# Integration Techniques – Improper Integrals

As we have learned, definite integrals represent the signed area of a function over a given interval. When the region over which the integral is taken is bounded, we refer to the integral as a *proper integral*. On the other hand, when the region is unbounded, we say it is an *improper integral*. We can identify two types of improper integrals:

1. One or more of the endpoints of the integration interval is infinite.
2. Integrand tends to infinity (i.e. contains vertical asymptotes) within the integration interval.

In this lesson we will learn how to deal with improper integrals , especially in determining when the unbounded region may indeed have a finite area. We'll start our lesson with improper integrals of the first type, i.e. infinite integration intervals.

**Improper Integrals Type 1: Infinite Integration Intervals:**

We can identify three separate ways for the integration interval to be infinite.

$$\int_a^\infty f(x)dx \qquad \int_{-\infty}^b f(x)dx \qquad \int_{-\infty}^\infty f(x)dx$$

In these cases, we define the area of the unbounded region using the concept of a limit as follows:

| Type 1 Improper Integral Definitions (Infinite Integration Intervals) |
|---|
| $$\int_a^\infty f(x)dx \stackrel{\text{def}}{=} \lim_{R\to\infty}\left(\int_a^R f(x)dx\right)$$ |
| $$\int_{-\infty}^b f(x)dx \stackrel{\text{def}}{=} \lim_{R\to-\infty}\left(\int_R^b f(x)dx\right)$$ |
| $$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx \stackrel{\text{def}}{=} \lim_{R_1\to-\infty}\left(\int_{R_1}^0 f(x)dx\right) + \lim_{R_2\to\infty}\left(\int_0^{R_2} f(x)dx\right)$$ |
| In all cases, we say that the improper integral *converges* if the limit exists (and is finite) and *diverges* if the limit does not exist. |

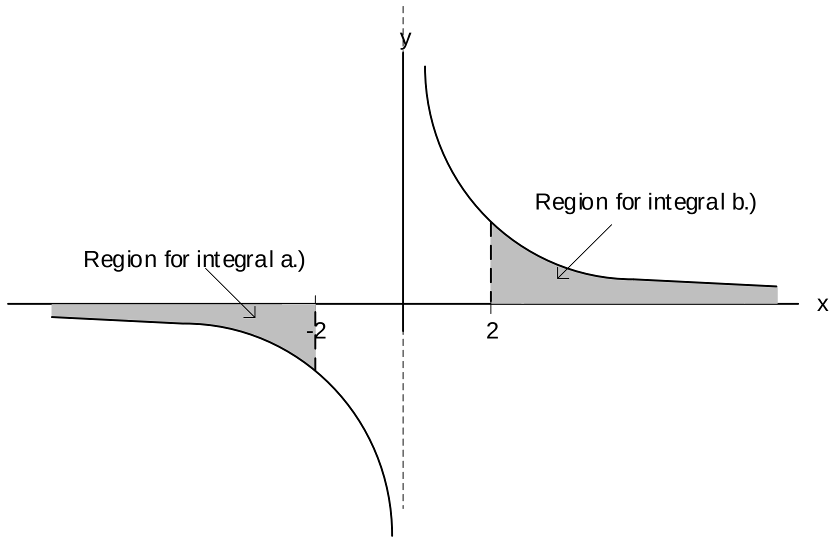
To get a better understanding of the definitions above let's do some examples.

**Example 1**

Evaluate the following improper integrals

a.) $\int_{-\infty}^{-2} \frac{1}{x^3} dx$ b.) $\int_{2}^{\infty} \frac{1}{x^3} dx$

Solution: Let's graph the function first to help visualize the different regions.




