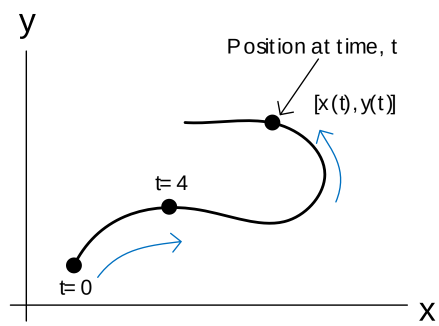
Parametric Calculus – Parametric Equations

Parametric equations are a group of functions that describe various quantities in terms of one or more independent variables, i.e. parameters. In this lesson we will focus on two quantities, e.g. x, y, and a single parameter, e.g. t. Parametric equations can be used, for example, to rewrite an implicit mathematical relationship as a set of explicit functions by introducing an additional variable/parameter. In some cases, the additional variable is simply an artificial variable with no specific meaning. However, in other cases, the additional variable may indeed provide additional information about the scenario. One of the most useful applications of parametric equations is in the analysis of motion. In this case the quantities are the space coordinates, e.g. x and y, and the additional variable is time, i.e. t. By introducing a time variable and creating parametric equations, e.g. $x = x(t)$ and $y = y(t)$, details about the motion of the particle along the path can now be known.

Parametric Equations:

As mentioned in the introduction, parametric equations are extremely useful for describing the motion of objects in space. Consider a particle moving along a path in a 2D plane as shown in the figure below.



Unfortunately, the path does not represent the graph of a function as it does not pass the vertical line test. Instead, we can represent the motion of the particle using parametric equations by introducing an additional *time* variable as below.

$$x = x(t) \ , \quad y = y(t)$$

In other words, the coordinates of the particle can be written as a function of time as below.

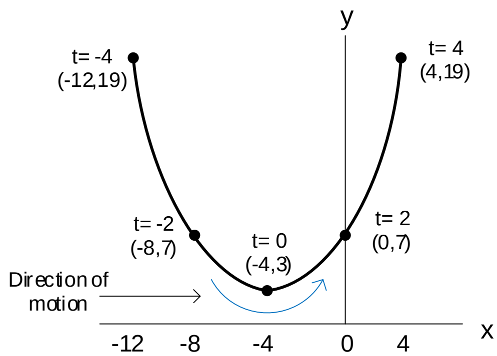
$$c(t) = [x(t), y(t)]$$

Example 1: Sketch the curve that is described by the following parametric equations.

$$c(t) = [2t - 4,\ t^2 + 3\ ]$$

Solution: One method we can use to sketch this curve is to start by computing the x and y coordinates for several time instances and plot the corresponding points. Then join the points by a smooth curve.

t	$x = 2t - 4$	$y = t^2 + 3$
-4	-12	19
-2	-8	7
0	-4	3
2	0	7
4	4	19



The curve represented by the parametric equations in example 1 is the graph of a function, as it passes the vertical line test. In this case, it may be possible to find the function, $y = f(x)$, by eliminating the parameter. Let's try this in the next example.

Example 2: Describe the parametric curve from the previous example by a function of the form $y = f(x)$.

$$c(t) = [2t - 4,\ t^2 + 3\ ]$$

Solution: We attempt to eliminate the parameter by solving the first parametric equation, $x(t)$, for t, and then substitute this into the second parameter equation, $y(t)$, to find y as a function of x, $y(x)$.

Step 1: Solve $x(t)$ for t.

$$2t - 4 = x$$
$$2t = x + 4$$
$$t = \frac{x+4}{2}$$
$$t(x) = \frac{1}{2}x + 2$$

Step 2: Substitute into $y(t)$.

$$y(t) = t^2 + 3$$
$$y(t(x)) = \left(\frac{1}{2}x + 2\right)^2 + 3$$
$$y(x) = \left(\frac{1}{2}x + 2\right)^2 + 3$$
$$y(x) = \frac{1}{4}x^2 + 2x + 7$$

*Note: Once we remove the time parameter the motion of a particle, (it's direction and speed) are lost.

