Integration Techniques – Method of Partial Fractions

The final integration method we will learn is used for rational functions:

$$f(x) = \frac{N(x)}{D(x)}$$

Where $N(x)$ and $D(x)$ are both polynomials.

We call this method the method of partial fractions because before we integrate, we perform a partial fraction decomposition on the integrand. By doing so, we generally render the integral much easier to evaluate. The main task of this lesson is therefore to learn how to perform partial fraction decomposition.

Partial Fraction Decomposition

To start we can split rational functions, similar to simple fractions, into two categories:

1. Proper Rational Functions
    - The degree of the numerator, $Deg(N)$, is less than the degree of the denominator, $Deg(D)$. i.e. $Deg(N) < Deg(D)$
2. Improper Rational Functions
    - $Deg(N) \geq Deg(D)$

Partial fraction decomposition allows us to express a single rational function as a sum of several simpler rational functions. For improper rational functions we need to add the initial step of long division before the main procedure. Because of this fact we will begin by assuming proper rational functions only. We will illustrate the main procedure for 4 cases shown below. We'll also illustrate a 5th case where the rational function is improper.

1. $D(x)$ contains distinct linear terms.

$$\frac{N(x)}{(x-a_1)(x-a_2)\ldots(x-a_N)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_N}{(x-a_N)}$$

2. $D(x)$ contains repeated linear terms.

$$\frac{N(x)}{(x-a_1)^M} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_M}{(x-a_1)^M}$$

3. $D(x)$ contains irreducible quadratic terms.

$$\frac{N(x)}{(a_1x^2+b_1x+c_1)\ldots(a_Nx^2+b_Nx+c_N)} = \frac{A_1+B_1x}{(a_1x^2+b_1x+c_1)} + \cdots + \frac{A_N+B_Nx}{(a_Nx^2+b_Nx+c_N)}$$

4. $D(x)$ contains repeated irreducible quadratic terms.

$$\frac{N(x)}{(ax^2+bx+c)^M} = \frac{A_1+B_1x}{(ax^2+bx+c)} + \cdots + \frac{A_M+B_Mx}{(ax^2+bx+c)^M}$$

The right-hand side of the equation for each case from above is referred to as the partial fraction decomposition of the original rational function as expressed on the left-hand side. Finding this decomposition requires us to find the constant terms in the numerators, e.g. $A_1, A_2, \dots A_N$. The procedure is illustrated below, staring with the first case.

Case 1 - (Distinct Linear Terms):

Evaluate the following integral:

$$\int \frac{3x+2}{(x-2)(x-5)}\,dx$$

The integrand is a *proper* rational function with two distinct linear terms, which from above, can be written as follows:

$$\frac{3x+2}{(x-2)(x-5)} = \frac{A_1}{(x-2)} + \frac{A_2}{(x-5)}$$

To find A_1 and A_2, we start by multiplying the equation by the denominator on the left-hand side. The result is as follows:

$$\cancel{(x-2)(x-5)}\left(\frac{3x+2}{\cancel{(x-2)(x-5)}}\right) = \frac{A_1\cancel{(x-2)}(x-5)}{\cancel{(x-2)}} + \frac{A_2(x-2)\cancel{(x-5)}}{\cancel{(x-5)}}$$

$$3x+2 = A_1(x-5) + A_2(x-2)$$

At this point there are two methods we can use to find A_1 and A_2. The first method relies on the fact that the equation holds for all x, including $x = 2$ and $x = 5$. Plugging either of these two values into the equation eliminates the corresponding constant term giving us an equation with one unknown. A second method, called the method of undetermined coefficients, relies on the fact that the coefficients of the powers of x, including x^0, on each side must be equal to each other. We illustrate both methods below.

<table>
<tr><th>Method 1</th><th>Method 2</th></tr>
<tr><td>In this method we set $x = 2$ and $x = 5$ consecutively to find both constants.

