Integration Techniques – Trigonometric Substitution

We have seen how the method of u-substitution can sometimes be used to evaluate integrals that initially seem not integrable. We also saw that, in general, there is no hard and fast rule for choosing the proper substitution for a particular integral. In this section however, we introduce a substitution technique that can be used for a certain class of functions. The class of functions involve square root expressions of the form $\sqrt{\pm a^2 \pm x^2}$, and the substitution uses the various forms of the Pythagorean identity to transform the integral into a trigonometric integral.

Let's start by taking a look at how the square root expression can be transformed into a trigonometric expression. The general square root expression from above can be split into three categories. For each of these categories the table below shows the original function, the substitution, the trigonometric identity used, and the new transformed function.

Original Function $\boldsymbol{f(x)}$	***Substitution***	***Trigonometric Identity***	***Transformed Function*** $\boldsymbol{f(\theta)}$
$\sqrt{a^2 - x^2}$	$x = a\,sin(\theta)$ $x^2 = a^2\,sin^2(\theta)$	$cos^2(\theta) = 1 - sin^2(\theta)$	$= \sqrt{a^2 - a^2\,sin^2(\theta)}$ $= \sqrt{a^2(1 - sin^2(\theta))}$ $= \sqrt{a^2(cos^2(\theta))}$ $= a\,cos(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\,sec(\theta)$ $x^2 = a^2\,sec^2(\theta)$	$tan^2(\theta) = sec^2(\theta) - 1$	$= \sqrt{a^2\,sec^2(\theta) - a^2}$ $= \sqrt{a^2(sec^2(\theta) - 1)}$ $= \sqrt{a^2(tan^2(\theta))}$ $= a\,tan(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\,tan(\theta)$ $x^2 = a^2\,tan^2(\theta)$	$sec^2(\theta) = tan^2(\theta) + 1$	$= \sqrt{a^2 + a^2\,tan^2(\theta)}$ $= \sqrt{a^2(1 + tan^2(\theta))}$ $= \sqrt{a^2(sec^2(\theta))}$ $= a\,sec(\theta)$

Let's start with a few examples to illustrate the integration procedure.

Example 1: Evaluate $\int \sqrt{4 - x^2}\,dx$

Solution: The square root term is in the form of $\sqrt{a^2 - x^2}$, with $a^2 = 4 \rightarrow a = 2$, therefore we substitute as follows:

$$x = 2\,sin(\theta) \qquad \rightarrow \qquad \frac{dx}{d\theta} = 2\,cos(\theta) \qquad \rightarrow \qquad dx = 2\,cos(\theta)d\theta$$

Appling the substitution we have

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-\underbrace{4\,sin^2(\theta)}_{x^2}} \cdot \underbrace{2\,cos(\theta)d\theta}_{dx}$$

$$= \int \sqrt{4\Big(\underbrace{1-sin^2(\theta)}_{cos^2(\theta)}\Big)} \cdot 2\,cos(\theta)d\theta$$

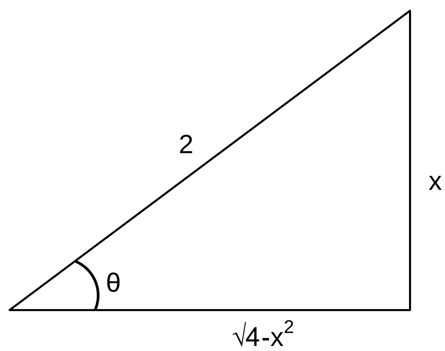
$$= \int 2\sqrt{cos^2(\theta)} \cdot 2\,cos(\theta)d\theta = 4\int cos^2(\theta)\,d\theta$$

The trigonometric integral can now be evaluated using a reduction formula as shown.

$$4\int cos^2(\theta)\,d\theta = 4\left(\frac{1}{2}cos(\theta)\,sin(\theta) + \frac{1}{2}\theta\right)$$

$$= 2(cos(\theta)\,sin(\theta) + \theta) + C$$

The final step is to convert the solution back to the original variable, x. To do this we return back to the original substitution and interpret the expression as a right triangle as shown below.



