Integration Techniques – Integration by Parts

The differentiation rules we developed are sufficient to differentiate just about any continuous function. On the other hand, we learned that there are no such hard and fast rules for integration, and that for many functions the integral is not even expressible in elementary terms. There are however certain integration "techniques" that may be employed to help solve certain classes of functions. As a matter of fact, we learned the first technique already in calculus 1; *The Substitution Method*. In the next few sections we will learn additional techniques. In this section we develop a technique referred to as *Integration by Parts, [IBP]*. Similar to substitution, it allows us, in certain cases, to convert an integral that we cannot immediately evaluate into one that we can. However, also similar to substitution, there are no definitive rules for when or how to apply this technique. We will develop some general guidelines but practicing problems will be the best way to learn this technique.

Integration by Parts

The Integration by Parts formula is an integral expression that is derived from the product rule of differentiation. Consider two function, $u(x)$ and $v(x)$. The product rule is stated as

$$\frac{d}{dx}\big(u(x)v(x)\big) = v(x)\frac{d}{dx}\big(u(x)\big) + u(x)\frac{d}{dx}\big(v(x)\big)$$

Next, we integrate both sides of this equation, apply the fundamental theorem of calculus to the left-hand side, and rearrange the terms.

$$\int \frac{d}{dx}\big(u(x)v(x)\big)dx = \int v(x)\frac{d}{dx}\big(u(x)\big)dx + \int u(x)\frac{d}{dx}\big(v(x)\big)dx \,|$$

$$u(x)v(x) = \int v(x)\frac{d}{dx}\big(u(x)\big)dx + \int u(x)\frac{d}{dx}\big(v(x)\big)dx$$

$$\int u(x)\frac{d}{dx}\big(v(x)\big)dx = u(x)v(x) - \int v(x)\frac{d}{dx}\big(u(x)\big)dx$$

To make things a bit clearer we can also remove the explicit x dependency notation of the functions, i.e., $u(x) \to u$, and $v(x) \to v$.

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

The last step is to cancel the dx terms, i.e., $\frac{dv}{dx}dx = dv$ and $\frac{du}{dx}dx = du$. Finally, we have the *Integration by Parts Formula* in a form that is most common and, as we'll see when we look at some examples, lends itself to much simpler execution.

Integration by Parts Formula
$\int udv = uv - \int vdu$

Example 1: Evaluate $\int xe^x dx$

Solution: Since the integrand is a product, we try "fitting" it to the IBP formula as follows:

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = uv - \int vdu$$

In order to finish the formula and determine if we can then evaluate the new integral, we need to determine v and du. Starting with u we have

Since $\quad u(x) = \boldsymbol{u} = \boldsymbol{x} \quad$ then $\quad \frac{du}{dx} = 1 \quad \rightarrow \quad \boldsymbol{du} = \boldsymbol{dx}$

Next, we begin with dv

Since $\quad dv = e^x dx \quad$ then $\quad \frac{dv}{dx} = e^x \quad \rightarrow \quad \int \frac{dv}{\cancel{dx}} \cancel{dx} = \int e^x dx$

$$\boldsymbol{v} = \boldsymbol{e^x}$$

Using these results, we complete the IBP formula as shown.

$$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \underbrace{u}_{x} \underbrace{v}_{e^x} - \int \underbrace{v}_{e^x} \underbrace{du}_{dx}$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

The great advantage of this method is that we replaced a difficult integral, $\int xe^x dx$, with an integral that we could easily evaluate, $\int e^x dx$. The key to this technique, and also the most difficult part, is to properly choose the u and dv. The only way this technique helps is if the integral on the right side, $\int vdu$, turns out to be easier to evaluate than the original integral, $\int udv$. With that we can give two general guidelines.

1. Choose dv so that $v = \int dv$ can be evaluated.
2. Choose u so that $\frac{du}{dx}$ is "simpler" that u itself.

An easier to remember "rule" resulting from the above is called "LIATE". The rule is stated as:

Choose u to be the function that comes first in the list:

L:	Logarithmic Function
I:	Inverse Trigonometric Function
A:	Algebraic Function
T:	Trigonometric Function
E:	Exponential Function

Note: The last two, (T and E), can actually be in either order.

To illustrate a "bad" choice of u versus a good choice let's return to example 1, $\int xe^{x}dx$. Let's see what would happen if we instead chose $u = e^{x}$. As a guide we can make the table below.