Let $x = 2$

$$\begin{aligned}3 \cdot 2 + 2 &= A_1(2-5) + A_2(2-2)\\ 8 &= -3A_1\\ -\frac{8}{3} &= A_1\end{aligned}$$

Let $x = 5$

$$\begin{aligned}3 \cdot 5 + 2 &= A_1(5-5) + A_2(5-2)\\ 17 &= 3A_2\\ \frac{17}{3} &= A_2\end{aligned}$$</td><td>In this method we start by expanding the right-hand side and combing like terms.

$$\begin{aligned}3x + 2 &= A_1(x-5) + A_2(x-2)\\ 3x + 2 &= A_1x - 5A_1 + A_2x - 2A_2\\ 3x + 2 &= (A_1 + A_2)x + (-5A_1 - 2A_2)\end{aligned}$$

By equating the coefficients for the terms on each side we get two simultaneous equations.

$$3 = A_1 + A_2 \qquad 2 = -5A_1 - 2A_2$$

Plugging $A_1 = 3 - A_2$ into the second equation we can find A_2.

$$\begin{aligned}2 &= -5(3 - A_2) - 2A_2\\ 2 &= -15 + 3A_2\\ \frac{17}{3} &= A_2\end{aligned}$$

Substituting back, we can find A_1.

$$\begin{aligned}A_1 &= 3 - \frac{17}{3}\\ A_1 &= \frac{9}{3} - \frac{17}{3} = -\frac{8}{3}\end{aligned}$$</td></tr>
</table>

The integral can now be rewritten and solved as follows.

$$\begin{aligned}\int \frac{3x+2}{(x-2)(x-5)}\,dx &= \int \frac{-\frac{8}{3}}{(x-2)} + \frac{\frac{17}{3}}{(x-5)}\,dx\\ &= -\frac{8}{3} ln|x-2| + \frac{17}{3} ln|x-5| + C\end{aligned}$$

Case 2 - (Repeated Linear Terms):

Evaluate the following integral:

$$\int \frac{3x-9}{(x-1)(x+2)^2}\,dx$$

The integrand is a *proper* rational function with one distinct linear term and one repeated linear term, which from above, can be written as follows:

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x+2)} + \frac{A_3}{(x+2)^2}$$

As before we start by multiplying through by the numerator on the left-hand side, resulting in

$$3x - 9 = A_1(x+2)^2 + A_2(x-1)(x+2) + A_3(x-1)$$

We'll use the first method to solve for the constants.

Let $x = 1$

$$\begin{aligned} 3 \cdot 1 - 9 &= A_1(1+2)^2 \\ -6 &= A_1 9 \\ -\frac{2}{3} &= A_1 \end{aligned}$$

Let $x = -2$

$$\begin{aligned} 3(-2) - 9 &= A_3(-2-1) \\ -15 &= A_3(-3) \\ 5 &= A_3 \end{aligned}$$

Lastly, we can use the values of A_1 and A_3 along with an arbitrary x value, e.g. $x = 0$ to solve for A_2.

$$\begin{aligned} 3 \cdot 0 - 9 &= -\frac{2}{3}(0+2)^2 + A_2(0-1)(0+2) + 5(0-1) \\ -9 &= -\frac{8}{3} - 2A_2 - 5 \\ \frac{9 - 5 - \frac{8}{3}}{2} &= A_2 \\ \frac{2}{3} &= A_2 \end{aligned}$$

The integral can now be rewritten and solved as follows.

$$\begin{aligned} \int \frac{3x-9}{(x-1)(x+2)^2}\,dx &= \int \frac{-\frac{2}{3}}{(x-1)}\,dx \int \frac{\frac{2}{3}}{(x+2)}\,dx + \int \frac{5}{(x+2)^2}\,dx \\ &= -\frac{2}{3} ln|x-1| + \frac{2}{3} ln|x+2| - \frac{5}{x+2} + C \end{aligned}$$

Case 3 - (Distinct Quadratic Terms):

Evaluate the following integral:

$$\int \frac{10}{(x^2+4x+10)(x^2-2x+6)}\,dx$$

The integrand has a denominator with two distinct, (irreducible), quadratic terms. Therefore, we can rewrite as follows:

$$\frac{10}{(x^2+4x+10)(x^2-2x+6)} = \frac{A_1+B_1x}{(x^2+4x+10)} + \frac{A_2+B_2x}{(x^2-2x+6)}$$