Parametric space curves, as described above, can be seen as representing the motion of a particle along a particular curve. Removing the time parameter leaves us with a simple curve void of any motion information. As a result, there is generally one way to describe any particular space curve using a function of the form $y(x)$. However, the same space curve can be represented in an infinite number of ways when we use a parametric representation. This is because the particle may be moving in either direction along the curve and at any speed. Of course, only one of the parametric representations truly describes the motion for a specific scenario. This concept is illustrated in the next example.

Example 3: Describe the motion of a particle moving along each of the following paths.

$$c_1(t) = [t^3, t^6] \qquad c_2(t) = [4t^3, 16t^6] \qquad c_3(t) = [t^2, t^4] \qquad c_4(t) = [cos(t), cos^2(t)]$$

Solution: Let's start by trying to re-express the above parametric equations as explicit functions by eliminating the parameter.

1. $x = t^3 \rightarrow t = x^{1/3},\ \ y = t^6 = \left(x^{1/3}\right)^6 \rightarrow \boldsymbol{y = x^2}$

2. $x = 4t^3 \rightarrow t = \left(\frac{x}{4}\right)^{1/3},\ \ y = 16t^6 = 16\left(\left(\frac{x}{4}\right)^{1/3}\right)^6 = 16\frac{x^2}{16} \rightarrow \boldsymbol{y = x^2}$

3. $x = t^2 \rightarrow t = x^{1/2},\ \ y = t^4 = \left(x^{1/2}\right)^4 \rightarrow \boldsymbol{y = x^2}$

4. $x = cos(t) \rightarrow t = cos^{-1}(x),\ \ y = (cos(cos^{-1}(x)))^2 = (x)^2 \rightarrow \boldsymbol{y = x^2}$

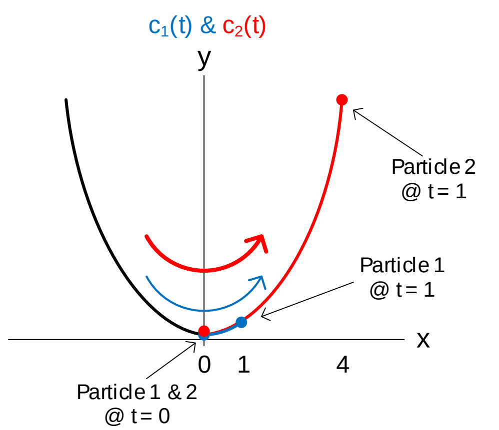
As you can see, in all cases the curve is the same. There is, however, a difference in the *how* the particle moves along this curve. This difference can only be understood by examining the parametric representation. Read through the explanations that follow while referring to the figures below.

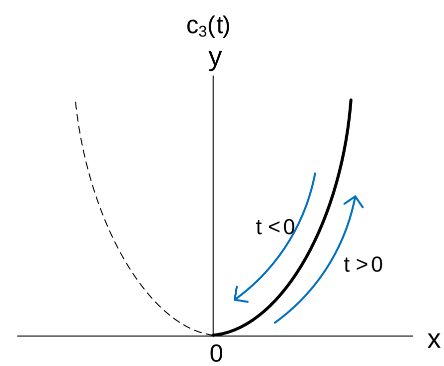
1. As t varies from $-\infty$ to ∞, t^3 also varies from $-\infty$ to ∞. Therefore, $c_1(t) = [t^3, t^6]$ describes a particle traveling from left to right as shown in the figure below.

