Parametric Calculus – Derivatives and Integrals

In the last lesson we learned some basic skills for working with parametric equations. In this lesson we extend our knowledge of parametric equations and learn how to use calculus, i.e., differentiation and integration, with parametric equations.

Differentiation:

For a curve represented by a function in the form of $y = f(x)$, the slope of the tangent line is given by the derivative y with respect to x as

$$\frac{dy}{dx} = f'(x)$$

For which we can use one of the many available rules to differentiate.

When a curve is represented parametrically, a shown below, we may still be interested in computing the slope of the tangent line for the curve.

$$c(t) = [x(t), y(t)]$$

One method, of course, is to eliminate the parameter and then compute the derivative directly. However, as we have learned, eliminating the variable is not always easy, or even possible. Therefore, we would like to find a way to compute dy/dx using $x(t)$ and $y(t)$ directly. We can derive a method using a simple "trick". We multiply the numerator and denominator by $1/dt$.

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \left(\frac{1/dt}{1/dt}\right)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

As an example, let's find the derivative, dy/dx, of the following parametric equation:

$$c(t) = [2t - 4,\ t^2 + 3\ ]$$

Let's start by eliminating the parameter and computing the derivative directly.

From $x(t)$, we find $t = \frac{1}{2}x + 2$. Therefore, $y(x) = \left(\frac{1}{2}x + 2\right)^2 + 3$. The derivative is then computed as

$$\frac{dy}{dx} = 2\left(\frac{1}{2}x + 2\right)\frac{1}{2}$$

$$= \frac{1}{2}x + 2$$

Now let's use our new formula and find dy/dx using the original parametric equations.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
$$= \frac{2t}{2}$$
$$= t$$

And since we have already shown that $t = \frac{1}{2}x + 2$, the answers are identical!

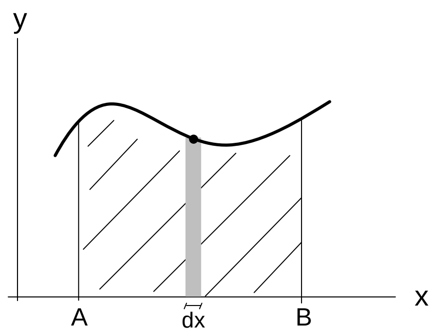
Slope of the Tangent Line for Parametric Equations

If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are both differentiable and $x'(t)$ is continuous and not equal to zero, then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Integration (Area Under the Curve):

As we know, the area under a curve, $y(x)$, when $y(x) \geq 0$ for $[A, B]$ is given as shown below.



$$A = \int_{x=A}^{x=B} y(x)dx$$

However, if we are given the curve in parametric form we would like to integrate with respect to the parameter, e.g. t. To do so we make the following substitutions.

- $dx = dx\left(\frac{dt}{dt}\right) = \frac{dx}{dt}dt = x'(t)dt$
- Use $t(x)$ to replace the limits of integration with $t(A)$ and $t(B)$
- Replace $y(x)$ with $y(t)$

Then we can write

$$A = \int_{t(A)}^{t(B)} y(t)x'(t)dt$$

Let's verify this formula by computing the area under the curve represented by the parametric equation below for $0 \leq x \leq 4$.

$$c(t) = [2t, t^2]$$

Again, let's start by eliminating the parameter and computing the area directly.

From $x(t)$, we find $t = \frac{1}{2}x$. Therefore, $y(x) = \left(\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$. The area is then computed as

$$\begin{aligned} A &= \int_0^4 \frac{1}{4}x^2 dx \\ &= \frac{1}{12}x^3\Big|_0^4 = \frac{64}{12} = \frac{16}{3} \end{aligned}$$

Now let's use our new formula.

Note: We can use the formula in this case since $y(x) \geq 0$ for $[0,4]$. For the limits we have $t(0) = \frac{1}{2}0 = 0$ and $t(4) = \frac{1}{2}4 = 2$.

$$\begin{aligned} A &= \int_{t(A)}^{t(B)} y(t)x'(t)dt \\ &= \int_0^2 t^2 \frac{d}{dt}(2t)dt \\ &= 2\int_0^2 t^2 dt \\ &= \frac{2}{3}t^3\Big|_0^2 = \frac{16}{3} \end{aligned}$$

Formally, we may write the following for the area under parametric curves.