We attempt to evaluate the integrals based on the definitions above.

| | |
|---|---|
| $\int_{-\infty}^{-2} \frac{1}{x^3} dx = \lim_{R\to-\infty} \left( \int_{R}^{-2} \frac{1}{x^3} dx \right)$ | $\int_{2}^{\infty} \frac{1}{x^3} dx = \lim_{R\to\infty} \left( \int_{2}^{R} \frac{1}{x^3} dx \right)$ |
| $= \lim_{R\to-\infty} \left( -\frac{1}{2x^2} \Big\|_{R}^{-2} \right)$ | $= \lim_{R\to\infty} \left( -\frac{1}{2x^2} \Big\|_{2}^{R} \right)$ |
| $= \lim_{R\to-\infty} \left( \frac{1}{2R^2} - \frac{1}{8} \right)$ | $= \lim_{R\to\infty} \left( \frac{1}{8} - \frac{1}{2R^2} \right)$ |
| $= \lim_{R\to-\infty} \left( \frac{1}{2R^2} \right) - \frac{1}{8}$ | $= \frac{1}{8} - \lim_{R\to\infty} \left( \frac{1}{2R^2} \right)$ |
| $= 0 - \frac{1}{8}$ | $= \frac{1}{8} - 0$ |
| $= -\frac{1}{8}$ | $= \frac{1}{8}$ |

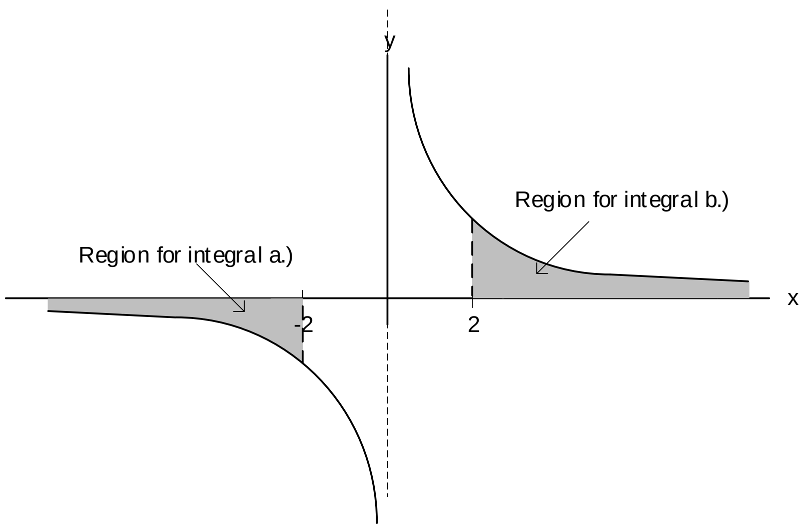
Therefore, even though the integration interval is infinite, the unbounded regions have finite area.

**Example 2**

Evaluate the following improper integrals

a.) $\int_{-\infty}^{-2} \frac{1}{x} dx$ b.) $\int_{2}^{\infty} \frac{1}{x} dx$

Solution: We can again graph the function to help visualize. We notice that the graph is very similar to example 1.




Let's evaluate as we did in the previous example.

| | |
|---|---|
| $\int_{-\infty}^{-2} \frac{1}{x} dx = \lim_{R\to-\infty} \left( \int_{R}^{-2} \frac{1}{x} dx \right)$ | $\int_{2}^{\infty} \frac{1}{x} dx = \lim_{R\to\infty} \left( \int_{2}^{R} \frac{1}{x} dx \right)$ |
| $= \lim_{R\to-\infty} (ln\|x\|\|_{R}^{-2})$ | $= \lim_{R\to\infty} (ln\|x\|\|_{2}^{R})$ |
| $= \lim_{R\to-\infty} (ln\|2\| - ln\|R\|)$ | $= \lim_{R\to\infty} (ln\|R\| - ln\|2\|)$ |
| $= ln(2) - \lim_{R\to-\infty} (ln(R))$ | $= \lim_{R\to\infty} (ln\|R\|) - ln(2)$ |
| $= ln(2) - \infty$ | $= \infty - ln(2)$ |
| $= -\infty$ | $= \infty$ |

In this case the integrals diverge, i.e. the regions have infinite area. After these two examples you may wonder why the integrals in example 1 have finite area, whereas the integrals in this example have infinite area? The answer is that the convergence of an improper integral depends on how rapidly $f(x)$ tends to zero as $x \to \pm\infty$. For the examples above we can conclude that $1/x^3$ tends to zero fast enough for convergence, while $1/x$ does not. The next example will make this observation more precise.