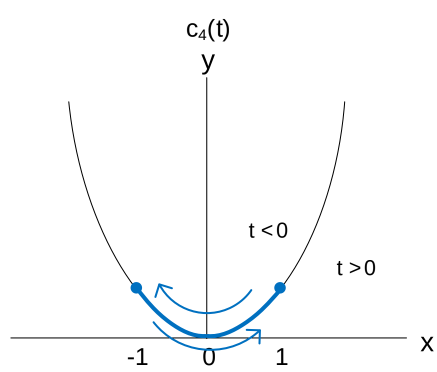
2. In this case t^3 also varies from $-\infty$ to ∞ as t varies from $-\infty$ to ∞. Therefore, the particle described by $c_2(t) = [4t^3, 16t^6]$ also travels from left to right along the curve. The difference, however, is the particle is traveling at a higher rate of speed. As an example, at $t = 0$, both particles are located at the point $[0,0]$. At $t = 1$ particle 1 is at the point $[1,1]$, while particle 2 has moved to the point $[4,16]$.

3. In this case, as t varies from $-\infty$ to 0, t^2 varies from ∞ to 0, and as t varies from 0 to ∞, t^2 varies from 0 to ∞. Therefore, the particle travels along the right side of the curve only. It travels in one direction for negative time and the other for positive time.

4. In this case, as t varies from $-\infty$ to ∞, $cos(t)$ oscillates between -1 to 1. Therefore, this particle will oscillate along a limited part of the curve, i.e. $-1 \leq x, y \leq 1$.







As you can see from example 3, one way to sketch a curve from a given set of parametric equations is to eliminate the parameter and find $y(x)$. If the parametric equations represent motion of a particle, we can then go back to the parametric representation to describe the motion of the particle along this curve. However, it is not always possible to find an explicit, or even implicit, relationship as desired. In these cases, we must work with the original parametric equations in order to sketch the curve. If the individual parametric functions display some sort of symmetry, i.e. they are either even or odd, sketching the curves can be made easier. We'll explore this in the next example.

Example 4: Sketch the parametric curve: $c(t) = [t^2 + 1, t^3 - 4t]$

Solution: Let's start by examining $x(t)$ and $y(t)$ for any symmetries.

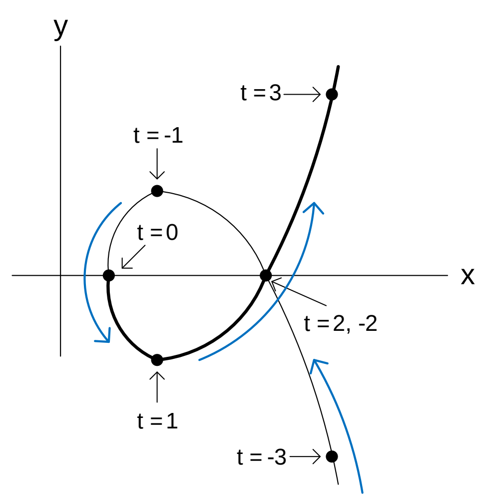
Note that $x(t)$ is an even function since $x(-t) = x(t)$, and $y(t)$ is an odd function since $y(-t) = -y(t)$. Therefore, we can write:

$$c(-t) = [x(t), -y(t)]$$

Which means that the x coordinates will be the same for negative and positive t, but the y coordinates will be negatives of each other. Therefore, the curve will be symmetric about the x axis, and we can create a table of coordinate values only for $t \geq 0$. Furthermore, since both parametric equations continue to increase after $t = 2$, we'll limit our table to $0 \leq t \leq 3$.

t	x	y
0	1	0
1	2	-3
2	5	0
3	10	15

Potting just these 4 points and connected with smooth curves is enough to provide a rough sketch of the curve, which is shown below. Note that the figure also indicates the direction of motion of a particle would take if the equations represented such.



The parametric equations for certain curves can be given in general forms. We review some of these next, starting with the parameterization of a line.

Parameterization of a Line:

Thinking again in terms of the motion of a particle in two-dimensional space, we can derive parametric equations for a line. If a particle is traveling at a constant speed, s, in one direction then the speed is given by the ratio of the distance to time as

$$s = \frac{\Delta d}{\Delta t} = \frac{p_1 - p_0}{t_1 - t_0}$$

Where, p_0 is the position at $t = t_0$, and p_1 is the position at $t = t_1$.

