Integration Techniques – Trigonometric Integrals

In calculus 1 we developed a table of the *basic* trigonometric integrals. We used the term *basic* because they were derived by simply reversing the derivative formulas. This list, however, did not include many important integrals, e.g., $\int tan(x)dx$, $\int cot(x)dx$, $\int sec(x)dx$, $\int csc(x)dx$, as well as various combinations of trigonometric functions. Many of these types of functions can be integrated by combining substitution and integration by parts. However, as the integrand becomes more complex the evaluation becomes tedious and prone to arithmetic errors. For these cases we usually rely on tables of integrals. The tables provide not only specific integrals, as the table of the basic trigonometric integrals does, but it also provides formulas that can be used to solve certain classes of integrands. The summary section provides such a table of trigonometric integrals. However, before referring to the table let's try to evaluate some of the most common trigonometric integrals.

Example 1: Integral of $tan(x)$ and $cot(x)$

For the integral of $tan(x)$ we first rewrite it in terms of sine and cosine and then use substitution.

$$\int tan(x)\,dx = \int \frac{sin(x)}{cos(x)}\,dx$$

Using the substitution

$$u = cos(x) \quad \rightarrow \quad du = -\,sin(x)\,dx$$

We have

$$\begin{aligned}\int \frac{sin(x)}{cos(x)}\,dx &= -\int \frac{1}{u}\,du\\ &= -\,ln|u|\\ &= -\,ln|cos(x)|\\ &= ln\left|\frac{1}{cos(x)}\right|\end{aligned}$$

$$\int tan(x)\,dx = ln|sec(x)| + C$$

The integral of $cot(x)$ can be evaluate in much the same way. We leave it as an exercise to prove the following results.

$$\int cot(x)\,dx = ln|sin(x)| + C$$

Example 2: Integral of $sec(x)$ and $csc(x)$

For the integral of $sec(x)$ we first employ a highly non-obvious trick. We multiply the integrand by

$$1 = \frac{sec(x) + tan(x)}{sec(x) + tan(x)}$$

Therefore, we can rewrite the integral as shown

$$\int sec(x)\,dx = \int sec(x) \cdot \left(\frac{sec(x) + tan(x)}{sec(x) + tan(x)}\right) dx = \int \frac{sec^2(x) + sec(x)\,tan(x)}{sec(x) + tan(x)}\,dx$$

Next, we substitute as follows

$$u = sec(x) + tan(x) \quad \rightarrow \quad du = (sec(x)\,tan(x) + sec^2(x))dx$$

$$\begin{aligned}\int sec(x)\,dx = &= \int \frac{sec^2(x) + sec(x)\,tan(x)}{sec(x) + tan(x)}\,dx \\ &= \int \frac{1}{u}\,du \\ &= ln|u| \\ &= ln|sec(x) + tan(x)| + C\end{aligned}$$

The integral of $csc(x)$ can be similarly evaluated, this time multiplying the integrand by

$$1 = \frac{csc(x) + cot(x)}{csc(x) + cot(x)}$$

Again, we leave it as an exercise to prove the following results.

$$\int csc(x)\,dx = -ln|csc(x) + cot(x)| + C$$

A table of integrals for the 6 trigonometric functions is given below.

Trigonometric Integrals	
$\int sin(x)\,dx = -cos(x) + C$	$\int csc(x)\,dx = -ln\|csc(x) + cot(x)\| + C$
$\int cos(x)\,dx = sin(x) + C$	$\int sec(x)\,dx = ln\|sec(x) + tan(x)\| + C$
$\int tan(x)\,dx = ln\|sec(x)\| + C$	$\int cot(x)\,dx = ln\|sin(x)\| + C$

It is also common to see trigonometric function raised to powers as integrands. Let's take a look at the three specific types shown below.

$$\int cos^n(x)\,dx \qquad \int sin^m(x)\,dx \qquad \int sin^m(x)\,cos^n(x)\,dx$$

There are various methods we can use to evaluate integrals of these types. One particular method uses integration by parts to derive a so-called *reduction formula*, which can then be used to evaluate the integrals. We illustrate by deriving the reduction formula for cosine.