**Example 3**

Evaluate the following improper integral, referred to as a *p-integral*.

$$\int_a^\infty \frac{1}{x^p}\,dx$$

Solution: We will assume $p \neq 1$, since we already know from example 2 that in this case the integral diverges.

$$\begin{aligned}\int_a^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty}\left(\int_a^R \frac{1}{x^p}\,dx\right)\\ &= \lim_{R\to\infty}\left(\left.\frac{x^{1-p}}{1-p}\right|_a^R\right)\\ &= \lim_{R\to\infty}\left(\frac{R^{1-p}}{1-p}-\frac{a^{1-p}}{1-p}\right)\\ &= \frac{1}{1-p}\left(\lim_{R\to\infty}(R^{1-p})-a^{1-p}\right)\end{aligned}$$

Let's look at the result for the following two cases:

1. $p > 1$
   - $1 - p < 0$ and $R^{1-p}$ tends to $0$ as $R \to \infty$, therefore we have.

$$\int_a^\infty \frac{1}{x^p}\,dx = -\frac{a^{1-p}}{1-p} = \frac{a^{1-p}}{p-1}, for\ p > 1$$

2. $p < 1$
   - $1 - p > 0$ and $R^{1-p}$ tends to $\infty$ as $R \to \infty$, therefore we have.

$$\int_a^\infty \frac{1}{x^p}\,dx = \infty, for\ p < 1$$

In general, we may state the following.

| **The *p-integral* over $[a,\infty)$, for $a > 0$** |
|---|
| $\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases}\frac{a^{1-p}}{1-p}, & p > 1\\ \infty, & p \leq 1\end{cases}$ |

**Example 4**

Determine if the following integral converges and, if so, compute its value.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Solution:

From the definitions above we split the integral into 2 integrals as follows:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to-\infty} \int_{R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to-\infty} \left(tan^{-1}(x)|_R^0\right) + \lim_{R\to\infty} \left(tan^{-1}(x)|_0^R\right)$$

$$= \lim_{R\to-\infty} \left(tan^{-1}(0) - tan^{-1}(R)\right) + \lim_{R\to\infty} \left(tan^{-1}(R) - tan^{-1}(0)\right)$$

$$= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right)$$

$$= \pi$$

Therefore, the integral converges, and the area of the region is $\pi$.

**Example 5**

Determine if the following integral converges and, if so, compute its value.

$$\int_{0}^{\infty} xe^{-x}dx$$

Solution:

We have seen this integrand before and should recognize that evaluation requires us to use the technique of integration by parts as shown.

$$u = x \qquad\qquad dv = e^{-x}dx$$
$$du = dx \qquad\qquad v = -e^{-x}$$

$$\int_0^{\infty} xe^{-x}dx = \lim_{R\to\infty}\int_0^R xe^{-x}dx$$

$$= \lim_{R\to\infty}\left(-xe^{-x}|_0^R + \int_0^R e^{-x}dx\right)$$

$$= \lim_{R\to\infty}(-xe^{-x} - e^{-x}|_0^R)$$

$$= \lim_{R\to\infty}(-e^{-x}(x+1)|_0^R)$$

$$= \lim_{R\to\infty}\left(-e^{-R}(R+1) + e^{-0}(0+1)\right)$$

$$= 1 - \lim_{R\to\infty}\left(\frac{R+1}{e^R}\right)$$

To evaluate this limit, we apply L' Hopital's Rule.