If we let $t_0 = 0$, and $t_1 = t$ we can write an equation for the position at any time t as

$$s = \frac{p_1 - p_0}{t}$$
$$p(t) = p_0 + st$$

Where we let $p_1 = p(t)$

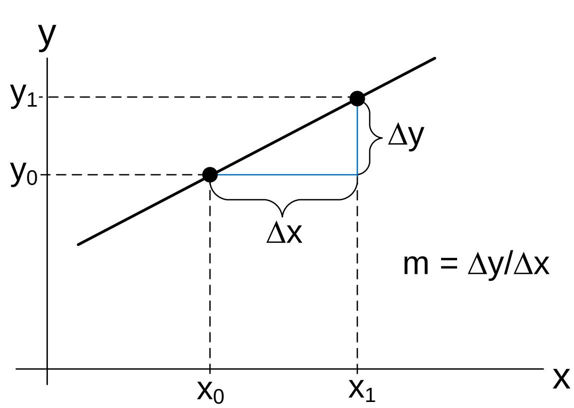
Subsequently, if the particle was traveling along a line in two-dimensional space, as shown in the figure below, we can write an equation that describes its x and y position separately as

$$x(t) = x_0 + s_x t \qquad\qquad y(t) = y_0 + s_y t$$

Where, s_x is the speed in the x direction and s_y is the speed in the y direction.

Finally, the speeds, s_x and s_y, can be related to the slope of the line as follows:

$$m = \frac{\Delta y}{\Delta x} = \frac{s_y}{s_x}$$



Therefore, given the slope of a line, $m = \frac{\Delta y}{\Delta x} = \frac{s_y}{s_x}$, that passes through the point $[x_0, y_0]$, the parametric equation can be given as

$$c(t) = \left[x_0 + s_x t, y_0 + s_y t\right]$$

Note, if we were instead were given two points, we could write the parametric equation as

$$c(t) = [x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t]$$

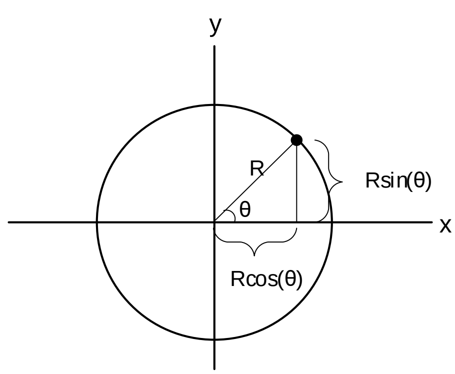
Where, $P_0 = [x_0, y_0]$, $P_1 = [x_1, y_1]$, and the slope is

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Furthermore, the line segment from P_0 to P_1c corresponds to $0 \leq t \leq 1$.

Parameterization of a Circle:

The figure below shows a circle of radius R, centered at the origin.



Using basic trigonometric definitions, we can give the $[x, y]$ coordinates for each point on the circle as

$$[x, y] = [R\,cos(\theta)\,, R\,sin(\theta)]$$

Where, θ varies over $[0, 2\pi)$ or $[-\pi, \pi)$.

Therefore, we can parameterize this circle in terms of θ as

$$c(\theta) = [R\,cos(\theta)\,, R\,sin(\theta)]$$

Furthermore, shifting this circle to be centered at the point $[x_c, y_c]$, we have

$$c(\theta) = [x_c + R\,cos(\theta)\,, y_c + \ R\,sin(\theta)]$$

Generalizing even further we can treat θ as a time parameter and define s as the speed, in for example radians per second, that a particle is traveling around this circle and write

$$c(t) = [x_c + R\,cos(st)\,, y_c + \;R\,sin(st)]$$

We can verify this parameterization by substituting $x = x_c + R\,cos(st)$ and $y = y_c + R\,sin(st)$ from the parameterization above into what we know to be the equation of a circle of radius R centered at $[x_c, y_c]$.

