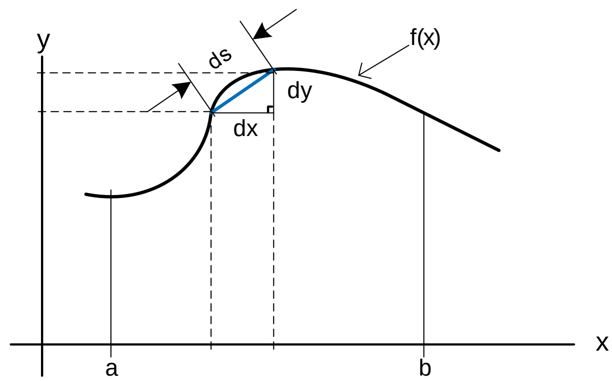
# Parametric Calculus – Arc Length and Speed

In a previous lesson we learned how to find the arc length of a curve when the curve was represented as a function. We have since gained the ability to represent curves using parametric equations. Therefore, in this lesson we will derive an arc length formula for parametric equations. Recall that parametric equations are extremely useful in motion analysis. In this sense the arc length formula can be used to represent the distance a particle has traveled along a curve, which can then be used to find the speed of the particle. Finally, we derive the surface area formula for parametric curves as well.

### Arc Length and Speed

The figure below was previously used to derive an arc length formula for explicit functions. We will use the same figure to derive an arc length for parametric equations.




We start, as we did previously, with the Pythagorean theorem.

$$ds^2 = dx^2 + dy^2$$

To find an expression for $ds$ in this case, we multiply the right hand side by $1 = \frac{dt^2}{dt^2}$.

$$ds^2 = (dx^2 + dy^2)\frac{dt^2}{dt^2}$$

$$ds^2 = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2$$

$$ds = \left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

Finally, integrating over $[a, b]$ the arc length is given as

$$s = \int_a^b \left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

| **Formula for Arc Length** |
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| If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are differentiable, the arc length for $a \leq t \leq b$ is<br><br>$$s = \int_a^b \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$ |

As an example, let's find the arc length for a circle of radius $R$.

As we learned in the previous lesson the parametric equations for a circle of radius $R$ can be written as

$$c(\theta) = [R\, cos(\theta)\, , R\, sin(\theta)]$$

To find the arc length we integrate for $0 \leq \theta \leq 2\pi$ as shown below.

$$\begin{aligned} s &= \int_0^{2\pi} \left(\sqrt{x'(t)^2 + x'(t)^2}\right) dt \\ &= \int_0^{2\pi} \left(\sqrt{R^2\, sin^2(\theta) + R^2\, cos^2(\theta)}\right) dt \\ &= R \int_0^{2\pi} \left(\sqrt{sin^2(\theta) + cos^2(\theta)}\right) dt \\ &= R \int_0^{2\pi} 1 dt \\ &= 2\pi R \end{aligned}$$

Which is the circumference formula that is likely familiar to many of us.

If we interpret the set of parametric equations as representing space coordinates, $x$ and $y$, and the parameter as representing time, $t$, we may interpret the equations as representing the position of a particle over time. In this case, the arc length does not simply represent the length of the curve, but rather it represents the distance the particle has traveled. Furthermore, we construct an integral which represents the distance the particle has traveled from some fixed time, e.g., $t_0$, to a variable time, $t$, which we can refer to as a distance function, $s(t)$.

$$s(t) = \int_{t_0}^{t} \left(\sqrt{x'(\tau)^2 + y'(\tau)^2}\right) dt$$

Moreover, since the speed is equal to the time derivative of distance, we can express a speed function as

$$|v(t)| = \frac{d}{dt}\big(s(t)\big)$$

$$|v(t)| = \frac{d}{dt}\left(\int_{t_0}^{t}\left(\sqrt{x'(\tau)^2 + y'(\tau)^2}\right)dt\right)$$

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

Where, we used the Fundamental Theorem of Calculus to take the derivative of an integral.

| **Distance Traveled Along a Parametric Curve** |
| --- |
| If $c(t) = [x(t), y(t)]$ represents the position of a particle in space over time, then the distance the particle has traveled along the curve is at time, $t$, is<br>$$s(t) = \int_{t_0}^{t}\left(\sqrt{x'(\tau)^2 + y'(\tau)^2}\right)dt$$<br>Furthermore, the speed of the particle at time, $t$, is<br>$$\lvert v(t)\rvert = \frac{d}{dt}\big(s(t)\big) = \sqrt{x'(t)^2 + y'(t)^2}$$ |

Let's do some examples to practice with our new formulas.