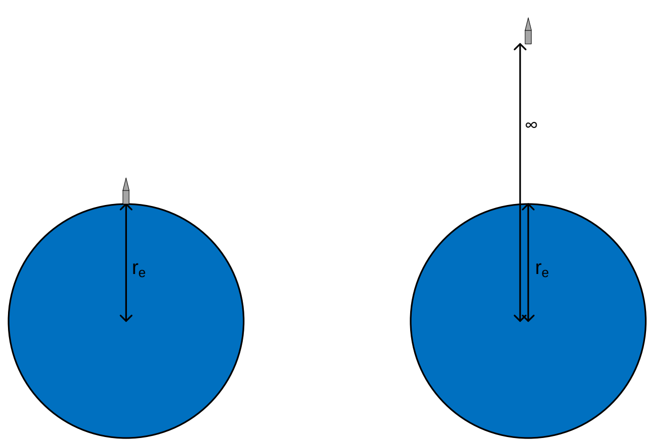
$$= 1 - \lim_{R\to\infty}\left(\frac{\frac{d}{dR}(R+1)}{\frac{d}{dR}(e^R)}\right)$$

$$= 1 - \lim_{R\to\infty}\left(\frac{1}{e^R}\right) = 1 - 0 = 1$$

Therefore, we can see that the integral converges, and the area of the region is 1.

**Example 6: Application**

If we launch an object away from the earth with an initial velocity it will travel a certain distance before it reverses direction and begins to fall back to the earth because of the force of gravity. The larger the velocity the further the object will travel before it reverses direction. *Escape velocity* is a term used to describe the initial velocity given to an object so that it "escapes" earths gravitational force and never reverses direction. This velocity can be found by first determining the work, (in the physics sense), required to move the object from the surface of the earth to an infinite distance from the earth.



In physics work is defined as the vector dot product of a force and the distance over which the force acts on an object. Note, we can drop the vector aspect if we assume the force and displacement of the object are along a single dimension. Therefore, we can write $W = F\Delta d$. However, if the force varies as a function of the distance, as it does with the gravitational force, we need to use an integral as shown below.

$$W = \int_{r_e}^{\infty} F_G\,(r)dr$$

Where the force of gravity, $F_G(r)$, is given as:

$$F_G(r) = -\frac{GmM_e}{r^2}$$

Where, G is the universal gravitational constant with a value of $6.67E^{-11}$, $m$ is the mass of the object, $M_e$ is the mass of the earth given as $5.98E^{24}\ kg$, and the radius of the earth, $r_e$ is $6.37E^{6}\ m$.

To find the work needed to move an object from the surface of the earth to infinity we need to compute the following integral.

$$W = \int_{r_e}^{\infty} -\frac{GmM_e}{r^2}dr = -GmM_e\int_{r_e}^{\infty}\frac{1}{r^2}dr$$

This is a *p-integral* with $p = 2$, and therefore converges with a solution given in example 3.

$$-GmM_e\int_{r_e}^{\infty}\frac{1}{r^2}dr = -GmM_e\left(\frac{{r_e}^{1-2}}{1-2}\right)$$

$$= GmM_e\left(\frac{{r_e}^{-1}}{-1}\right) = \frac{GmM_e}{r_e}$$

This is the amount of work the mass needs to "do" in order to escape earths gravitational force. In other words, the object needs to be given, at a minimum, this mount of kinetic energy, where $K = \frac{1}{2}mv^2$. Therefore, we can find the minimum velocity as follows

$$K \geq \frac{GmM_e}{r_e}$$

$$\frac{1}{2}\cancel{m}v^2 \geq \frac{G\cancel{m}M_e}{r_e}$$

$$v \geq \sqrt{2GM_e/r_e}$$

Finally, we define the escape velocity, $v_e$, as

$$v_e = \sqrt{2GM_e/r_e}$$

Substituting the values given above and converting to $mph$ we find an escape velocity of approximately $25{,}000\ mph$!