The original substitution, $x = 2\,sin(\theta)$, implies that $sin(\theta) = \frac{x}{2}$. The right triangle above corresponds to this relationship, where we used the Pythagorean Theorem to find the adjacent side. Using this triangle, we can easily develop expressions for θ and $cos(\theta)$ as follows.

$$\theta = \sin^{-1}\left(\frac{x}{2}\right) \qquad cos(\theta) = \frac{\sqrt{4-x^2}}{2}$$

Resubstituting, we can write the final solution as a function of x.

$$2(cos(\theta)\,sin(\theta) + \theta) + C = 2\left(\frac{\sqrt{4-x^2}}{2} \cdot \frac{x}{2} + sin^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$= \frac{x\sqrt{4-x^2}}{2} + 2\,sin^{-1}\left(\frac{x}{2}\right) + C$$

Example 2: Evaluate $\int \frac{1}{x^4\sqrt{x^2-4}}\,dx$

Solution: Even though the square root term is in the denominator and is multiplied by x^4, we still attempt to use the substitution as specified above.

$$x = 2\,sec(\theta) \qquad \rightarrow \qquad dx = 2\,sec(\theta)\,tan(\theta)\,d\theta$$

Appling the substitution we have

$$\int \frac{1}{x^4\sqrt{x^2-4}}\,dx = \int \frac{1}{16\,sec^4(\theta)\cdot\sqrt{4\Big(\underbrace{sec^2(\theta)-1}_{tan^2(\theta)}\Big)}}\cdot 2\,sec(\theta)\,tan(\theta)\,d\theta$$

$$= \int \frac{2\,sec(\theta)\,tan(\theta)}{32\,sec^4(\theta)\,tan(\theta)}\,d\theta$$

$$= \frac{1}{16}\int \frac{1}{sec^3(\theta)}\,d\theta = \frac{1}{16}\int cos^3(\theta)\,d\theta$$

We'll evaluate this integral using the Pythagorean Identity method from the previous lesson.

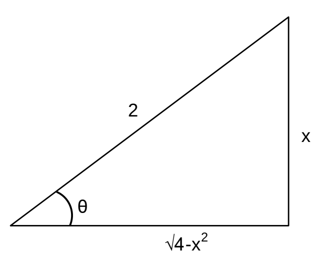
$$\frac{1}{16}\int cos^3(\theta)\,d\theta = \frac{1}{16}\int (1-sin^2(\theta))\,cos(\theta)\,d\theta$$

Now we let $u = sin(\theta) \ \rightarrow du = cos(\theta)\,d\theta$.

$$= \frac{1}{16}\int (1-u^2)\,du = \frac{1}{16}\left(u - \frac{1}{3}u^3\right)$$

$$= \frac{1}{16}\left(sin(\theta) - \frac{1}{3}sin^3(\theta)\right) + C$$

Once again, we convert back to a function of x. Based on the original substitution we can draw a right triangle and derive an expression for $sin(\theta)$ as follows.



$$\rightarrow \qquad sin(\theta) = \frac{\sqrt{x^2-4}}{x}$$

Therefore, we have

$$\int \frac{1}{x^4\sqrt{x^2-4}}\,dx = \frac{1}{16}\left(sin(\theta) - \frac{1}{3}sin^3(\theta)\right) + C$$

$$= \frac{\sqrt{x^2-4}}{16x} - \frac{(x^2-4)^{3/2}}{48x^3} + C$$

Example 3: Evaluate $\int \frac{1}{x\sqrt{x^2+16}}\,dx$

Solution: In this case we use the following substitution:

$$x = 4\,tan(\theta) \qquad \rightarrow \qquad dx = 4\,sec^2(\theta)\,d\theta$$

Appling the substitution we have

$$\int \frac{1}{x\sqrt{x^2+16}}\,dx = \int \frac{1}{4\,tan(\theta)\cdot\sqrt{16\Big(\underbrace{tan^2(\theta)+1}_{sec^2(\theta)}\Big)}}\cdot 4\,sec^2(\theta)\,d\theta$$

$$= \int \frac{sec^{\cancel{2}}(\theta)}{4\,tan(\theta)\,\cancel{sec(\theta)}}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{tan(\theta)}\cdot sec(\theta)\,d\theta$$

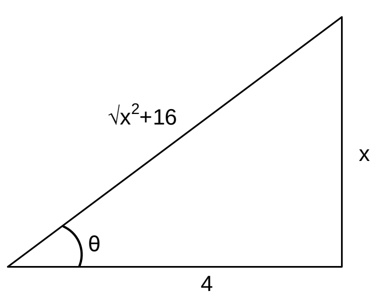
$$= \frac{1}{4}\int \frac{\cancel{cos(\theta)}}{sin(\theta)}\cdot\frac{1}{\cancel{cos(\theta)}}\,d\theta$$

$$= \frac{1}{4}\int csc(\theta)\,d\theta$$

$$= \frac{1}{4}(ln|csc(\theta) - cot(\theta)|) + C$$

Where, we used a table of integrals to evaluate $\int csc(\theta)\,d\theta$.