**Example 1:** Suppose the position of a particle is given in parametric form as shown below. Find the distance the particle travels for , $0 \le t \le 4$.

$$c(t) = [2t^2, 3t^2 - 1]$$

Solution:

$$s = \int_0^4\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

$$= \int_0^4\left(\sqrt{(4t)^2 + (6t)^2}\right)dt$$

$$= \int_0^4 t\left(\sqrt{16 + 36}\right)dt$$

$$= \sqrt{52}\int_0^4 t dt$$

$$= \frac{\sqrt{52}}{2}t^2\Big|_0^4 = \frac{\sqrt{52}}{2}\cdot 16 = 16\sqrt{13}$$

**Example 2:** Find the distance a particle will travel over one arc of a cycloid given below.

$$c(t) = [Rt - R\,sin(t)\,, R - R\,cos(t)]$$

Solution:

Applying the arc length formula, we have

$$\begin{aligned}
s &= \int_0^{2\pi} \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt \\
&= \int_0^{2\pi} \left(\sqrt{(R - R\,cos(t))^2 + (R\,sin(t))^2}\right) dt \\
&= \int_0^{2\pi} \left(\sqrt{R^2 - 2R^2\,cos(t) + R^2\,cos^2(t) + R^2\,sin^2(t)}\right) dt \\
&= R\int_0^{2\pi} \left(\sqrt{1 - 2\,cos(t) + cos^2(t) + sin^2(t)}\right) dt \\
&= R\int_0^{2\pi} \left(\sqrt{2 - 2\,cos(t)}\right) dt \\
&= R\int_0^{2\pi} \left(\sqrt{2(1 - cos(t))}\right) dt \\
&= R\int_0^{2\pi} \left(\sqrt{2\left(2\,sin^2\left(\frac{t}{2}\right)\right)}\right) dt \\
&= 2R\int_0^{2\pi} sin\left(\frac{t}{2}\right) dt \\
&= 2R\int_0^{\pi} 2\,sin(u)\,du \\
&= 4R\left(-cos(u)|_0^{\pi}\right) \\
&= 4R(-cos(\pi) + cos(0)) \\
&= 8R
\end{aligned}$$

**Example 3:** Find the speed of a particle traveling along a path given by $c(t) = [5t + 1, 4t - 3]$ at $t = 9$ .

The speed of the particle is given by the equation from above.

$$\begin{aligned}
|v(t)| &= \sqrt{x'(t)^2 + y'(t)^2} \\
&= \sqrt{5^2 + 4^2} \\
&= \sqrt{41}
\end{aligned}$$

Therefore, the speed of the particle is constant and equal to $\sqrt{41}$ for all time.

**Example 4:** Find the speed of a particle moving along the first arch in a cycloid given below when the tangent line is horizontal, (assume meters per second).

$$c(t) = [4t - 4\,sin(t)\,, 4 - 4\,cos(t)]$$

Solution:

$$\begin{aligned} |v(t)| &= \sqrt{x'(t)^2 + y'(t)^2} \\ &= \sqrt{(4 - 4\,cos(t))^2 + (4\,sin(t))^2} \\ &= \sqrt{16\ - 32\,cos(t) + 16\,cos^2(t) + 16\,sin^2(t)} \\ &= \sqrt{32\ - 32\,cos(t)} \\ &= \sqrt{32}\sqrt{2\,sin^2\left(\frac{t}{2}\right)} \\ |v(t)| &= 8\,sin\left(\frac{t}{2}\right) \end{aligned}$$

In this case the speed is a function of $t$. The tangent line of a cycloid is horizontal when $t = \pi$. Therefore, the speed at this location along the curve is

$$|v(\pi)| = 8\,sin\left(\frac{\pi}{2}\right) = 8$$

**Example 5:** Find the minimum speed, (assume meters per second), of a particle with trajectory $c(t) = [t^3, t^{-2}]$ for $t \geq 0.5$

Solution: The speed of the particle is

$$\begin{aligned} |v(t)| &= \sqrt{x'(t)^2 + y'(t)^2} \\ |v(t)| &= \sqrt{(3t^2)^2 + (-2t^{-3})^2} \\ |v(t)| &= \sqrt{9t^4 + 4t^{-6}} \end{aligned}$$

To find the minimum speed we differentiate $|v(t)|$ and set to zero. However, we note that $|v(t)|$ will be minimum at the same time that $|v(t)|^2$ is, which is much easier to differentiate. Therefore, we find the time of the minimum speed, $t_m$, as follows:

$$\begin{aligned} \frac{d}{dt}(|v(t)|^2) &= 0 \\ 36t^3 - 24t^{-7} &= 0 \\ t^{10} &= \frac{24}{36} \\ t_m &= \left(\frac{2}{3}\right)^{1/10} \end{aligned}$$

Where, we took the positive time only.