Example 3: Reduction Formula for the Integral of $cos^n(x)$

We start by splitting off one power from the function and writing the integral as follows.

$$\int cos^n(x)\,dx = \int cos^{n-1}(x)\,cos(x)\,dx$$

We then make the following IBP substitutions

$u = cos^{n-1}(x)$	$dv = cos(x)\,dx$
$du = -(n-1)\,cos^{n-2}(x)\,sin(x)\,dx$	$v = sin(x)$

$$\int cos^n(x)\,dx = \underbrace{cos^{n-1}(x)\,sin(x)}_{uv} - \int \underbrace{-(n-1)\,cos^{n-2}(x)\ \underbrace{sin^2(x)}_{(1-cos^2(x))}\ dx}_{vdu}$$

$$\int cos^n(x)\,dx = cos^{n-1}(x)\,sin(x) + (n-1)\int cos^{n-2}(x)\,(1-cos^2(x))dx$$

$$\int cos^n(x)\,dx = cos^{n-1}(x)\,sin(x) + (n-1)\int cos^{n-2}(x)\,dx - (n-1)\int cos^n(x)\,dx$$

Where, we used the Pythagorean identity, $sin^2(x) + cos^2(x) = 1$.

Next, we add the $(n-1)\int cos^n(x)\,dx$ to both sides.

$$n\int cos^n(x)\,dx = cos^{n-1}(x)\,sin(x) + (n-1)\int cos^{n-2}(x)\,dx$$

Lastly, we divide the entire equation by n, giving us the cosine reduction formula shown below.

Cosine Integral Reduction Formula
$\int cos^n(x)\,dx = \frac{1}{n} cos^{n-1}(x)\,sin(x) + \frac{n-1}{n}\int cos^{n-2}(x)\,dx$

To illustrate let's apply this formula to $\int cos^5(x)\,dx$.

$$\int cos^5(x)\,dx = \frac{1}{5}cos^4(x)\,sin(x) + \frac{4}{5}\underbrace{\int cos^3(x)\,dx}_{apply\ reduction\ again}$$

$$\int cos^5(x)\,dx = \frac{1}{5}cos^4(x)\,sin(x) + \frac{4}{5}\left(\frac{1}{3}cos^2(x)\,sin(x) + \frac{2}{3}\underbrace{\int cos(x)\,dx}_{directly\ evaluate}\right)$$

$$\int cos^5(x)\,dx = \frac{1}{5}cos^4(x)\,sin(x) + \frac{4}{5}\left(\frac{1}{3}cos^2(x)\,sin(x) + \frac{2}{3}sin(x)\right)$$

$$\int cos^5(x)\,dx = \frac{1}{5}cos^4(x)\,sin(x) + \frac{4}{15}cos^2(x)\,sin(x) + \frac{8}{15}sin(x) + C$$

A reduction formula for the integral of $sin^m(x)$ can be derived in much the same way with the result shown below.

Sine Integral Reduction Formula
$\int sin^m(x)\,dx = -\frac{1}{m}sin^{m-1}(x)\,cos(x) + \frac{m-1}{m}\int sin^{m-2}(x)\,dx$

Reduction formulas can also be developed for the third type of integral shown, however instead of deriving these we provide them in a table in the summary section.

Alternate techniques can also be used to solve these types of integrals, which we will discuss below. There are two main techniques that we will discuss, one we refer to as the "*Pythagorean Identity Method*", and the other the "*Double Angle Formula Method*". The first method is used when at least on the powers, m or n, is odd and the second method is used when only even powers are present. The table below summarizes.

$\boldsymbol{m}$	$\boldsymbol{n}$	***Method***
Odd	Even	*Pythagorean Identity Method*
Even	Odd	
Odd	Odd	
Odd	0	
0	Odd	
Even	Even	Double Angle Formula Method
Even	0	
0	Even	

Pythagorean Method

We'll look at two separate examples first and then outline the method for the general case.

