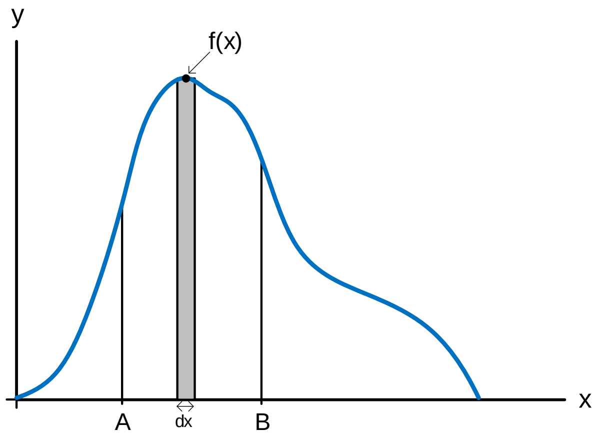
Polar Coordinates – Area and Arc Length

In the previous lesson we saw that it is sometimes more convenient to represent curves using polar coordinates as oppose to rectangular coordinates. We are also aware from previous lessons that it is sometimes necessary to find the area under a curve or the arc length of a curve, and we have learned techniques for computing these features. However, the techniques we learned applied to curves that were represented using rectangular coordinates. In this lesson we learn how to compute these features for curves represented in polar coordinates.

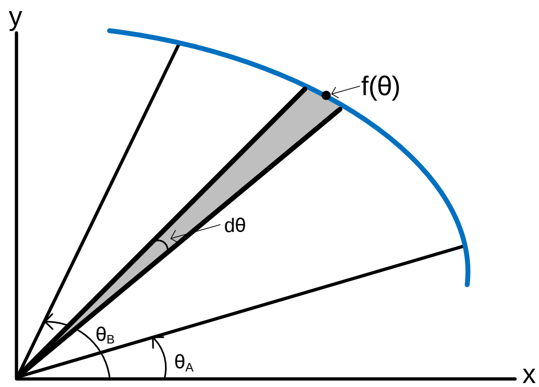
Area in Polar Coordinates

When computing the area *under* a curve in rectangular coordinates we used rectangles with infinitesimal width, dx, as shown in the figure below. The use of rectangles is facilitated by the grid lines associated with the rectangular coordinate system.

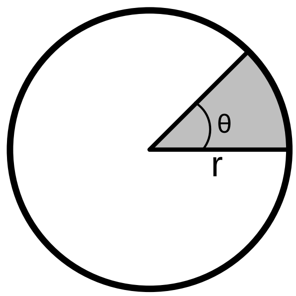


$$A = \int_A^B f(x)dx$$

We purposely italicized the word "*under*" in the above description to call attention to the fact that for polar coordinates we **do not** compute the area *under* the curve but rather *the area of a sector bounded by the curve*. Because of the grid lines associated with polar coordinates a more suitable shape for computing the area are narrow sectors as shown in the figure below.



Therefore, instead of summing infinitesimal *rectangles* to find the *area under a curve* as we did in the rectangular coordinate system, we will sum infinitesimally narrow *sectors* to find the *area bounded by a curve*. To write the integral for computing this quantity we'll first derive the formula for the area of a sector of a circle. We can do this using a ratio as shown below.



$$\frac{Area\ of\ a\ Sector}{Area\ of\ A\ Circle} = \frac{radians\ in\ a\ sector}{radians\ in\ circle}$$
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$
$$A = \frac{1}{2} r^2 \theta$$

To apply this formula to the infinitesimally narrow sectors from above we need only make the following substitutions:

$$r = f(\theta) \qquad\qquad \theta = d\theta$$

Therefore, the area of the infinitesimal sector, dA, is given as

$$dA = \frac{1}{2} f^2(\theta) d\theta$$

Integrating both sides over $[\theta_A, \theta_B]$, we have

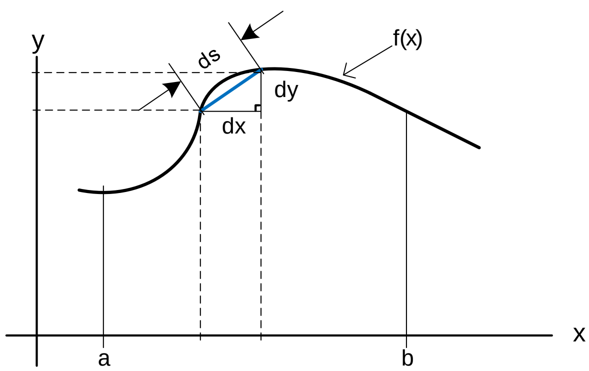
$$A = \frac{1}{2} \int_{\theta_A}^{\theta_B} f^2(\theta) d\theta$$