We start again by multiplying through by the denominator from the left-hand side, resulting in

$$10 = (A_1+B_1x)(x^2-2x+6) + (A_2+B_2x)(x^2+4x+10)$$

In this case we'll use the method of undetermined coefficients. Expanding first we have

$$\begin{aligned} 10 &= (A_1x^2 - 2A_1x + 6A_1 + B_1x^3 - 2B_1x^2 + 6B_1x) \\ &\qquad + (A_2x^2 + 4A_2x + 10A_2 + B_2x^3 + 4B_2x^2 + 10B_2x) \\ 10 &= (B_1+B_2)x^3 + (A_1+A_2-2B_1+4B_2)x^2 + (-2A_1+4A_2+6B_1+10B_2)x \\ &\qquad + (6A_1+10A_2) \end{aligned}$$

Equating coefficients we obtain four equations for the four unknowns.

$$\begin{aligned} B_1+B_2 &= 0 \\ A_1+A_2-2B_1+4B_2 &= 0 \\ -2A_1+4A_2+6B_1+10B_2 &= 0 \\ 6A_1+10A_2 &= 10 \end{aligned}$$

Solving this system of equations, we get the following results:

$$A_1 = \frac{5}{7} \qquad A_2 = \frac{4}{7} \qquad B_1 = \frac{3}{14} \qquad B_2 = -\frac{3}{14}$$

The integral can now be rewritten as follows.

$$\int \frac{10}{(x^2+4x+10)(x^2-2x+6)}dx = \frac{1}{7}\int \frac{5+4x}{(x^2+4x+10)}dx + \frac{3}{14}\int \frac{1-x}{(x^2-2x+6)}dx$$

Completing the squares in the denominator of each integral we have

$$= \frac{1}{7}\int \frac{5+4x}{((x+2)^2+6)}dx + \frac{3}{14}\int \frac{1-x}{((x-1)^2+5)}dx$$

Next, we use the following u-substitutions

$$u = x + 2 \rightarrow x = u - 2 \qquad u = x - 1 \rightarrow x = u + 1$$
$$du = dx \qquad du = dx$$

$$= \frac{1}{7}\int \frac{4u-3}{(u^2+6)}du + \frac{3}{14}\int \frac{-u}{(u^2+5)}du$$
$$= \frac{4}{7}\int \frac{u}{(u^2+6)}du - \frac{3}{7}\int \frac{1}{(u^2+6)}du - \frac{3}{14}\int \frac{u}{(u^2+5)}du$$

At this point an integral table can be used to evaluate each of the three integrals, however for illustration purposes we will evaluate each integral below.

The first and the third integral are similarly evaluated as shown.

$\frac{4}{7}\int \frac{u}{(u^2+6)}du$	$\frac{3}{14}\int \frac{u}{(u^2+5)}du$
We start with the following substitution $v = u^2 + 6$ $dv = 2udu \rightarrow udu = \frac{1}{2}dv$ $\frac{4}{7}\int \frac{u}{(u^2+6)}du = \frac{4}{7}\cdot\frac{1}{2}\int \frac{1}{v}dv$ $= \frac{2}{7}ln\lvert v\rvert$ $= \frac{2}{7}ln\lvert u^2+6\rvert$ $= \frac{2}{7}ln\lvert (x+2)^2+6\rvert$ $= \frac{2}{7}ln\lvert x^2+4x+10\rvert$	We start with the following substitution $v = u^2 + 5$ $dv = 2udu \rightarrow udu = \frac{1}{2}dv$ $\frac{3}{14}\int \frac{u}{(u^2+5)}du = \frac{3}{14}\cdot\frac{1}{2}\int \frac{1}{v}dv$ $= \frac{3}{28}ln\lvert v\rvert$ $= \frac{3}{28}ln\lvert u^2+5\rvert$ $= \frac{3}{28}ln\lvert (x-1)^2+6\rvert$ $= \frac{3}{28}ln\lvert x^2-2x+6\rvert$