The speed at this time is

$$|v(t_m)| = \sqrt{9\left(\left(\frac{2}{3}\right)^{1/10}\right)^4 + 4\left(\left(\frac{2}{3}\right)^{1/10}\right)^{-6}}$$

$$= \sqrt{9\left(\frac{2}{3}\right)^{2/5} + 4\left(\frac{3}{2}\right)^{3/5}}$$

$$|v(t_m)| \cong 3.57\ {}^{m}/_{s}$$

**Example 6:** A particle travels along a path given by $c(t) = \left[4t, 1 + t^{3/2}\right]$. Find the distance traveled and the displacement of the particle during the interval $0 \leq t \leq 4$.

Solution: Note that distance and displacement can be very different. For example, if you walked 10 feet, turned around and walked back to where you began the distance you covered would be 20 feet. However, your displacement would be zero!

The distance is computed using the arc length formula as follows:

$$s = \int_0^4 \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$

$$= \int_0^4 \left(\sqrt{(4)^2 + \left(\frac{3}{2}t^{1/2}\right)^2}\right) dt$$

$$= \int_0^4 \left(\sqrt{16 + \frac{9}{4}t}\right) dt$$

To evaluate this integral we use substitution as follows:

$$u = 16 + \frac{9}{4}t \qquad\qquad du = \frac{9}{4}dt \rightarrow \frac{4}{9}du = dt$$

$$s = \frac{4}{9}\int_{16}^{25} u^{1/2} du$$

$$= \frac{4}{9}\left(\frac{2}{3}u^{3/2}\Big|_{16}^{25}\right)$$

$$= \frac{8}{27}\left(25^{3/2} - 16^{3/2}\right)$$

$$\cong 18.1\ m$$

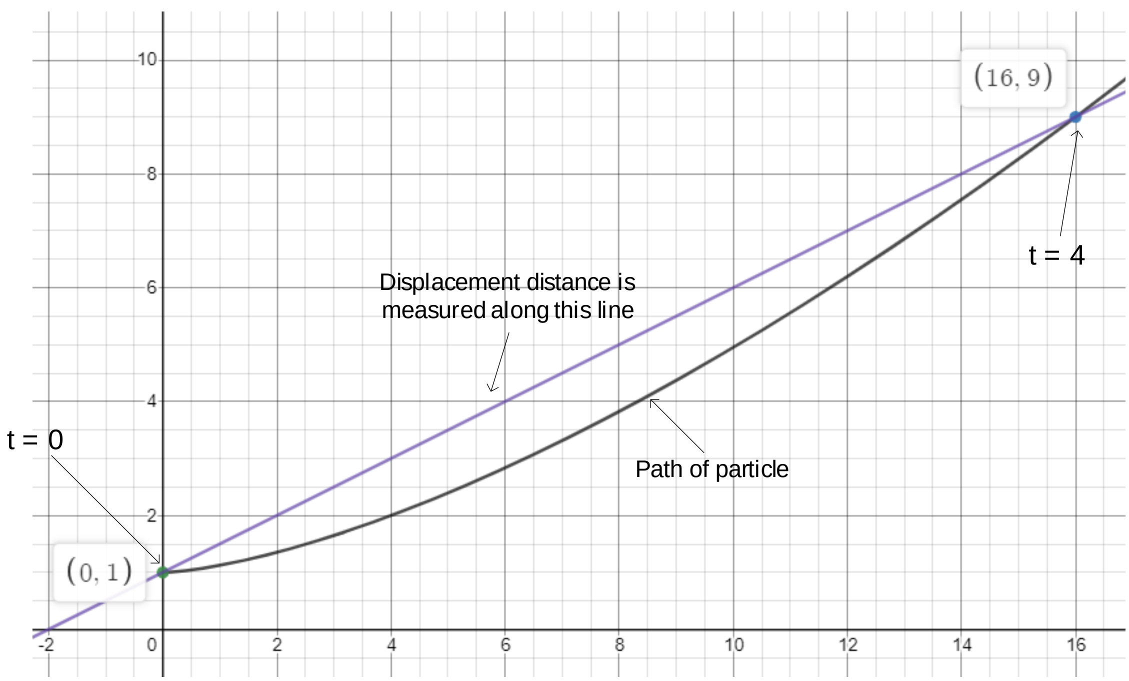
Since $c(t)$ represents the position at time $t$, the displacement is computed as follows:

$$\begin{aligned} displacement &= c(4) - c(0) \\ &= \left[4 \cdot 4, 1 + (4)^{3/2}\right] - \left[4 \cdot 0,1 + (0)^{3/2}\right] \\ &= [16, 9] - [0,1] \\ &= [18, 8] \end{aligned}$$

The particle was displaced by 18 meters in the $x$ direction and 8 meters in the $y$ direction. The magnitude of the displacement is computed with the Pythagorean theorem as

$$\begin{aligned} |displacement| &= \sqrt{18^2 + 8^2} \\ &\cong 19.7\ m \end{aligned}$$

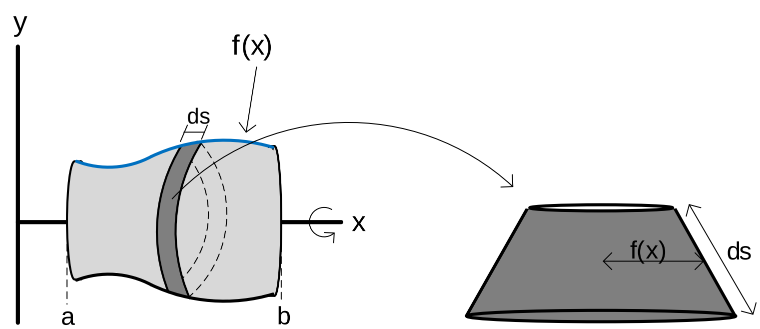
For illustration, a graph of $c(t)$ is shown below highlighting the distance traveled versus the displacement.



**Surface Area:**

Using explicit functions, we previously derived the following formula for the infinitesimal surface area, $dA$, based on the surface area of a frustum as shown in the figure below.

$$dA = 2\pi f(x)ds$$




Assuming the curve is instead described by a parametric equation we can replace $f(x)$ with $y(t)$. Furthermore, the arc length, $ds$, for parametric equations is $\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$. We can now rewrite the above formula as

$$dA = 2\pi y(t)\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

Finally, the surface area of the surface of revolution in the interval $a \le t \le b$, is found by integrating.

$$A_S = \int_a^b 2\pi y(t)\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

Where we assume $y(t) \ge 0$ so that the curve stays above the $x$-axis.

Let's do a few examples.

**Example 1:** Calculate the surface area of the surface obtained by rotating the parametric equation $c(t) = [t, t^3]$ about the $x$-axis for $0 \le t \le 1$.

Solution: Since $y(t) \ge 0$ we can directly use the formula from above.

$$A_S = \int_0^1 2\pi y(t)\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

$$A_S = 2\pi \int_0^1 t^3\left(\sqrt{1 + 9t^4}\right)dt$$

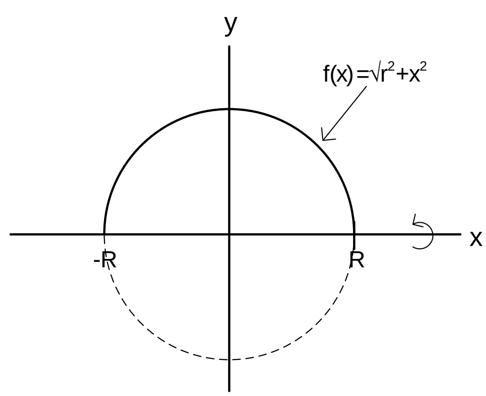
We use the following substitution to evaluate.

$$u = 1 + 9t^4 \qquad du = 36t^3 dt \rightarrow \frac{1}{36} du = t^3 dt$$

$$\begin{aligned} 2\pi \int_0^1 t^3 \left(\sqrt{1+9t^4}\right) dt &= \frac{2\pi}{36} \int_1^{10} u^{1/2} du \\ &= \frac{\pi}{18}\left(\frac{2}{3} u^{3/2}\Big|_1^{10}\right) \\ &= \frac{\pi}{27}\left(10^{3/2} - 1^{3/2}\right) \\ &\cong 3.56 \end{aligned}$$

**Example 2:** Calculate the surface area for a sphere with radius $R$.

Solution: We computed this in a previous section using an explicit function to describe the top half of a circle, as shown in the figure below.