To convert the solution to a function of x we again start by drawing a right triangle, which is defined by relationship established with the initial substitution, i.e. $x = 4\tan(\theta)$.



$$\rightarrow \qquad csc(\theta) = \frac{hyp}{opp} = \frac{\sqrt{x^2+16}}{x}$$

$$cot(\theta) = \frac{adj}{opp} = \frac{4}{x}$$

Therefore, we have

$$\int \frac{1}{x\sqrt{x^2+16}}\,dx = \frac{1}{4}(ln|csc(\theta) - cot(\theta)|) + C$$

$$= \frac{1}{4}\left(ln\left|\frac{\sqrt{x^2+16}-4}{x}\right|\right) + C$$

The next example requires one extra step to start.

Example 4: Evaluate $\int \frac{1}{\sqrt{25x^2+2}}\,dx$

Solution: The first thing we notice in this case is that the term inside the square root has a coefficient in front of the x^2 term, and therefore does not precisely match one of our three categories. However, we can rewrite the integral as follows:

$$\int \frac{1}{\sqrt{25x^2+2}}\,dx = \int \frac{1}{\sqrt{(5x)^2+\left(\sqrt{2}\right)^2}}\,dx$$

Next, we make a u-substitution as follows.

$$u = 5x \qquad \rightarrow \qquad du = 5dx$$

Therefore, we have:

$$\int \frac{1}{\sqrt{(5x)^2+\left(\sqrt{2}\right)^2}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{u^2+\left(\sqrt{2}\right)^2}}\,du$$

We can now make the trigonometric substitution and evaluate the resulting integral as before.

$$u = \sqrt{2}\,tan(\theta) \qquad \rightarrow \qquad du = \sqrt{2}\,sec^2(\theta)\,d\theta$$

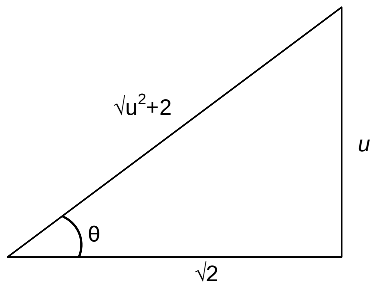
$$\frac{1}{5}\int \frac{1}{\sqrt{u^2+\left(\sqrt{2}\right)^2}}\,du = \frac{1}{5}\int \frac{1}{\sqrt{\left(\sqrt{2}\,tan(\theta)\right)^2+\left(\sqrt{2}\right)^2}}\cdot\sqrt{2}\,sec^2(\theta)\,d\theta$$

$$= \frac{1}{5}\int \frac{\cancel{\sqrt{2}}\,sec^2(\theta)}{\cancel{\sqrt{2}}\sqrt{\underbrace{tan^2(\theta)+1}_{sec^2(\theta)}}}\,d\theta$$

$$= \frac{1}{5}\int sec(\theta)\,d\theta$$

$$= \frac{1}{5}\left(ln|sec(\theta)+tan(\theta)|\right) + C$$

As usual we use a right triangle to convert the solution, this time back to u.



$$\rightarrow \qquad sec(\theta) = \frac{hyp}{adj} = \frac{\sqrt{u^2+2}}{\sqrt{2}}$$
$$tan(\theta) = \frac{opp}{adj} = \frac{u}{\sqrt{2}}$$

Therefore, we have

$$\frac{1}{5}\int \frac{1}{\sqrt{u^2+\left(\sqrt{2}\right)^2}}du = \frac{1}{5}(ln|sec(\theta)+tan(\theta)|) + C$$

$$= \frac{1}{5}\left(ln\left|\frac{\sqrt{u^2+2}+u}{\sqrt{2}}\right|\right) + C$$

For the final solution we let $u = 5x$

$$\int \frac{1}{\sqrt{25x^2+2}}dx = \frac{1}{5}\left(ln\left|\frac{\sqrt{(5x)^2+2}+5x}{\sqrt{2}}\right|\right) + C$$

$$= \frac{1}{5}\left(ln\left|\frac{\sqrt{25x^2+2}+5x}{\sqrt{2}}\right|\right) + C$$

The technique outlined can also be used for integrands involving $(\pm a^2 \pm x^2)^{n/2}$, since we can rewrite this expression as $\left(\sqrt{\pm a^2 \pm x^2}\right)^n$. Let's do an example of this type to illustrate.