To evaluate the second integral, we start by multiplying the integrand by $\frac{1/(\sqrt{6})^2}{1/(\sqrt{6})^2}$

$$\frac{3}{7}\int \frac{1}{(u^2+6)}du = \frac{3}{7(\sqrt{6})^2}\int \frac{1}{\left(\frac{u}{\sqrt{6}}\right)^2+1}du$$

Then we substitute as follows:

$$v = u/\sqrt{6} \qquad\qquad dv = 1/\sqrt{6}\,du$$
$$du = \sqrt{6}dv$$

$$\frac{3}{7\left(\sqrt{6}\right)^2}\int \frac{1}{v^2+1}\cdot\sqrt{6}dv = \frac{3}{7\sqrt{6}}\int \frac{1}{v^2+1}dv$$
$$= \frac{3}{7\sqrt{6}}tan^{-1}(v)$$
$$= \frac{3}{7\sqrt{6}}tan^{-1}\left(u/\sqrt{6}\right)$$
$$= \frac{3}{7\sqrt{6}}tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)$$

The final solution is then written as follows:

$$\int \frac{10}{(x^2+4x+10)(x^2-2x+6)}dx = \frac{2}{7}ln|x^2+4x+10| - \frac{3}{7\sqrt{6}}tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right) - \frac{3}{28}ln|x^2-2x+6|$$

Case 4 - (Repeated Quadratic Terms):

Evaluate the following integral:

$$\int \frac{4-x}{x(x^2+2)^2}dx$$

The integrand has a denominator with one distinct linear term and one repeated quadratic term. Therefore, we can rewrite as follows:

$$\frac{4-x}{x(x^2+2)^2} = \frac{A_1}{x} + \frac{A_2+B_2x}{(x^2+2)} + \frac{A_3+B_3x}{(x^2+2)^2}$$

We again multiply through by the denominator from the left-hand side, expand, and use the method of undetermined coefficients for find the unknown constants.

$$4-x = A_1(x^2+2)^2 + (A_2+B_2x)(x)(x^2+2) + x(A_3+B_3x)$$
$$4-x = (A_1x^4+4A_1x^2+4A_1) + (A_2x^3+2A_2x+B_2x^4+2B_2x^2) + (A_3x+B_3x^2)$$
$$4-x = x^4(A_1+B_2) + x^3(A_2) + x^2(4A_1+2B_2+B_3) + x(2A_2+A_3) + (4A_1)$$

Equating coefficients we obtain five equations for the five unknowns.

$$A_1 + B_2 = 0$$

$$A_2 = 0$$

$$4A_1 + 2B_2 + B_3 = 0$$

$$2A_2 + A_3 = -1$$

$$4A_1 = 4$$

Solving this system of equations, we get the following results:

$$A_1 = 1 \qquad A_2 = 0 \qquad A_3 = -1 \qquad B_2 = -1 \qquad B_3 = -2$$

The integral can now be rewritten as follows.

$$\int \frac{4-x}{x(x^2+2)^2}dx = \int \frac{1}{x}dx - \int \frac{x}{(x^2+2)}dx - \int \frac{2x+1}{(x^2+2)^2}dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{(x^2+2)}dx - \int \frac{2x}{(x^2+2)^2}dx - \int \frac{1}{(x^2+2)^2}dx$$

The first integral is directly solved as

$$\int \frac{1}{x}dx = ln|x|$$

The second and third integral are evaluated with substitution as shown below.