$$\begin{aligned}
(x - x_c)^2 + (y - y_c)^2 &= R^2 \\
\left((\cancel{x_c} + R\,cos(st)) \cancel{- x_c}\right)^2 + \left((\cancel{y_c} + \;R\,sin(st)) \cancel{- y_c}\right)^2 &= R^2 \\
R^2\,cos^2(st) + R^2\,sin^2(st) &= R^2 \\
R^2(cos^2(st) + sin^2(st)) &= R^2 \\
R^2(1) &= R^2 \\
R^2 &= R^2
\end{aligned}$$

Parameterization of an Ellipse:

The parametric equation for an ellipse, with major axis of $2a$ and minor axis $2b$, is given as

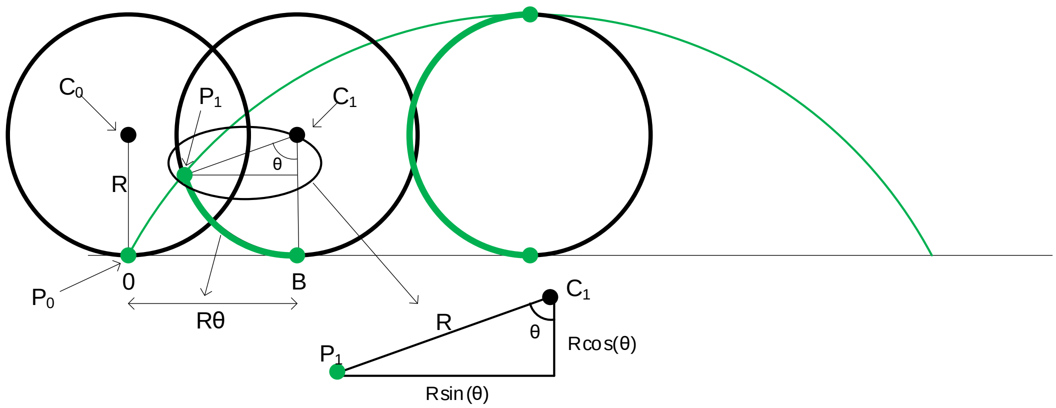
$$c(t) = [a\,cos(t)\,, b\,sin(t)]\;, \qquad for\; -\pi \le t \le \pi$$

Let's verify, again using what we know to be the equation of this ellipse.

$$\begin{aligned}
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= 1 \\
\left(\frac{\cancel{a}\,cos(t)}{\cancel{a}}\right)^2 + \left(\frac{\cancel{b}\,sin(t)}{\cancel{b}}\right)^2 &= 1 \\
cos^2(t) + sin^2(t) &= 1 \\
1 &= 1
\end{aligned}$$

Parameterization of a Cycloid:

A cycloid is a curve that is traced out by a point on the rim of a circular wheel as it rolls along a straight line. We'll derive the parametric equations using a circle of radius R as shown in the figure below.



We can start by noticing that as the wheel rolls the x distance traveled is equal to the arc length traced out by an initial point, P_0. Recall, that the arc length of a circle is given as the radius of the circle times the angular measure traced by the arc. In the case above, the arc length traced out when the initial point on the wheel moves to P_1 is $R\theta$. Therefore, as the wheel rolls the position of the center of the wheel can be expressed as

$$C(\theta) = [R\theta, R]$$

Our goal, however, is to find the position of the point, P_1. We can do this by translating the point C_1 to P_1. To do this we use the triangle shown in the figure. To get from C_1 to P_1 we must travel in the negative y direction by $R\,cos(\theta)$ and in the negative x direction by $R\,sin(\theta)\,0$. Therefore, we can describe the position of a point on the wheel as a function of the angle in parameterized form as

$$P(\theta) = [R\theta - R\,sin(\theta)\,, R - R\,cos(\theta)]$$

Before providing a summary of this lesson, let's do a few more examples.