Example 5: Evaluate $\int \frac{x^2}{(x^2+1)^{3/2}}\,dx$

Solution: We start by rewriting the integral as follows:

$$\int \frac{x^2}{(x^2+1)^{3/2}}\,dx = \int \frac{x^2}{\left(\sqrt{x^2+1}\right)^3}\,dx$$

Next, we substitute as follows

$$x = tan(\theta) \qquad \rightarrow \qquad dx = sec^2(\theta)\,d\theta$$

$$\int \frac{x^2}{\left(\sqrt{x^2+1}\right)^3}\,dx = \int \frac{tan^2(\theta)}{\left(\sqrt{tan^2(\theta)+1}\right)^3} \cdot sec^2(\theta)\,d\theta$$

$$= \int \frac{tan^2(\theta)\,sec^2(\theta)}{sec^3(\theta)}\,d\theta = \int \frac{tan^2(\theta)}{sec(\theta)}\,d\theta$$

This integral does not look straightforward to evaluate; therefore, we should try to use trigonometric identities to rewrite the integrand in a form we can more easily evaluate.

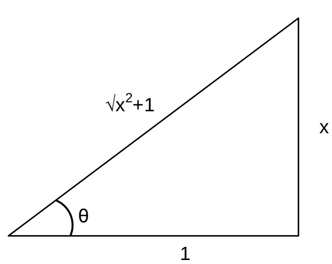
$$\frac{tan^2(\theta)}{sec(\theta)} = \frac{sec^2(\theta)-1}{sec(\theta)} = \left(sec(\theta) - \frac{1}{sec(\theta)}\right) = (sec(\theta) - cos(\theta))$$

Note that which identities to use is not always obvious and there is sometimes more than one method. In this case we changed the integral into two integrals that can directly be solved.

$$\int \frac{tan^2(\theta)}{sec(\theta)}\,d\theta = \int sec(\theta)\,d\theta - \int cos(\theta)\,d\theta$$

$$= ln|sec(\theta) + tan(\theta)| - sin(\theta) + C$$

Finally, we draw our right triangle and write the solution as a function of x.



$$sec(\theta) = \sqrt{x^2+1}$$

$$tan(\theta) = x$$

$$sin(\theta) = \frac{x}{\sqrt{x^2+1}}$$

$$\int \frac{x^2}{(x^2+1)^{3/2}}\,dx = \left(ln\left|\sqrt{x^2+1} + x\right|\right) - \frac{x}{\sqrt{x^2+1}} + C$$

Square Roots of General Quadratics:

What if we the integrand contained a quadratic function in standard form as follows?

$$\sqrt{ax^2 + bx + c}$$

Although the expression is not in the form needed for the technique used above, we can re-write the quadratic in vertex from by completing the square. After completing the square, we then perform a u-substitution, which results in an expression in the form needed for our trigonometric substitution method from above. Let's see how the procedure works with the next two examples.

Example 6: Evaluate $\int \frac{1}{\sqrt{x^2+4x+13}} dx$

Solution:

Step 1: Complete the square.

$$x^2 + 4x + 13 = (x+2)^2 - 4 + 13 = (x+2)^2 + 9$$

Therefore, we have

$$\int \frac{1}{\sqrt{x^2 + 4x + 13}} dx = \int \frac{1}{\sqrt{(x+2)^2 + 9}} dx$$

Step 2: Use u-substitution

$$u = x + 2 \qquad \rightarrow \qquad du = dx$$

$$\int \frac{1}{\sqrt{(x+2)^2 + 9}} dx = \int \frac{1}{\sqrt{u^2 + 9}} du$$

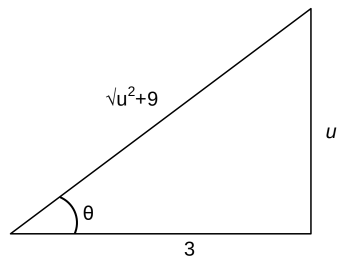
Step 3: Use Trigonometric substitution

$$u = 3\,tan(\theta) \qquad \rightarrow \qquad du = 3\,sec^2(\theta)\,d\theta$$

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+9}} du &= \int \frac{1}{3\sqrt{tan^2(\theta)+1}} \cdot 3\,sec^2(\theta)\,d\theta \\ &= \int sec(\theta)\,d\theta \\ &= ln|sec(\theta) + tan(\theta)| + C\end{aligned}$$

Step 4: Convert the solution back to the original variable.