$\int \frac{x}{(x^2+2)}dx$	$\int \frac{2x}{(x^2+2)^2}dx$
We start with the following substitution $u = x^2 + 2$ $du = 2xdx \rightarrow xdx = \frac{1}{2}du$ $\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}ln\|u\|$ $= \frac{1}{2}ln\|x^2+2\|$	We start with the following substitution $u = x^2 + 2$ $du = 2xdx$ $\int \frac{1}{u^2}du = -\frac{1}{u}$ $= -\frac{1}{x^2+2}$

Finally, for the fourth integral we can use trigonometric substitution as follows.

$$x = \sqrt{2}\,tan(\theta) \qquad dx = \sqrt{2}\,sec^2(\theta)\,d\theta$$

$$\int \frac{1}{(x^2+2)^2}\,dx = \int \frac{\sqrt{2}\,sec^2(\theta)}{(2\,tan^2(\theta)+2)^2}\,d\theta$$

$$= \int \frac{\sqrt{2}\,sec^2(\theta)}{4(tan^2(\theta)+1)^2}\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int \frac{sec^2(\theta)}{sec^4(\theta)}\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int cos^2(\theta)\,d\theta$$

$$= \frac{\sqrt{2}}{8}\left(\int (1+cos(2\theta))d\theta\right)$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}sin(2\theta)\right)$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{\cancel{1}}{\cancel{2}}(\cancel{2}sin(\theta)\,cos(\theta))\right)$$

$$= \frac{\sqrt{2}}{8}(\theta + sin(\theta)\,cos(\theta))$$

To convert this solution back to the original variable, x, we create a right triangle using the original substitution, $x = \sqrt{2}\,tan(\theta)$, as we learned in the previous section. Doing this we find

$$\theta = tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \qquad sin(\theta) = \frac{x}{\sqrt{x^2+2}} \qquad cos(\theta) = \frac{\sqrt{2}}{\sqrt{x^2+2}}$$

Plugging this in to the above we have

$$= \frac{\sqrt{2}}{8}\left(tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \left(\frac{x}{\sqrt{x^2+2}}\right)\left(\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)\right)$$

$$= \frac{\sqrt{2}}{8}\left(tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}x}{x^2+2}\right)$$

The final solution can now be written below

$$\int \frac{4-x}{x(x^2+2)^2}\,dx = ln|x| - \frac{1}{2}ln|x^2+2| + \frac{1}{x^2+2} - \frac{\sqrt{2}}{8}\left(tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}x}{x^2+2}\right) + C$$

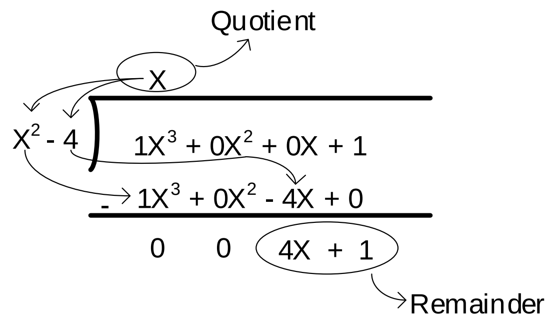
$$\int \frac{4-x}{x(x^2+2)^2}\,dx = ln|x| - \frac{1}{2}ln|x^2+2| - \frac{\sqrt{2}}{8}tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{4-x}{4(x^2+2)} + C$$

Case 5: Improper Rational Function

Evaluate the following integral:

$$\int \frac{x^3+1}{x^2-4}\,dx$$

The integrand is an improper rational function since $Deg(N) \geq Deg(D)$, therefore we need to divide the polynomials first. We use long division as illustrated below.



The integrand can now be written as follows:

$$\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} = x + \frac{4x+1}{(x-2)(x+2)}$$

The first term is easily integrated. The second term is now a proper rational function with two distinct terms, for which we can perform a partial fraction decomposition.

$$\frac{4x+1}{(x-2)(x+2)} = \frac{A_1}{(x-2)} + \frac{A_2}{(x+2)}$$

We find the unknown constants as we have done in previous examples.

$$4x+1 = A_1(x+2) + A_2(x-2)$$

Using the techniques from the previous problems we find the constants to be as shown.

$$A_1 = 9/4 \qquad\qquad A_2 = 7/4$$

Let's rewrite the integral now.

$$\int \frac{x^3+1}{x^2-4}\,dx = \int x + \frac{{}^{9}/_{4}}{(x-2)} + \frac{{}^{7}/_{4}}{(x+2)}\,dx$$

$$= \int x dx + \frac{9}{4}\int \frac{1}{(x-2)}\,dx + \frac{7}{4}\int \frac{1}{(x+2)}\,dx$$

$$= \frac{1}{2}x^2 + \frac{9}{4}ln|x-2| + \frac{7}{4}ln|x+2| + C$$

Before ending this section and summarizing the techniques we have learned let's do two more examples.

