Circle of radius a centered at $(a, 0)$	$r(\theta) = 2a\,cos(\theta)$
Circle of radius b centered at $(0, b)$	$r(\theta) = 2b\,sin(\theta)$
Circle of radius $\sqrt{a^2 + b^2}$ centered at (a, b)	$r(\theta) = 2a\,cos(\theta) + 2b\,sin(\theta)$

By: ferrantetutoring