Example 4: Evaluate $\int cos^5(x)\,dx$

Solution: This method starts by splitting of one power of the cosine function and using the Pythagorean identity as shown below.

$$\int cos^5(x)\,dx = \int cos(x)\,cos^4(x)\,dx$$

$$= \int cos(x)\,(cos^2(x))^2\,dx$$

$$= \int cos(x)\Big(\underbrace{1 - sin^2(x)}_{Pythagorean\ Identity}\Big)^2\,dx$$

Next, we can now use the following substitution.

$$u = sin(x) \qquad\qquad du = cos(x)\,dx$$

The integral then becomes a simple polynomial in u and is evaluated as follows

$$= \int (1 - u^2)^2\,du$$

$$= \int (1 - 2u^2 + u^4)\,du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \; sin(x) - \frac{2}{3}sin^3(x) + \frac{1}{5}sin^5(x) + C$$

Example 5: Evaluate $\int sin^3(x)\,cos^4(x)\,dx$

Solution: In this case we take the same basic approach as the previous example using the sinusoidal term with the odd power.

$$\int sin^3(x)\,cos^4(x)\,dx = \int sin(x)\,sin^2(x)\,cos^4(x)\,dx$$

$$= \int sin(x)\,(1 - cos^2(x))\,cos^4(x)\,dx$$

As before, we next make the following substitution

$$u = cos(x) \qquad\qquad du = -\,sin(x)\,dx$$

$$\begin{aligned}\int sin(x)\,(1 - cos^2(x))\,cos^4(x)\,dx &= -\int (1-u^2)u^4\,du \\ &= \int (u^6 - u^4)\,du \\ &= \frac{1}{7}cos^7(x) - \frac{1}{5}cos^5(x) + C\end{aligned}$$

The general method can be summarized as follows

- Split off one power of the sinusoidal function that is raised to the **odd** power.
  - E.g., $cos^{2k+1}(x) = \; cos(x) \cdot cos^{2k}(x)$
- Pull out the power of k.
  - E.g., $cos(x) \cdot cos^{2k}(x) = \; cos(x) \cdot (cos^2(x))^k$
- Apply the Pythagorean identity to the square term.
  - $\text{E.g.}, cos(x) \cdot (cos^2(x))^k = \; cos(x) \cdot (1 - sin^2(x))^k$
- Use a u-substitution with the new sinusoid term from the Pythagorean identity.
  - E.g., $u = sin(x),\; du = cos(x)\,dx$
- Apply the u-substitution and expand the resulting polynomial.
  - E.g., $cos(x) \cdot (1 - sin^2(x))^k dx = \; (1-u^2)^k du$

Double Angle Formula Method

For this case we will again look at two separate examples first and then outline the method for the general case.

Example 6: Evaluate $\int cos^4(x)\,dx$

Solution: This method starts by writing the sinusoid as a squared term raised to a power and apply the double angle formula to the squared sinusoid as shown below.

$$\begin{aligned}\int cos^4(x)\,dx &= \int (cos^2(x))^2\,dx \\ &= \int \left(\frac{1 + cos(2x)}{2}\right)^2 dx\end{aligned}$$

Next, we expand, use the double angle formula repeatedly as required, then integrate.

$$\int \left(\frac{1+cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int \left(1 + 2\,cos(2x) + \underbrace{cos^2(2x)}_{apply\ double\ angle\ again}\right) dx$$

$$= \frac{1}{4}\int 1 + 2\,cos(2x) + \frac{1}{2} + \frac{cos(4x)}{2}\,dx$$

$$= \frac{1}{4}\left(\frac{3}{2}\int dx + \frac{1}{2}\int cos(2x)\,dx + \frac{1}{2}\int cos(4x)\right)$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \frac{1}{4}sin(2x) + \frac{1}{8}sin(4x)\right)$$

$$= \frac{3}{8}x + \frac{1}{16}sin(2x) + \frac{1}{32}sin(4x) + C$$

Example 7: Evaluate $\int sin^2(x)\,cos^4(x)\,dx$

Solution: In this case we take the same basic approach as the previous example using the double angle formula on both sinusoidal terms as shown below.

$$\int sin^2(x)\,cos^4(x)\,dx = \int \left(\frac{1}{2} - \frac{1}{2}cos(2x)\right)\left(\frac{1}{2} + \frac{1}{2}cos(2x)\right)^2 dx$$

Next, we expand the terms while again applying the double angle formula again as needed.