Example 1:

Evaluate the following integral:

$$\int \frac{9}{(x+1)(x^2-2x+6)}dx$$

The integrand has a denominator with one distinct linear term and one distinct, (irreducible), quadratic term. Therefore, we can rewrite as follows:

$$\frac{9}{(x+1)(x^2-2x+6)} = \frac{A_1}{(x+1)} + \frac{A_2+B_2x}{(x^2-2x+6)}$$

Multiplying through by the denominator from the left-hand side results in the following.

$$9 = A_1(x^2-2x+6) + (A_2+B_2x)(x+1)$$

In this case we'll use the method of undetermined coefficients.

$$9 = (A_1x^2 - 2A_1x + 6A_1) + (A_2x + A_2 + B_2x^2 + B_2x)$$
$$9 = (A_1+B_2)x^2 + (A_2+B_2-2A_1)x + (6A_1+A_2)$$

Equating coefficients we obtain three equations for the three unknowns.

$$A_1 + B_2 = 0 \qquad A_2 + B_2 - 2A_1 = 0 \qquad 6A_1 + A_2 = 9$$

Solving this system of equation results in the following.

$$A_1 = 1 \qquad A_2 = 3 \qquad B_2 = -1$$

The integral can now be rewritten as follows.

$$\int \frac{9}{(x+1)(x^2-2x+6)}dx = \int \frac{1}{(x+1)}dx + \int \frac{3-x}{(x^2-2x+6)}dx$$

The first integral is straightforward, but the second integral will require some work. Let's begin there, where we will start by completing the square for the denominator.

$$(x^2-2x+6) = (x-1)^2 - 1 + 6 = (x-1)^2 + 5$$

Therefore, we now have

$$\int \frac{3-x}{(x-1)^2+5}dx$$

To solve we start with the following u-substitution.

$$u = x - 1 \rightarrow x = u + 1 \qquad\qquad du = dx$$

$$\begin{aligned}\int \frac{3-x}{(x-1)^2+5}dx &= \int \frac{3-(u+1)}{u^2+5}du \\ &= \int \frac{2-u}{u^2+5}du \\ &= 2\int \frac{1}{u^2+5}du + \int \frac{u}{u^2+5}du\end{aligned}$$

Although we can use an integral table for these, we illustrate their evaluation below.

$2\int \frac{1}{u^2+5}du$	$\int \frac{u}{u^2+5}du$
Start by multiplying the integrand by $\frac{1/(\sqrt{5})^2}{1/(\sqrt{5})^2}$. $$2\int \frac{1}{u^2+5}du = \frac{2}{(\sqrt{5})^2}\int \frac{1}{\left(\frac{u}{\sqrt{5}}\right)^2+1}du$$ Then we substitute as follows: $$v = u/\sqrt{5} \qquad dv = 1/\sqrt{5}\,du \qquad du = \sqrt{5}dv$$ $$\begin{aligned}\frac{2}{5}\int \frac{1}{v^2+1}\cdot\sqrt{5}dv &= \frac{2}{\sqrt{5}}\int \frac{1}{v^2+1}dv \\ &= \frac{2}{\sqrt{5}}tan^{-1}(v)\end{aligned}$$ Finally, resubstituting for v and u we have: $$= \frac{2}{\sqrt{5}}tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right)$$	We start with the following substitution: $$v = u^2 \qquad dv = 2udu \qquad udu = \frac{1}{2}dv$$ $$\begin{aligned}\int \frac{u}{u^2+5}du &= \frac{1}{2}\int \frac{1}{v+5}dv \\ &= \frac{1}{2}ln(v+5)\end{aligned}$$ Finally, resubstituting for v and u we have: $$\begin{aligned}&= \frac{1}{2}ln((x-1)^2+5) \\ &= \frac{1}{2}ln(x^2-2x+6)\end{aligned}$$