We start with the right triangle as usual.



$$tan(\theta) = \frac{u}{3}$$

$$sec(\theta) = \frac{\sqrt{u^2+9}}{3}$$

$$\int \frac{1}{\sqrt{u^2+9}} du = ln\left|\frac{\sqrt{u^2+9}+u}{3}\right| + C$$

Next, we re-substitute for x, i.e. $u = x + 2$.

$$\int \frac{1}{\sqrt{x^2+4x+13}} dx = ln\left|\frac{\sqrt{(x+2)^2+9}+(x+2)}{3}\right| + C$$

$$= ln\left|\frac{\sqrt{x^2+4x+13}+(x+2)}{3}\right| + C$$

$$= ln\left|\sqrt{x^2+4x+13}+(x+2)\right| - ln|3| + C$$

$$= ln\left|\sqrt{x^2+4x+13}+(x+2)\right| + C$$

Where, we absorbed the $(-\,ln|3|)$ term into the constant to simplify the expression.

Example 7: Evaluate $\int \sqrt{x^2-4x+3}\,dx$

Step 1: Complete the square.

$$x^2 - 4x + 3 = (x-2)^2 - 4 + 3 = (x-2)^2 - 1$$

Therefore, we have

$$\int \sqrt{x^2-4x+3}\,dx = \int \sqrt{(x-2)^2-1}\,dx$$

Step 2: Use u-substitution

$$u = x - 2 \qquad \rightarrow \qquad du = dx$$

$$\int \sqrt{(x-2)^2-1}\,dx = \int \sqrt{u^2-1}\,du$$

Step 3: Use Trigonometric substitution

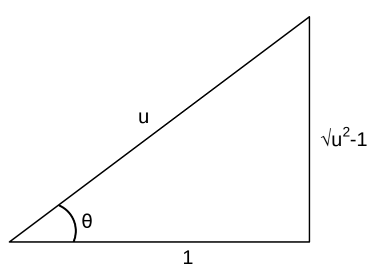
$$u = sec(\theta) \quad \rightarrow \quad du = sec(\theta)\,tan(\theta)\,d\theta$$

$$\begin{aligned}\int \sqrt{u^2-1}\,du &= \int \left(\sqrt{sec^2(\theta)-1}\right) sec(\theta)\,tan(\theta)\,d\theta \\ &= \int sec(\theta)\,tan^2(\theta)\,d\theta \\ &= \int sec(\theta)\,(sec^2(\theta)-1)\,d\theta \\ &= \left(\int sec^3(\theta)\,d\theta\right) - \left(\int sec(\theta)\,d\theta\right)\end{aligned}$$

We will use the reduction formula for the first integral.

$$\begin{aligned}&= \left(\frac{1}{2}sec(\theta)\,tan(\theta) + \frac{1}{2}\int sec(\theta)\,d\theta\right) - \left(\int sec(\theta)\,d\theta\right) \\ &= \frac{1}{2}\left(sec(\theta)\,tan(\theta) - \int sec(\theta)\,d\theta\right) \\ &= \frac{1}{2}(sec(\theta)\,tan(\theta) - ln|sec(\theta)+tan(\theta)|) + C\end{aligned}$$

Step 4: Convert solution back to the original variable.



$$sec(\theta) = u$$

$$tan(\theta) = \sqrt{u^2-1}$$

$$\begin{aligned}\int \sqrt{u^2-1}\,du &= \frac{1}{2}(sec(\theta)\,tan(\theta) - ln|sec(\theta)+tan(\theta)|) + C \\ &= \frac{1}{2}\left(u\sqrt{u^2-1} - ln\left|u+\sqrt{u^2-1}\right|\right) + C\end{aligned}$$

Next, we re-substitute for x, i.e. $u = x - 2$.

$$\begin{aligned}\int \sqrt{x^2-4x+3}\,dx &= \frac{1}{2}\left((x-2)\sqrt{(x-2)^2-1} - ln\left|x+2+\sqrt{(x-2)^2-1}\right|\right) + C \\ &= \frac{(x-2)\sqrt{x^2-4x+3} - ln|x-2+\sqrt{x^2-4x+3}|}{2} + C\end{aligned}$$

<u>Final Summary for Integration Techniques – Trigonometric Substitution</u>

Trigonometric Substitution

Trigonometric substitution may be used on the class of functions involving square root expressions of the form $\sqrt{\pm a^2 \pm x^2}$. The substitution involves the following three steps.