Example 5: Express the following in the form $y = f(x)$ by eliminating the parameter.

$$c_a(t) = [t + 3, 4t] \qquad c_b(t) = [e^{-2t}, 6e^{4t}] \qquad c_c(t) = [ln(t)\,, 2 - t]$$

Solution: For all cases we solve for t in $x(t)$ and substitute into $y(t)$.

a.

$$x = t + 3 \;\rightarrow t = x - 3$$

$$y = 4t$$
$$y = 4(x - 3)$$
$$y = 4x - 12$$

b.

$$x = e^{-2t}$$
$$ln(x) = -2t$$
$$-\frac{ln(x)}{2} = t$$

$$y = 6e^{4t}$$
$$y = 6e^{-4\left(\frac{ln(x)}{2}\right)}$$
$$y = 6\left(e^{ln(x)}\right)^{-2}$$
$$y = 6(x)^{-2}$$
$$y = \frac{6}{x^2}$$

c.

$$x = ln(t)$$
$$e^x = t$$

$$y = 2 - t$$
$$y = 2 - e^x$$

Example 6: Sketch the curve for the following and draw an arrow specifying the direction of motion.

$$c_a(t) = \left[\frac{1}{2}t, 2t^2\right] \qquad c_b(t) = [t^3 - 4t, t^2]$$

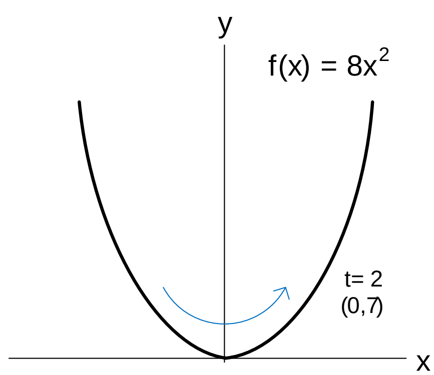
Solution:

a. One method we can use is to eliminate the parameter to find the curve, and then analyze the direction of motion. Eliminating the parameter, we have:

$$x = \frac{1}{2}t$$
$$2x = t$$

$$y = 2t^2$$
$$y = 2(2x)^2$$
$$y = 8x^2$$

Next, we notice that as t varies from $-\infty$ to ∞, $\frac{1}{2}t$ also varies from $-\infty$ to ∞. Therefore, a particle travels from left to right along the curve as shown below.



b. In this case, since it's impossible to eliminate the parameter and obtain an explicit function, we instead look for symmetries and plug in some values.

$x(t) = t^3 - 4t$: Odd Function

$$\begin{aligned} x(-t) &= (-t)^3 - 4(-t) \\ &= -t^3 + 4t \\ &= -(t^3 - 4t) \\ &= -x(t) \end{aligned}$$

$y(t) = t^2$: Even Function

$$\begin{aligned} y(-t) &= (-t)^2 \\ &= t^2 \\ &= y(t) \end{aligned}$$

Therefore, we can write:

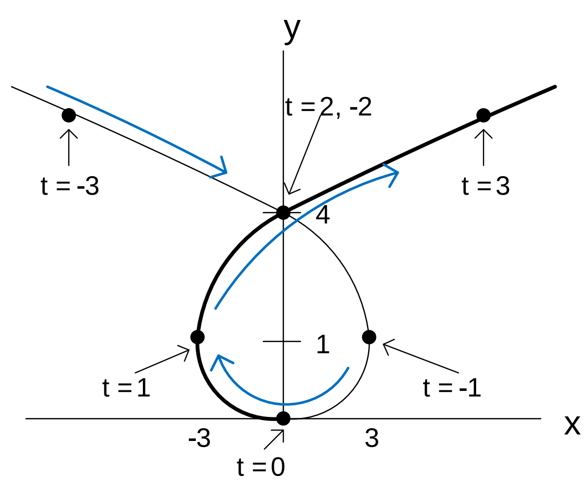
$$c(-t) = [-x(t), y(t)]$$

Which means that the y coordinates will be the same for negative and positive t, but the x coordinates will be negatives of each other. Therefore, the curve will be symmetric about the y axis. Furthermore, $x = 0$ @ $t = -2,0,2$.