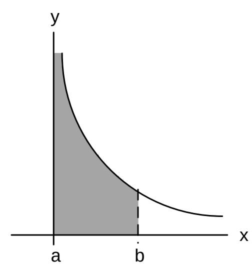
**Improper Integrals Type 2: Unbounded Functions:**

Improper integrals of the type 1 have integration interval(s) that are infinite, which created the possibility that the integral diverges. However, even if the integration interval is finite a definite integral may still diverge. This is possible when the function itself tends to infinity within the integration interval. We refer to these as type 2 improper integrals. As an example, $\int_0^4 \frac{1}{\sqrt{x}} dx$ is improper because the function, $f(x) = \frac{1}{\sqrt{x}}$, tends to infinity as $x \to 0^+$. Improper integrals of this type are defined as a one-sided limit. The formal definition is given below.

**Type 2 Improper Integral Definitions (Unbounded Integrands)**

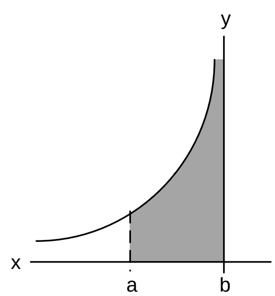
If $f$ is continuous on $(a,b]$ and $\lim\limits_{x \to a^+} \left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R \to a^+} \left( \int_R^b f(x)dx \right)$$




If $f$ is continuous on $[a,b)$ and $\lim\limits_{x \to b^-} \left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R \to b^-} \left( \int_a^R f(x)dx \right)$$




In both cases, we say that the improper integral converges if the limit exists (and is finite) and that it diverges if the limit does not exist.

A third version exists if the unbounded point lies within the interval of integration. For these cases we state the following: If there is a single point $c$ in the interval $[a, b]$ such that $\lim\limits_{x \to c^-} \left(f(x)\right) = \pm\infty$ or $\lim\limits_{x \to c^+} \left(f(x)\right) = \pm\infty$ and if both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge then we may define the following

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let's now do a few examples of type 2 improper integrals.

**Example 7**

Evaluate the following integrals:

$$\int_0^8 \frac{1}{x^{1/3}}\,dx \qquad\qquad \int_0^8 \frac{1}{x^2}\,dx$$

Both integrals are improper integrals because they are unbounded at $x = 0$. We evaluate both integrals using the definitions from above.

$$\begin{aligned}\int_0^8 \frac{1}{x^{1/3}}\,dx &= \lim_{R\to 0^+}\left(\int_R^8 x^{-1/3}dx\right)\\ &= \lim_{R\to 0^+}\left(\frac{3}{2}x^{2/3}\Big|_R^8\right)\\ &= \frac{3}{2}\lim_{R\to 0^+}\left(8^{2/3} - R^{2/3}\right)\\ &= \frac{3}{2}\left(4 - \lim_{R\to 0^+}\left(R^{2/3}\right)\right)\\ &= \frac{3}{2}(4-0) = 6\end{aligned}$$

$$\begin{aligned}\int_0^8 \frac{1}{x^2}\,dx &= \lim_{R\to 0^+}\left(\int_R^8 x^{-2}dx\right)\\ &= \lim_{R\to 0^+}\left(-\frac{1}{x}\Big|_R^8\right)\\ &= \lim_{R\to 0^+}\left(-\frac{1}{8}+\frac{1}{R}\right)\\ &= \left(-\frac{1}{8} + \lim_{R\to 0^+}\left(\frac{1}{R}\right)\right)\\ &= \left(-\frac{1}{8}+\infty\right) = \infty\end{aligned}$$

You may notice that these integrals have the same form as the p-integral we defined above, expect, the limits of integration are different. Also notice that for these cases the divergence/convergence behavior as a function of $p$ seems to be opposite to what we defined earlier, e.g. the integral diverges for $p > 1$. This behavior comes from the fact that the limits of integration are different. With this we can define another rule for p-integrals with limits according to the problem above.