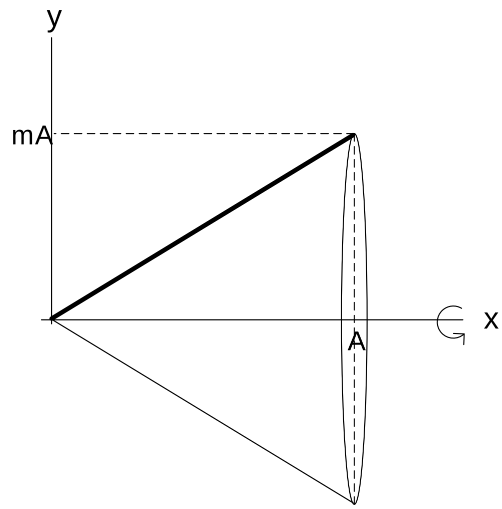

In this case we represent the top half for the circle parametrically as

$$c(t) = [R\,cos(t)\,, R\,sin(t)], 0 \le t \le \pi$$

The surface area is then

$$\begin{aligned} A_S &= \int_0^{\pi} 2\pi y(t)\left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt \\ &= 2\pi \int_0^{\pi} R\,sin(t)\left(\sqrt{R^2\,sin^2(t) + R^2\,cos^2(t)}\right) dt \\ &= 2\pi R^2 \int_0^{\pi} sin(t)\,dt \\ &= 2\pi R^2(-\,cos(\pi) + cos(0)) = 4\pi R^2 \end{aligned}$$

**Example 3:** Calculate the surface area for a cone generated by revolving $c(t) = [t, mt]$ about the $x$-axis for $0 \leq t \leq A$.




Solution: In a previous section we derived the surface area of a cone with radius $r$, height $h$, and lateral height $L$, as

$$A_S = \pi r L = \pi r\sqrt{h^2 + r^2}$$

Using the values from the cone above, the surface area is

$$A_S = \pi m A\sqrt{A^2 + m^2A^2} = \pi m A^2\sqrt{1 + m^2}$$

Let's see if we obtain the same value using the surface area formula

$$A_S = \int_0^A 2\pi y(t)\left(\sqrt{x'(t)^2 + y'(t)^2}\right)dt$$

$$= 2\pi \int_0^A mt\left(\sqrt{1 + m^2}\right)dt$$

$$= 2\pi m\sqrt{1 + m^2}\int_0^A tdt$$

$$= 2\pi m\sqrt{1 + m^2}\left(\frac{1}{2}A^2\right)$$

$$= \pi m A^2\sqrt{1 + m^2}$$

***Final Summary for Parametric Calculus – Arc Length and Speed***

| **Formula for Arc Length** |
|---|
| If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are differentiable, the arc length for $a \leq t \leq b$ is<br><br>$$s = \int_a^b \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$ |
| **Distance Traveled Along a Parametric Curve** |
| If $c(t) = [x(t), y(t)]$ represents the position of a particle in space, then the distance the particle has traveled along the curve at time, $t$, is<br><br>$$s(t) = \int_{t_0}^t \left(\sqrt{x'(\tau)^2 + y'(\tau)^2}\right) dt$$<br><br>Furthermore, the speed of the particle at time, $t$, is<br><br>$$\lvert v(t) \rvert = \frac{d}{dt}\big(s(t)\big) = \sqrt{x'(t)^2 + y'(t)^2}$$<br><br>The displacement of the particle, $\Delta c$, in each direction during a time $t_a \leq t \leq t_b$ is given as<br><br>$$\begin{aligned} \Delta c &= c(t_b) - c(t_a) \\ &= \left[\Delta c_x, \Delta c_y\right] \end{aligned}$$<br><br>The magnitude of the displacement is then<br><br>$$\lvert \Delta c \rvert = \sqrt{(\Delta c_x)^2 + \left(\Delta c_y\right)^2}$$ |
| **Formula for Surface Area of a Surface of Revolution** |
| If $c(t) = [x(t), y(t)]$ and $y(t) \geq 0$ on $[a, b]$, then the surface area, $A_S$, of the surface obtained by rotating the curve about the $x$-axis for $a \leq t \leq b$ is equal to<br><br>$$A_S = \int_a^b 2\pi y(t) \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$ |

*By:* ferrantetutoring