Area Under the Curve for Parametric Equations
For a parametric curve, $c(t) = [x(t), y(t)]$, that stays above the x-axis for $t_0 \leq t \leq t_1$ and represents a function in the same interval, i.e., passes the vertical line test, the area under this curve is given as $$A = \int_{t_0}^{t_1} y(t)x'(t)dt$$

Let's practice using our differentiation and integration formulas with some examples.

Example 1: Find dy/dx at the given point.

1.	2.	3.
$[t^3, t^2 - 1], t = -4$	$[s^{-1} - 3s, s^3], s = -1$	$[sin^3(\theta), cos(\theta)], \theta = \frac{\pi}{4}$

Solution:

1.	2.	3.
$\left.\frac{dy}{dx}\right\|_{t=-4} = \left.\frac{y'(t)}{x'(t)}\right\|_{t=-4}$ $= \left.\frac{2t}{3t^2}\right\|_{t=-4}$ $= \frac{2(-4)}{3(-4)^2}$ $= -\frac{1}{6}$	$\left.\frac{dy}{dx}\right\|_{s=-1} = \left.\frac{y'(s)}{x'(s)}\right\|_{s=-1}$ $= \left.\frac{3s^2}{-\frac{1}{s^2} - 3}\right\|_{s=-1}$ $= \frac{3(-1)^2}{-\frac{1}{(-1)^2} - 3}$ $= -\frac{3}{4}$	$\left.\frac{dy}{dx}\right\|_{\theta=-1} = \left.\frac{y'(\theta)}{x'(\theta)}\right\|_{\theta=\frac{\pi}{4}}$ $= \left.\frac{-sin(\theta)}{3\,sin^2(\theta)\,cos(\theta)}\right\|_{\theta=\frac{\pi}{4}}$ $= \left.\frac{-\sin\left(\frac{\pi}{4}\right)}{3\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)}\right\|_{\theta=\frac{\pi}{4}}$ $= \frac{-\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}}$ $= -\frac{\sqrt{2}}{3}$

Example 2: Given $c(t) = [3t^2 - 2t, t^3 - 6t]$, find

a. The equation of the tangent line at $t = 3$
b. Points where the tangent is horizontal

Solution: To find the equation of the tangent line we start by finding the slope at $t = 3$.

$$\left.\frac{dy}{dx}\right|_{t=3} = \left.\frac{y'(t)}{x'(t)}\right|_{t=3}$$
$$= \left.\frac{3t^2 - 6}{6t - 2}\right|_{t=3}$$
$$= \frac{3(3)^2 - 6}{6(3) - 2}$$
$$= \frac{21}{16}$$

Next, we find the x-y coordinate at $t = 3$.

$$[x_3, y_3] = [3(3)^2 - 2 \cdot 3, (3)^3 - 6 \cdot 3] = [21,9]$$

Finally, we use the point slope formula to find the tangent line.

$$y - y_3 = m(x - x_3)$$
$$y - 9 = \frac{21}{16}(x - 21)$$
$$y = \frac{21}{16}(x - 21) + 9$$

To find when the tangent line is horizontal, we find when the derivative is equal to zero.

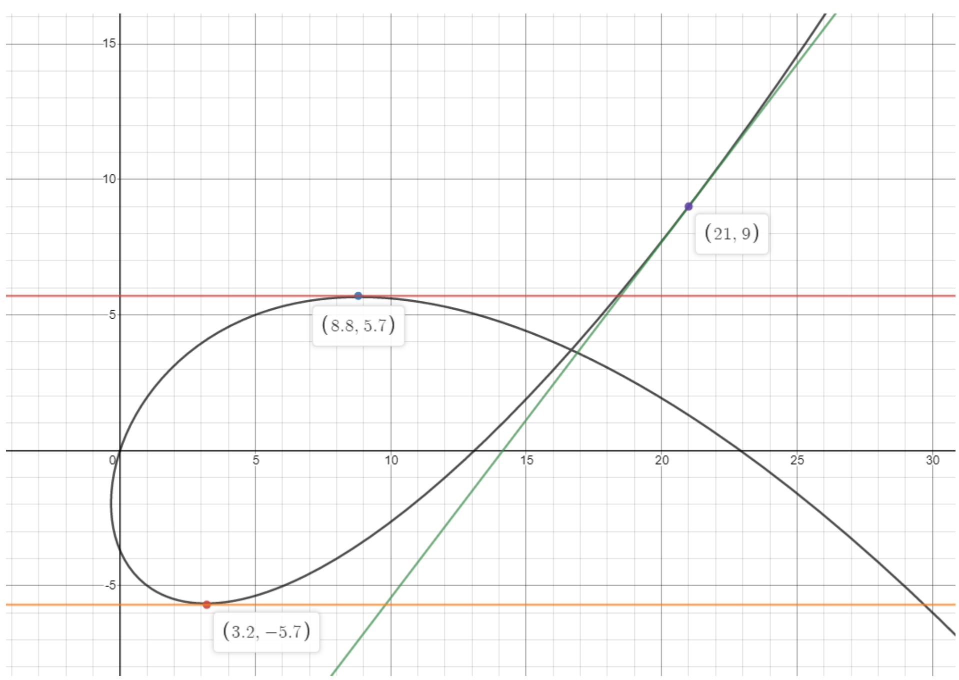
$$\frac{3t^2 - 6}{6t - 2} = 0$$
$$3t^2 = 6$$
$$t = \pm\sqrt{2}$$