The result is stated formally below.

Area in Polar Coordinates
The area bounded by the curve, $r = f(\theta)$, and the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$A = \frac{1}{2} \int_{\theta_A}^{\theta_B} f^2(\theta) d\theta$$

Arc Length in Polar Coordinates

When computing the arc length for a curve in rectangular coordinates we created an infinitesimal right triangle and used the Pythagorean theorem as shown below.



$$ds^2 = (dx^2 + dy^2)\frac{dx^2}{dx^2}$$

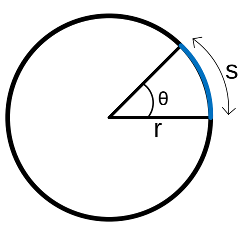
$$ds^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)dx^2$$

$$ds = \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right)dx$$

$$ds = \left(\sqrt{1 + \left(f'(x)\right)^2}\right)dx$$

$$s = \int_a^b \left(\sqrt{1 + \left(f'(x)\right)^2}\right)dx$$

We can derive an expression for the arc length of a curve in polar coordinates using a similar method. However, in this case we will make use of the formula for the arc length of a circle, which we derive below using a ratio as we did for the area of a sector above.

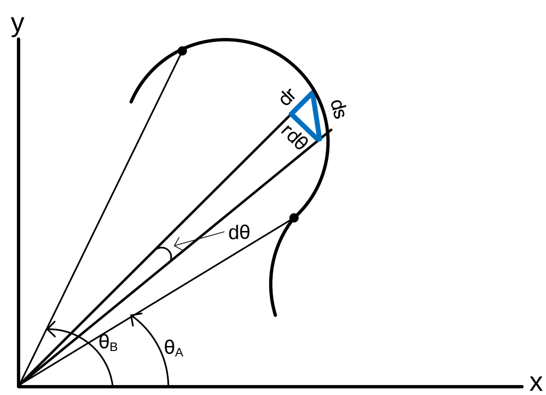


$$\frac{Sector\ Arc\ Length}{Cirumference} = \frac{radians\ in\ a\ sector}{radians\ in\ circle}$$

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

$$s = r\theta$$

We can now derive the arc length of a curve using the figure below.



The infinitesimal arc length is shown as the hypotenuse of a right triangle with side lengths of dr and $rd\theta$. Therefore, from the Pythagorean theorem we can write the following.

$$ds^2 = dr^2 + r^2 d\theta^2$$

Using a similar "trick" as we did for the rectangular coordinate case, we have

$$ds^2 = (dr^2 + r^2 d\theta^2)\frac{d\theta^2}{d\theta^2}$$

$$ds^2 = \left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) d\theta^2$$

$$ds = \sqrt{\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)} d\theta$$

And since $r = f(\theta)$ we can write the infinitesimal arc length as

$$ds = \sqrt{f'(\theta)^2 + f(\theta)^2} d\theta$$

Finally, to find the total arc length we integrate.

$$s = \int_{\theta_A}^{\theta_B} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta$$

The result is stated formally below.

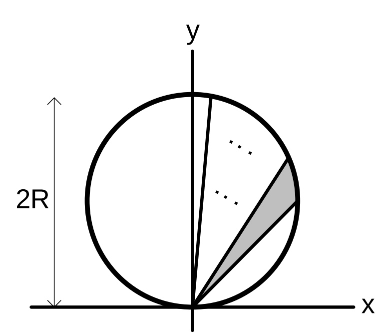
Arc Length in Polar Coordinates
The arc length of a curve, $r = f(\theta)$, between the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$s = \int_{\theta_A}^{\theta_B} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta$$

Let's do some examples utilizing our newly derived formulas.