$$= \int \left(\frac{1}{2} - \frac{1}{2}cos(2x)\right)\left(\frac{1}{4} + \frac{1}{2}cos(2x) + \frac{1}{4}\underbrace{cos^2(2x)}_{double\ angle}\right) dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}cos(2x)\right)\left(\frac{1}{4} + \frac{1}{2}cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}cos(4x)\right)\right) dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}cos(2x)\right)\left(\frac{3}{8} + \frac{1}{2}cos(2x) + \frac{1}{8}cos(4x)\right) dx$$

Unfortunately, we need to continue with the simplification and multiply the two factors as shown.

$$= \int \left(\frac{3}{16} + \frac{1}{4}cos(2x) + \frac{1}{16}cos(4x) - \frac{3}{16}cos(2x) - \frac{1}{4}cos^2(2x) - \frac{1}{16}cos(4x)\,cos(2x)\right) dx$$

$$= \int \left(\frac{3}{16} + \frac{1}{16}cos(2x) + \frac{1}{16}cos(4x) - \frac{1}{4}\underbrace{cos^2(2x)}_{double\ angle} - \frac{1}{16}\underbrace{cos(4x)\,cos(2x)}_{product\ to\ sum}\right) dx$$

Using the double angle formula one last time along with the product to sum formula we have

$$= \int \left(\frac{3}{16} + \frac{1}{16}cos(2x) + \frac{1}{16}cos(4x) - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}cos(4x)\right) - \frac{1}{16}\left(\frac{1}{2}cos(2x) + \frac{1}{2}cos(6x)\right)\right) dx$$

$$= \int \left(\frac{3}{16} + \frac{1}{16}cos(2x) + \frac{1}{16}cos(4x) - \frac{1}{8} - \frac{1}{8}cos(4x) - \frac{1}{32}cos(2x) - \frac{1}{32}cos(6x)\right) dx$$

$$= \int \left(\frac{1}{16} + \frac{1}{32}cos(2x) - \frac{1}{16}cos(4x) - \frac{1}{32}cos(6x)\right) dx$$

$$= \frac{1}{16}x + \frac{1}{64}sin(2x) - \frac{1}{64}sin(4x) - \frac{1}{192}sin(6x) + C$$

The general method can be summarized as follows

- Write the sinusoid(s) as a squared term raised to a power.
  - E.g., $cos^{2k}(x) = (cos^2(x))^k$
- Use the double angle formula to the squared terms(s).
  - E.g., $(cos^2(x))^k = \left(\frac{1}{2} + \frac{1}{2}cos(2x)\right)^k$
- Expand and use the double angle formula (and possible the product to sum formula), repeatedly as required until all terms are raised to a power of one.
- Combine and integrate as required.

Table of Integrals:

Other than using the reduction formulas, the general technique for these types of integrals is to look for various trigonometric identities that can be used to rewrite the integrands in a form that can be directly evaluated. However, as you can see, the integrals become quite time consuming (and prone to arithmetic errors), as the powers get larger. For these cases it is usually better to rely on a table of integrals. The tables provide not only specific integrals, but they also provide formulas that can be used to solve certain classes of integrands. We provide a table of some common trigonometric integrals in the summary section. Note that reduction formulas exist for the other trigonometric functions as well.

To illustrate how to use integral tables let's do two more few more examples.

Example 8: Evaluate $\boldsymbol{\int tan^4(x)\, dx}$

Solution: In this case we directly use the formula from the table of integrals.

The formula from the table is as follows:

$$\int tan^n(x)\, dx = \frac{tan^{n-1}(x)}{n-1} - \int tan^{n-2}(x)\, dx$$

Applying this formula, we have

$$\begin{aligned}
\int tan^4(x)\, dx &= \frac{tan^3(x)}{3} - \underbrace{\int tan^2(x)\, dx}_{apply\ formula\ again} \\
&= \frac{tan^3(x)}{3} - \left(\frac{tan(x)}{1} - \int dx\right) \\
&= \frac{tan^3(x)}{3} - tan(x) + x + C
\end{aligned}$$

Final Summary for Integration Techniques – Trigonometric Integrals

Integration Techniques

Integrals of the following form

$$\int sin^m(x)\, cos^n(x)\, dx$$

Can be solved using two different methods depending on the polarity of the powers as specified on the table below.