Returning to the original integral we have

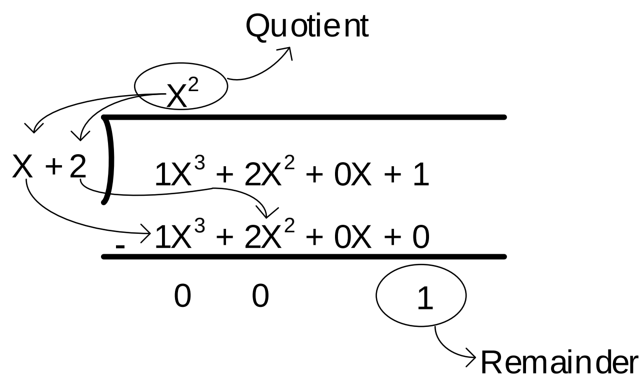
$$\begin{aligned}\int \frac{9}{(x+1)(x^2-2x+6)}dx &= \int \frac{1}{(x+1)}dx + \int \frac{3-x}{(x^2-2x+6)}dx \\ &= ln(x+1) + \frac{2}{\sqrt{5}}tan^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + \frac{1}{2}ln(x^2-2x+6) + C\end{aligned}$$

Example 2:

Evaluate the following integral:

$$\int \frac{(x^3 + 2x^2 + 1)}{(x + 2)} dx$$

The integrand is an improper rational function since $Deg(N) \geq Deg(D)$, therefore we need to divide the polynomials first. We use long division as illustrated below.



The integral can now be rewritten and easily evaluated as follows:

$$\int \frac{(x^3 + 2x^2 + 1)}{(x + 2)} dx = \int \left(x^2 + \frac{1}{x + 2}\right) dx$$

$$= \frac{1}{3}x^3 + ln|x + 2| + C$$

Final Summary for Integration Techniques – Method of Partial Fractions

The Method of Partial Fractions

The method of partial fractions may be applied when the integrand is a rational function.

$$f(x) = \frac{N(x)}{D(x)}$$

We can identify the following two categories:

1. Proper Rational Functions
    - The degree of the numerator, $Deg(N)$, is less than the degree of the denominator, $Deg(D)$. i.e. $Deg(N) < Deg(D)$
    - In this case we perform a partial fraction decomposition on the integrand before we attempt to evaluate the integral.
2. Improper Rational Functions
    - $Deg(N) \geq Deg(D)$
    - In this case we need to perform long division first, which will result in two terms; a non-fraction term and a *proper* rational function term.
        - We then perform partial fraction decomposition on the second term if required before attempting to evaluate.

Partial Fraction Decomposition

The partial fraction decomposition for 4 different cases is illustrated below.

1. $D(x)$ contains distinct linear terms.

$$\frac{N(x)}{(x-a_1)(x-a_2)\ldots(x-a_N)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_N}{(x-a_N)}$$

2. $D(x)$ contains repeated linear terms.

$$\frac{N(x)}{(x-a_1)^M} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_M}{(x-a_1)^M}$$

3. $D(x)$ contains irreducible quadratic terms.

$$\frac{N(x)}{(a_1x^2+b_1x+c_1)\ldots(a_Nx^2+b_Nx+c_N)} = \frac{A_1+B_1x}{(a_1x^2+b_1x+c_1)} + \cdots + \frac{A_N+B_Nx}{(a_Nx^2+b_Nx+c_N)}$$

4. $D(x)$ contains repeated irreducible quadratic terms.

$$\frac{N(x)}{(ax^2+bx+c)^M} = \frac{A_1+B_1x}{(ax^2+bx+c)} + \cdots + \frac{A_M+B_Mx}{(ax^2+bx+c)^M}$$

To find the unknown constant we multiply by the denominator on the left-hand side and either:

1. Consecutively set x value such that one or more of the terms is removed.
2. Expand equation and set the coefficients of the powers of x equal to get a system of equations, i.e. method of undetermined coefficients.

By: ferrantetutoring