Example 1: Using the polar representation of circle derive an equation for the area of a circle of radius R.

Solution: The figure below is described by the following polar equation.

$$f(\theta) = 2R\, sin(\theta)$$



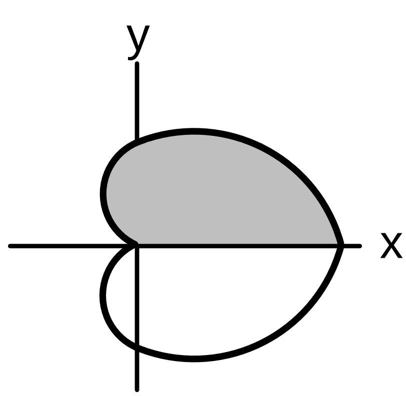
Next, we note that the circle is traced out in only a half cycle of the sine wave, therefore our integration limits are from 0 to π.

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} f^2(\theta)d\theta \\
&= \frac{1}{2}\int_0^{\pi} (2R)^2\, sin^2(\theta)\, d\theta \\
&= 2R^2 \int_0^{\pi} \left(\frac{1 - cos(2\theta)}{2}\right) d\theta \\
&= \int_0^{\pi} (1 - cos(2\theta))d\theta \\
&= R^2\left(\theta - \frac{1}{2}sin(2\theta)\right)\Big|_0^{\pi} \\
&= R^2\left(\left(\pi - \frac{1}{2}sin(2\pi)\right) - \left(0 - \frac{1}{2}sin(0)\right)\right) \\
&= R^2\big((\pi - 0) - (0 - 0)\big) \\
&= \pi R^2
\end{aligned}
$$

Which agrees with area of a circle formula we are already familiar with.

Example 2: Find the area of the top half of the cardioid shown in the figure below and given by the following equation.

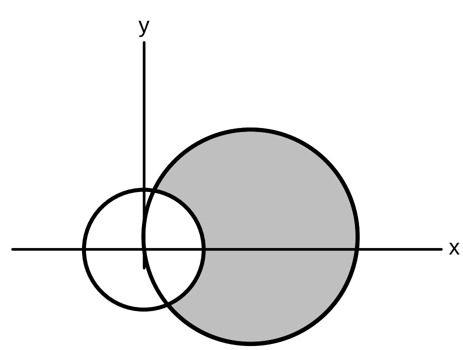
$$f(\theta) = 3 + 3cos(\theta)$$



Solution: The top half of the cardioid is traced out as θ varies from 0 to π.

$$A = \frac{1}{2}\int_0^{\pi} f^2(\theta)d\theta$$
$$= \frac{1}{2}\int_0^{\pi} 3^2(1 + cos(\theta))^2 d\theta$$
$$= \frac{9}{2}\int_0^{\pi} (1 + 2\,cos(\theta) + cos^2(\theta))d\theta$$
$$= \frac{9}{2}\int_0^{\pi} \left(1 + 2\,cos(\theta) + \frac{1}{2} - \frac{1}{2}sin(2\theta)\right) d\theta$$
$$= \frac{9}{2}\left(\theta + 2\,sin(\theta) + \frac{1}{2}\theta + \frac{1}{4}cos(2\theta)\right)\Big|_0^{\pi}$$
$$= \frac{9}{2}\left(\pi + 2\,sin(\pi) + \frac{1}{2}\pi + \frac{1}{4}cos(2\pi)\right) - \left(0 + 2\,sin(0) + \frac{1}{2}0 + \frac{1}{4}cos(0)\right)$$
$$= \frac{9}{2}\left(\pi + 0 + \frac{1}{2}\pi + \frac{1}{4}\cdot 1\right) - \left(0 + 0 + 0 + \frac{1}{4}\cdot 1\right)$$
$$= \frac{9}{2}\left(\frac{3}{2}\pi\right) = \frac{27}{4}\pi$$

Example 3: Find the area that lies inside $f(\theta) = 2cos(\theta)$ and outside $f(\theta) = 1$