The coordinates are then

$$= \left[3(\sqrt{2})^2 - 2\sqrt{2}, (\sqrt{2})^3 - 6\sqrt{2}\right]$$
$$= \left[6 - 2\sqrt{2}, -4\sqrt{2}\right]$$
$$\cong [3.2, -5.7]$$

$$= \left[3(-\sqrt{2})^2 - 2(-\sqrt{2}), (-\sqrt{2})^3 - 6(-\sqrt{2})\right]$$
$$= \left[6 + 2\sqrt{2}, 4\sqrt{2}\right]$$
$$\cong [8.8, 5.7]$$

The curve along with the tangent line and horizontal lines are shown below for illustration.



Example 3: Compute the area under the curve $c(t) = [e^t, t]$ for $0 \leq t \leq 1$

Solution: Since the y coordinate is positive in the interval under consideration we can find the area using the formula derived from above.

$$A = \int_0^1 y(t)x'(t)dt = \int_0^1 te^t dt$$

To evaluate this integral we use integration by parts.

$$u = t \qquad dv = e^t dt$$
$$du = dt \qquad v = e^t$$

$$\int_0^1 te^t dt = te^t - \int_0^1 e^t dt$$
$$= e^t(t-1)|_0^1$$
$$A = 1$$

Example 4: Compute the area under the curve $c(t) = [sin(t)\,, cos^2(t)]$ for $0 \leq t \leq \pi/2$

Solution: We can again directly us the formula since $cos^2(t) \geq 0$.

$$A = \int_0^{\pi/2} y(t)x'(t)dt$$
$$= \int_0^{\pi/2} cos^2(t)\, cos(t)\, dt$$
$$= \int_0^{\pi/2} (1 - sin^2(t))\, cos(t)\, dt$$

This integral can be solved using the following substitution.

$$u = sin(t) \qquad du = cos(t)\, dt$$

$$= \int_0^1 (1 - u^2)du$$
$$= u - \frac{1}{3}u^3\Big|_0^1 = \frac{2}{3}$$

Example 5: Find the area under one arch of a cycloid generated by a circle of radius R.

Solution: We derived the parametric representation of a cycloid in the previous lesson as

$$c(\theta) = [R\theta - R\,sin(\theta)\,, R - R\,cos(\theta)]$$

When the wheel rotates by 2π radians it traces out one arch. Therefore, we have

$$A = \int_0^{2\pi} (R - R\,cos(\theta))(R\theta - R\,sin(\theta))'d\theta$$

$$= \int_0^{2\pi} (R - R\,cos(\theta))(R - R\,cos(\theta))d\theta$$

$$= R^2 \int_0^{2\pi} (1 - cos(\theta))^2 d\theta$$

$$= R^2 \left(\left(\int_0^{2\pi} 1 d\theta \right) - \left(\int_0^{2\pi} 2\,cos(\theta)\,d\theta \right) + \left(\int_0^{2\pi} cos^2(\theta)\,d\theta \right) \right)$$

$$= R^2 \left((2\pi) - (0) + \left(\frac{1}{2} \int_0^{2\pi} (1 + cos(\theta))d\theta \right) \right)$$

$$= R^2 \left(2\pi + \frac{1}{2}(2\pi + 0) \right) = 3\pi R^2$$

<u>Final Summary for Parametric Calculus – Derivates and Integrals</u>

Slope of the Tangent Line for Parametric Equations
If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are both differentiable and $x'(t)$ is continuous and not equal to zero, then $$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
Area Under the Curve for Parametric Equations
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