1. Substitute to eliminate the square root and convert to a trigonometric integral in θ.
2. Evaluate the trigonometric integral.
3. Convert the solution back to the original variable using the appropriate right triangle.

The general procedure is shown below for the three different cases.

General Quadratic Functions

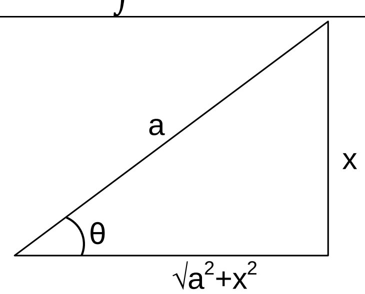
If the expression contains instead a general quadratic function as follows:

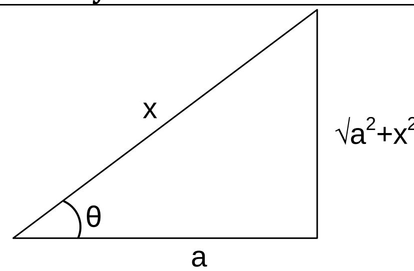
$$\sqrt{ax^2 + bx + c}$$

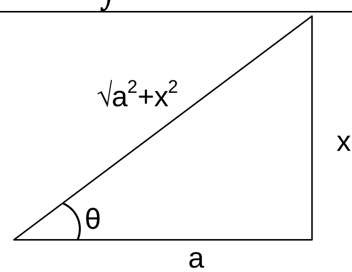
We can convert the standard form quadradic into vertex form by completing the square and then use u-substitution. The example below illustrates the procedure when $a = 1$.

$$\sqrt{\left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} \qquad \rightarrow \qquad u = x + \frac{b}{2} \;\rightarrow\; \sqrt{u^2 + \left(c - \frac{b^2}{4}\right)}$$

Once in this form we return to the original three steps from above to evaluate.

Case 1: $\sqrt{a^2 - x^2}$	
Original Integral	$\int \sqrt{a^2 - x^2}\, dx$
Substitution	$x = a\, sin(\theta) \;\rightarrow\; dx = a\, cos(\theta)\, d\theta$
New Integral	$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2 sin^2(\theta)} \cdot a\, cos(\theta)\, d\theta$ $= \int \sqrt{a^2 \left(\underbrace{1 - sin^2(\theta)}_{cos^2(\theta)}\right)} \cdot a\, cos(\theta)\, d\theta$ $= \int a\, cos(\theta) \cdot a\, cos(\theta)\, d\theta$ $= a^2 \int cos^2(\theta)\, d\theta$
Associated Right Triangle	

Case 2: $\sqrt{x^2 - a^2}$	
Original Integral	$\int \sqrt{x^2 - a^2}\, dx$
Substitution	$x = a\, sec(\theta) \;\rightarrow\; dx = a\, sec(\theta)\, tan(\theta)\, d\theta$
New Integral	$\begin{aligned} \int \sqrt{x^2 - a^2}\, dx &= \int \sqrt{a^2\, sec^2(\theta) - a^2} \cdot a\, sec(\theta)\, tan(\theta)\, d\theta \\ &= \int \sqrt{a^2 \Big(\underbrace{sec^2(\theta) - 1}_{tan^2(\theta)} \Big)} \cdot a\, sec(\theta)\, tan(\theta)\, d\theta \\ &= \int a\, tan(\theta) \cdot a\, sec(\theta)\, tan(\theta)\, d\theta \\ &= a^2 \int sec(\theta)\, tan^2(\theta)\, d\theta \end{aligned}$
Associated Right Triangle	

Case 3: $\sqrt{a^2 + x^2}$	
Original Integral	$\int \sqrt{a^2 + x^2}\, dx$
Substitution	$x = a\, tan(\theta) \;\rightarrow\; dx = a\, sec^2(\theta)\, d\theta$
New Integral	$\begin{aligned} \int \sqrt{a^2 + x^2}\, dx &= \int \sqrt{a^2 + a^2\, tan^2(\theta)} \cdot a\, sec^2(\theta)\, d\theta \\ &= \int \sqrt{a^2 \Big(\underbrace{1 + tan^2(\theta)}_{sec^2(\theta)} \Big)} \cdot a\, sec^2(\theta)\, d\theta \\ &= \int a\, sec(\theta) \cdot a\, sec^2(\theta)\, d\theta \\ &= a^2 \int sec^3(\theta)\, d\theta \end{aligned}$
Associated Right Triangle	

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