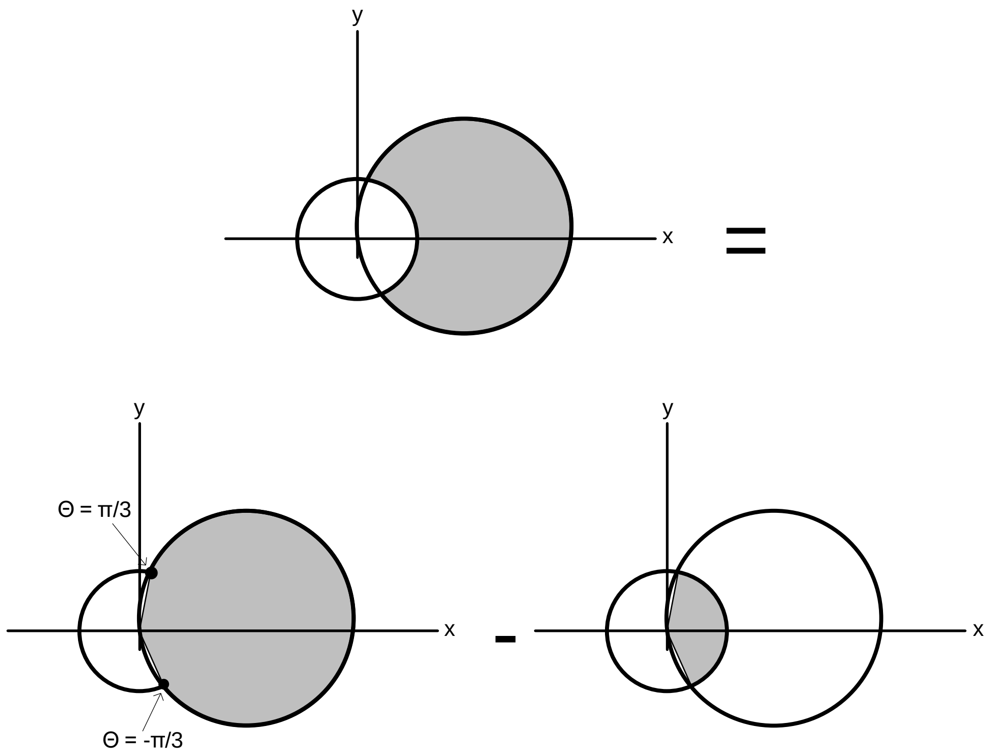
Solution: The first task is to find where the two circles intersect by equating the two equations.

$$2cos(\theta) = 1 \rightarrow cos(\theta) = \frac{1}{2}$$

Which gives us the following intersection points.

$$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$$

The shaded area is found by computing the area bounded by $f(\theta) = 2cos(\theta)$ and then subtracting the area bounded by $f(\theta) = 1$. The figure below illustrates the procedure.



$$A = \left(\frac{1}{2}\int_{-\pi/3}^{\pi/3} 4cos^2(\theta)\, d\theta\right) - \left(\frac{1}{2}\int_{-\pi/3}^{\pi/3} 1 d\theta\right)$$

$$= \int_{-\pi/3}^{\pi/3} 1 + cos(2\theta)\, d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/3} 1 d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2} + cos(2\theta)\, d\theta$$

$$= \left(\frac{1}{2}\theta + \frac{1}{2}sin(2\theta)\right)\Bigg|_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}(\theta + sin(2\theta))\Bigg|_{-\pi/3}^{\pi/3}$$

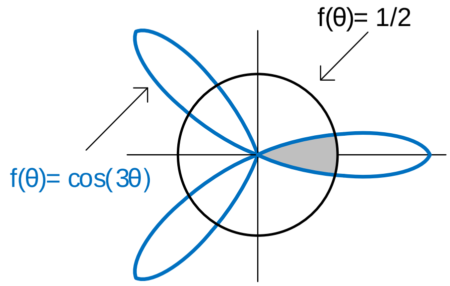
$$= \frac{1}{2}\left(\left(\pi/3 + sin\left(2\frac{\pi}{3}\right)\right) - \left(-\pi/3 + sin\left(-2\frac{\pi}{3}\right)\right)\right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} + 2\, sin\left(2\frac{\pi}{3}\right)\right)$$