| **The *p-integral* over $[0, a]$, for $a > 0$** |
|---|
| $\int_0^a \frac{1}{x^p}\,dx = \begin{cases}\frac{a^{1-p}}{1-p}, & p < 1\\ \infty, & p \geq 1\end{cases}$ |

**Example 8**

Evaluate the following integrals:

$$\int_{-1}^{2} \frac{1}{x^2}\,dx \qquad\qquad \int_{-1}^{2} \frac{1}{x^{1/3}}\,dx$$

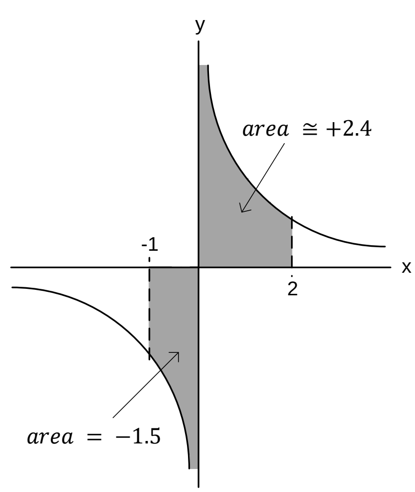
For both of these integrals the point of discontinuity falls between the integration intervals. Based on the above summary these integrals can be split into two as long as the each of the integrals converge. We also know from the previous example that the first integral diverges, and therefore this integral will diverge. One the other hand, the second integral can be split since each of the individual integrals will converge.

$$\int_{-1}^{2} \frac{1}{x^{1/3}}\,dx = \int_{-1}^{0} \frac{1}{x^{1/3}}\,dx + \int_{0}^{2} \frac{1}{x^{1/3}}\,dx$$

We know that both of these integrals converge since $p < 1$, so let's evaluate.

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{x^{1/3}}\,dx + \int_{0}^{2} \frac{1}{x^{1/3}}\,dx &= \lim_{R\to 0^+}\left(\int_{-1}^{R} \frac{1}{x^{1/3}}\,dx\right) + \lim_{R\to 0^-}\left(\int_{R}^{2} \frac{1}{x^{1/3}}\,dx\right) \\
&= \lim_{R\to 0^+}\left(\frac{3}{2}x^{2/3}\Big|_{-1}^{R}\right) + \lim_{R\to 0^-}\left(\frac{3}{2}x^{2/3}\Big|_{R}^{2}\right) \\
&= \left(\frac{3}{2}\lim_{R\to 0^+}\left(R^{2/3} - (-1)^{2/3}\right)\right) + \left(\frac{3}{2}\lim_{R\to 0^-}\left(2^{2/3} - R^{2/3}\right)\right) \\
&= \left(\frac{3}{2}\lim_{R\to 0^+}\left(R^{2/3}\right) - \frac{3}{2}\right) + \left(\frac{3\sqrt[3]{4}}{2} - \lim_{R\to 0^-}\left(R^{2/3}\right)\right) \\
&= \left(0 - \frac{3}{2}\right) + \left(\frac{3\sqrt[3]{4}}{2} - 0\right) \\
&\cong (-1.5 + 2.4) \cong 0.88
\end{aligned}$$

The figure below shows the graph for illustration.



**Example 9**

Evaluate the following integral:

$$\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx$$

This integral is also improper with a discontinuity at $x = 1$. Therefore, we have

$$\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$$

Let's look at each integral individually.