m	n	***Method***
Odd	Even	*Pythagorean Identity Method*
Even	Odd	
Odd	Odd	
Odd	0	
0	Odd	
Even	Even	Double Angle Formula Method
Even	0	
0	Even	

The **Pythagorean Identity Method** can be outlined as follows

- Split off one power of the sinusoidal function that is raised to the **odd** power.
  - E.g., $cos^{2k+1}(x) = cos(x) \cdot cos^{2k}(x)$
- Pull out the power of k.
  - E.g., $cos(x) \cdot cos^{2k}(x) = cos(x) \cdot (cos^2(x))^k$
- Apply the Pythagorean identity to the square term.
  - $\text{E.g.}, cos(x) \cdot (cos^2(x))^k = cos(x) \cdot (1 - sin^2(x))^k$
- Use a u-substitution with the new sinusoid term from the Pythagorean identity.
  - E.g., $u = sin(x),\ du = cos(x)\, dx$
- Apply the u-substitution and expand the resulting polynomial.
  - E.g., $cos(x) \cdot (1 - sin^2(x))^k dx = (1 - u^2)^k du$

The **Double Angle Formula Method** can be outlined as follows

- Write the sinusoid(s) as a squared term raised to a power.
  - E.g., $cos^{2k}(x) = (cos^2(x))^k$
- Use the double angle formula to the squared terms(s).
  - E.g., $(cos^2(x))^k = \left(\frac{1}{2} + \frac{1}{2}cos(2x)\right)^k$
- Expand and use the double angle formula (and possible the product to sum formula), repeatedly as required until all terms are raised to a power of one.
- Combine and integrate as required.

** These integrals, and others, can also be solved using reduction formulas provided on the table of integrals below.

Table of Trigonometric Integrals

$$\int sin(x)\,dx = -cos(x) + C$$

$$\int csc(x)\,dx = -ln|csc(x) + cot(x)| + C$$

$$\int cos(x)\,dx = sin(x) + C$$

$$\int sec(x)\,dx = ln|sec(x) + tan(x)| + C$$

$$\int tan(x)\,dx = ln|sec(x)| + C$$

$$\int cot(x)\,dx = ln|sin(x)| + C$$

$$\int sin^n(x)\,dx = -\frac{1}{n} sin^{n-1}(x)\,cos(x) + \frac{n-1}{n}\int sin^{n-2}(x)\,dx$$

$$\int cos^n(x)\,dx = \frac{1}{n} cos^{n-1}(x)\,sin(x) + \frac{n-1}{n}\int cos^{n-2}(x)\,dx$$

$$\int tan^n(x)\,dx = \frac{1}{n-1} tan^{n-1}(x) - \int tan^{n-2}(x)\,dx$$

$$\int cot^n(x)\,dx = -\frac{1}{n-1} cot^{n-1}(x) - \int cot^{n-2}(x)\,dx$$

$$\int sec^n(x)\,dx = \frac{1}{n-1} sec^{n-2}(x)\,tan^n(x) + \frac{n-2}{n-1}\int sec^{n-2}(x)\,dx$$

$$\int csc^n(x)\,dx = -\frac{1}{n-1} csc^{n-2}(x)\,cot(x) + \frac{n-2}{n-1}\int csc^{n-2}(x)\,dx$$

$$\int sin^m(x)\,cos^n(x)\,dx = \frac{sin^{m+1}(x)\,cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int sin^m(x)\,cos^{n-2}(x)\,dx$$

$$\int sin(mx)\,sin(nx)\,dx = \frac{sin\big((m-n)x\big)}{2(m-n)} - \frac{sin\big((m+n)x\big)}{2(m+n)} + C, \qquad (m \neq n)$$

$$\int cos(mx)\,cos(nx)\,dx = \frac{sin\big((m-n)x\big)}{2(m-n)} + \frac{sin\big((m+n)x\big)}{2(m+n)}, \qquad (m \neq n)$$

$$\int sin(mx)\,cos(nx)\,dx = -\frac{cos\big((m-n)x\big)}{2(m-n)} - \frac{cos\big((m+n)x\big)}{2(m+n)} + C + C, \qquad (m \neq n)$$

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