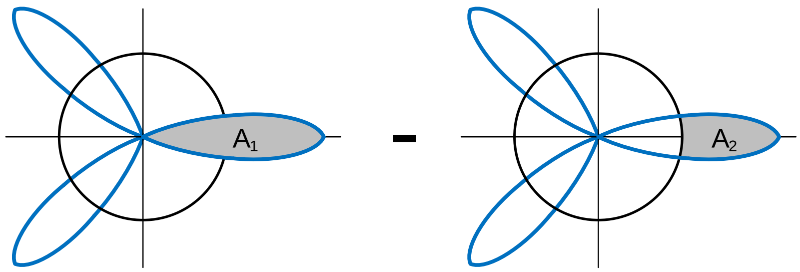
$$= \frac{1}{2}\left(\left(\frac{2\pi}{3} + \cancel{2}\frac{\sqrt{3}}{\cancel{2}}\right)\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \cong 1.91$$

Example 4: Find the area of the shaded region in the figure below enclosed by $f(\theta) = 1/2$ and $f(\theta) = cos(3\theta)$.



Solution: For this case we need to compute the area for the entire pedal and subtract the area of the pedal that lies outside the circle. The figure below illustrates this.



To compute A_1 we need to determine the angle through which the given rose pedal is traced. For this we can use techniques from the previous lesson. Using $r = cos(\varphi)$ we find that the top half of the pedal is traced out as φ varies from 0 to $\pi/2$. The bottom half is traced out as φ varies from $\pi/2$ to π, or similarly from $-\pi/2$ to 0. For the current problem, where $\varphi = 3\theta$, the top trace becomes 0 to $\pi/6$, and the bottom trace becomes $\pi/6$ to $\pi/3$ or $-\pi/6$ to 0, respectively. Integrating using the positive angles region we have

$$\begin{aligned}
A_1 &= \left(\frac{1}{2}\int_0^{\pi/3} cos^2(3\theta)\, d\theta\right) \\
&= \left(\frac{1}{4}\int_0^{\pi/3} (1 + cos(6\theta))d\theta\right) \\
&= \frac{1}{4}\left(\theta + \frac{1}{6}sin(6\theta)\right)\Bigg|_0^{\pi/3} \\
&= \frac{1}{4}\left(\frac{\pi}{3} + \frac{1}{6}(sin(2\pi) - sin(0))\right) \\
&= \frac{1}{4}\left(\frac{\pi}{3} + \frac{1}{6}(0 - 0)\right) \\
&= \frac{\pi}{12}
\end{aligned}$$

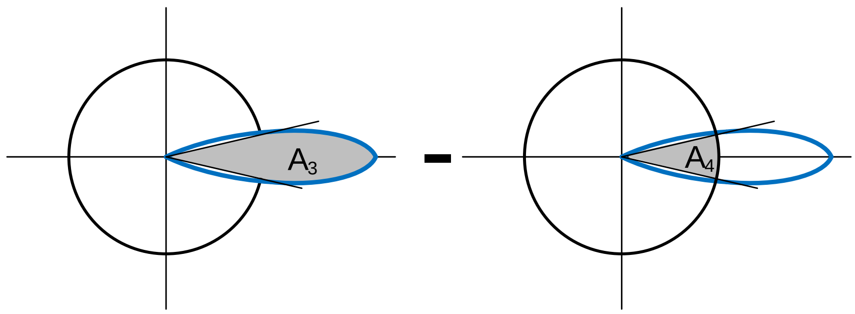
For the second region we need first to find when circle intersects with the first pedal of the rose function. To find this we set the two functions equal as follows:

$$cos(3\theta) = 1/2$$

The cosine function evaluates to $1/2$, i.e. $\pm\pi/3$. Therefore, the points of intersection, and hence the integration limits are computed as follows:

$$3\theta = \pm\frac{\pi}{3} \rightarrow \theta = \pm\frac{\pi}{9}$$

Now to compute A_2 we need to subtract two regions as shown in the figure below.