$$\begin{aligned}
\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx &= \lim_{R\to 1^-}\left(\int_0^R \frac{1}{(x-1)^{2/3}}\,dx\right) \\
&= \lim_{R\to 1^-}\left(\int_{0-1}^{R-1} u^{-2/3}\,du\right) \\
&= \lim_{R\to 1^-}\left(3u^{1/3}\Big|_{-1}^{R-1}\right) \\
&= 3\lim_{R\to 1^-}\left((R-1)^{1/3} - (-1)^{1/3}\right) \\
&= 3(0+1) = 3
\end{aligned}$$

$$\begin{aligned}
\int_1^2 \frac{1}{(x-1)^{2/3}}\,dx &= \lim_{R\to 1^+}\left(\int_R^2 \frac{1}{(x-1)^{2/3}}\,dx\right) \\
&= \lim_{R\to 1^+}\left(\int_{R-1}^{2-1} u^{-2/3}\,du\right) \\
&= \lim_{R\to 1^+}\left(3u^{1/3}\Big|_{R-1}^{1}\right) \\
&= 3\lim_{R\to 1^+}\left((1)^{1/3} - (R-1)^{1/3}\right) \\
&= 3(1-0) = 3
\end{aligned}$$

Therefore, we obtain the following.

$$\begin{aligned}
\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx &= \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx \\
&= 3 + 3 = 6
\end{aligned}$$

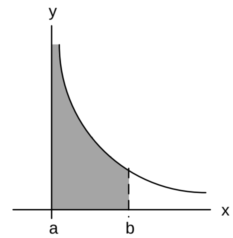
## ***Final Summary for Integration – Improper Integrals***

| **Improper Integrals** |
|---|
| When the region over which the integral is taken is unbounded, we refer to the integral as an *improper integral*. We can identify two types of improper integrals:<br>1. One or more of the endpoints of the integration interval is infinite.<br>2. Integrand tends to infinity within the integration interval. |
| **Type 1 Improper Integral Definitions (Infinite Integration Intervals)** |
| $$\int_a^\infty f(x)dx \stackrel{\text{def}}{=} \lim_{R\to\infty}\left(\int_a^R f(x)dx\right)$$ $$\int_{-\infty}^b f(x)dx \stackrel{\text{def}}{=} \lim_{R\to-\infty}\left(\int_R^b f(x)dx\right)$$ $$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx \stackrel{\text{def}}{=} \lim_{R_1\to-\infty}\left(\int_{R_1}^0 f(x)dx\right) + \lim_{R_2\to\infty}\left(\int_0^{R_2} f(x)dx\right)$$ In all cases, we say that the improper integral *converges* if the limit exists (and is finite) and *diverges* if the limit does not exist. |

**Type 2 Improper Integral Definitions (Unbounded Integrands)**

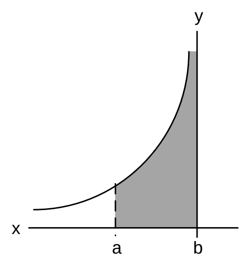
If $f$ is continuous on $(a,b]$ and $\lim_{x \to a^+} \left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R \to a^+} \left( \int_R^b f(x)dx \right)$$




If $f$ is continuous on $[a,b)$ and $\lim_{x \to b^-} \left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R \to b^-} \left( \int_a^R f(x)dx \right)$$




In both cases, we say that the improper integral converges if the limit exists (and is finite) and that it diverges if the limit does not exist.

A third version exists if the unbounded point lies within the interval of integration. For these cases we state the following: If there is a single point $c$ in the interval $[a, b]$ such that $\lim_{x \to c^-} \left(f(x)\right) = \pm\infty$ or $\lim_{x \to c^+} \left(f(x)\right) = \pm\infty$ and if both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge then we may define the following

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**The *p-integral* over $[a,\infty)$, for $a > 0$**

$$\int_a^\infty \frac{1}{x^p} dx = \begin{cases} \dfrac{a^{1-p}}{1-p}, & p > 1 \\ \infty, & p \leq 1 \end{cases}$$

**The *p-integral* over $[0, a]$, for $a > 0$**

$$\int_0^a \frac{1}{x^p} dx = \begin{cases} \dfrac{a^{1-p}}{1-p}, & p < 1 \\ \infty, & p \geq 1 \end{cases}$$

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