Next let's create a table of coordinate values for $t \geq 0$. Similar to the previous example we can stop at $t = 3$, since x and y both continue to increase after this time.

t	x	y
0	0	0
1	-3	1
2	0	4
3	15	9

We can start by plotting the above points and connect them with a smooth curve. Then based on symmetry, we can reflect this portion of the curve about the y axis to sketch the entire curve.



Example 7:

Find parametric equations for the given curves.

1.) The line through $[3,1]$ and $[-5,4]$

2.) A circle of radius 4 centered at $[3,9]$

3.) An ellipse with major axis of 10, minor axis of 24, and centered at $[7,4]$

Solution: In all cases, we can directly use the formulas derived above.

1) We start by finding the slope as

$$m = \frac{4-1}{-5-3} = \frac{3}{-8} = \frac{\Delta y}{\Delta x}$$

Note that we can use either point as the starting location. Using the first point will describe a particle that moves from right to left and using the second point will describe a particle that moves from left to right.

Particle moving to right, $[x_0, y_0] = [-5,4]$	Particle moving to right, $[x_0, y_0] = [3,1]$
$c(t) = [-5 - 8t, 4 + 3t]$	$c(t) = [3 - 8t, 1 + 3t]$

2) We can directly substitute in the formula.

$$c(t) = [3 + 4\,cos(t)\,, 9 + \;4\,sin(t)]$$

3) We can directly substitute in the formula. Recall that $a = major\ axis/2$ and $b = minor\ axis/2$

$$c(t) = [7 + 5\,cos(t)\,, 4 + 12\,sin(t)]$$

Final Summary for Parametric Calculus – Parametric Equations

Parametric Equations
Parametric equations are a group of functions that describe various quantities in terms of one or more independent variables, i.e. parameters. In the most general sense, a set of parametric equations is as follows.

$$\begin{cases} f_1(\alpha_1, \alpha_2, \ldots, \alpha_M) \\ f_2(\alpha_1, \alpha_2, \ldots, \alpha_M) \\ \cdot \\ \cdot \\ f_N(\alpha_1, \alpha_2, \ldots, \alpha_M) \end{cases}$$

Where, N is the number of quantities and α_i are the M parameters.

If we let the quantities represent $2D$ space coordinates, i.e., x and y, and the parameter represent time, t, then the parametric equations can be said to represent the position of an object over time in a $2D$ coordinate plane. In this case we say the object moves along the curve, $c(t)$, which can be written as

$$c(t) = [x(t), y(t)]$$

- The curve, $c(t)$, is not unique, i.e., all curves can be parameterized in infinitely many ways. However, the motion of a particle along this path will follow a particular parameterization.
- If the graph of the curve represents an explicit function, we can usually eliminate the parameter by solving for one of the equations and substituting it into the other to find $y(x)$ or $x(y)$.

Standard Parameterizations

- Line with slope $m = \frac{\Delta y}{\Delta x}$, that passes through the point, $[x_0, y_0]$.

$$c(t) = [x_0 + \Delta x t, y + \Delta y t]$$

- Circle with radius, R, centered at $[x_c, y_c]$.

$$c(t) = [x_c + R\,cos(t)\,, y_c + \ R\,sin(t)]$$

- Ellipse with major axis $2a$, and minor axis $2b$, centered at $[x_c, y_c]$.

$$c(t) = [x_c + a\,cos(t)\,, y_c + \ b\,sin(t)]$$

- Cycloid generated by a circle of radius, R.

$$c(t) = [Rt - R\,sin(t)\,, R - R\,cos(t)]$$

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