$$\begin{aligned}
A_2 &= (A_3) - (A_4) \\
&= \left(\frac{1}{2}\int_{-\pi/9}^{\pi/9} cos^2(3\theta)\, d\theta\right) - \left(\frac{1}{2}\int_{-\pi/9}^{\pi/9} \left(\frac{1}{2}\right)^2 d\theta\right) \\
&= \left(\frac{1}{4}\int_{-\pi/9}^{\pi/9} (1 + cos(6\theta))d\theta\right) - \left(\frac{1}{8}\cdot\frac{2\pi}{9}\right) \\
&= \frac{1}{4}\left(\frac{2\pi}{9} + \frac{1}{6}\left(sin\left(\frac{2\pi}{3}\right) - sin\left(-\frac{2\pi}{3}\right)\right)\right) - \left(\frac{\pi}{36}\right) \\
&= \frac{1}{4}\left(\frac{2\pi}{9} + \frac{1}{6}\left(2\, sin\left(\frac{2\pi}{3}\right)\right)\right) - \left(\frac{\pi}{36}\right) \\
&= \frac{1}{4}\left(\frac{2\pi}{9} + \frac{\sqrt{3}}{6}\right) - \left(\frac{\pi}{36}\right) \\
&= \frac{\pi}{36} + \frac{\sqrt{3}}{24}
\end{aligned}$$

Finally, the desired area is given as

$$\begin{aligned}
A &= A_1 - A_2 \\
&= \frac{\pi}{12} - \left(\frac{\pi}{36} + \frac{\sqrt{3}}{24}\right) \cong 0.1024
\end{aligned}$$

Example 5: Find the length of the spiral curve $f(\theta) = \theta^2$ for $[0 \leq \theta \leq \pi]$.

Solution: Using the arc length formula from above we have

$$s = \int_0^{\pi} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta = \int_0^{\pi} \left(\sqrt{4\theta^2 + \theta^4}\right) d\theta = \int_0^{\pi} \theta \left(\sqrt{4 + \theta^2}\right) d\theta$$

To solve this integral we use the following substitution:

$$u = 4 + \theta^2 \qquad\qquad du = 2\theta d\theta \rightarrow \theta d\theta = du/2$$

$$= \frac{1}{2}\int_4^{4+\pi^2} \sqrt{u} du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{4+\pi^2} = \frac{1}{3}\left((4+\pi^2)^{3/2} - 8\right) \cong 14.55$$

Example 7: Find the length of the cardioid given by the equation, $f(\theta) = a - a\,cos(\theta)$.

Solution: The cardioid is traced out in the interval $[0 \leq \theta \leq 2\pi]$.

$$s = \int_0^{2\pi} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta$$

$$= \int_0^{2\pi} \left(\sqrt{a^2\,sin^2(\theta) + a^2(1 - cos(\theta))^2}\right) d\theta$$

$$= a\int_0^{2\pi} \left(\sqrt{sin^2(\theta) + 1 - 2\,cos(\theta) + cos^2(\theta)}\right) d\theta$$

$$= a\int_0^{2\pi} \left(\sqrt{2 - 2\,cos(\theta)}\right) d\theta$$

$$= \sqrt{2}a\int_0^{2\pi} \left(\sqrt{1 - cos(\theta)}\right) d\theta$$

$$= \sqrt{2}a\int_0^{2\pi} \left(\sqrt{2\,sin^2\left(\frac{\theta}{2}\right)}\right) d\theta$$

$$= 2a\int_0^{2\pi} \left(sin\left(\frac{\theta}{2}\right)\right) d\theta$$

$$= 2a\left(-2\,cos(\pi) - (-2cos(0))\right)$$

$$= 8a$$

Final Summary for Polar Coordinates – Area and Arc Length

Area in Polar Coordinates
The area bounded by the curve, $r = f(\theta)$, and the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$A = \frac{1}{2}\int_{\theta_A}^{\theta_B} f^2(\theta)d\theta$$
Arc Length in Polar Coordinates
The arc length of a curve, $r = f(\theta)$, between the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$s = \int_{\theta_A}^{\theta_B} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta$$

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