$u = e^{x}$	$dv = xdx$
$\frac{du}{dx} = e^{x}$ $du = e^{x}dx$	$v = \int xdx$ $v = \frac{1}{2}x^{2}$

$$\int \underbrace{e^{x}}_{u}\underbrace{xdx}_{dv} = \underbrace{u}_{e^{x}}\underbrace{v}_{\frac{1}{2}x^{2}} - \int \underbrace{v}_{\frac{1}{2}x^{2}}\underbrace{du}_{e^{x}dx}$$

$$\int e^{x}xdx = \frac{e^{x}x^{2}}{2} - \frac{1}{2}\int \boldsymbol{e^{x}x^{2}dx}$$

As you can see the new integral is more complex than the original, which indicates this is not a good choice for u. Let's also take a look at the LIATE rule. Our integrand contains two functions, an algebraic, x, and an exponential, e^{x}. The rule says that we should choose u to be the algebraic function, which was the choice we originally used and resulted in an easy evaluation of the new integral.

Example 2 (Repeated Use): Evaluate $\int x^{2}\, sin(x)\, dx$

Solution: According to LIATE we should choose $u = x^{2}$. We construct a table below.

$u = x^{2}$	$dv = sin(x)\, dx$
$\frac{du}{dx} = 2x$ $du = 2xdx$	$v = -cos(x)$

$$\int x^{2}\, sin(x)\, dx = x^{2}(-cos(x)) - \int (-cos(x))2xdx$$

$$= -x^{2}\, cos(x) + 2\int x\, cos(x)\, dx$$

Although we still cannot evaluate the new integral, we notice that the integrand has gotten "simpler", so we try applying IBP again to this new integral. The new choice for u and dv are shown below.

$u = x$	$dv = cos(x)\, dx$
$du = 1\, dx$	$v = sin(x)$

Continuing with the original integral from above we have

$$\begin{aligned}\int x^2\, sin(x)\, dx &= -x^2\, cos(x) + 2\left(\int x\, cos(x)\, dx\right)\\ &= -x^2\, cos(x) + 2\left(x\, sin(x) - \int sin(x)\, dx\right)\\ &= -x^2\, cos(x) + 2\big(x\, sin(x) - (-\,cos(x))\big)\\ &= -x^2\, cos(x) + 2x\, sin(x) + 2\, cos(x) + C\end{aligned}$$

Example 3 (Going in Circles): Evaluate $\int e^x\, sin(x)\, dx$

Solution: In this case, as the note says in the LIATE rules, either choice for u will suffice.

$u = e^x$	$dv = sin(x)\, dx$
$du = e^x dx$	$v = -\,cos(x)$

$$\begin{aligned}\int e^x\, sin(x)\, dx &= e^x(-\,cos(x)) - \int(-\,cos(x))e^x dx\\ &= -e^x\, cos(x) + \int e^x\, cos(x)\, dx\end{aligned}$$

As you can see the new integral is very similar to the original. Let's see what happens if we apply IBP again to this integral.

$u = e^x$	$dv = cos(x)\, dx$
$du = e^x dx$	$v = sin(x)$

$$\int e^x\, sin(x)\, dx = -e^x\, cos(x) + \left(\int e^x\, cos(x)\, dx\right)$$
$$\int e^x\, sin(x)\, dx = -e^x\, cos(x) + \left(e^x\, sin(x) - \int e^x\, sin(x)\, dx\right)$$

This time our new integral is *exactly identical* to the integral we started with in the problem. Furthermore, since the other terms do not contain integral expressions, we can rearrange this equation and solve the original integral!

$$2\int e^x\, sin(x)\, dx = -e^x\, cos(x) + e^x\, sin(x)$$
$$\int e^x\, sin(x)\, dx = \frac{e^x(sin(x) - cos(x))}{2} + C$$

Example 4 (Definite Integral): Evaluate $\int_1^2 x\,ln(x)\,dx$

Solution: When evaluating a definite integral using IBP, the limits need to be used for all terms.

Integration by Parts Formula for Definite Integrals
$\int_a^b udv = uv\Big\|_a^b - \int_a^b vdu = \left(uv - \int vdu\right)\Big\|_a^b$

Continuing with our example we choose the following substitutions

$u = ln(x)$	$dv = xdx$
$du = \frac{1}{x}dx$	$v = \frac{1}{2}x^2$

$$\int_1^2 x\,ln(x) = \left(\frac{ln(x)\,x^2}{2} - \frac{1}{2}\int x^{\cancel{2}}\frac{1}{\cancel{x}}dx\right)\Bigg|_1^2$$

$$= \left(\frac{ln(x)\,x^2}{2} - \frac{1}{2}\int xdx\right)\Bigg|_1^2$$

$$= \left(\frac{ln(x)\,x^2}{2} - \frac{1}{4}x^2\right)\Bigg|_1^2$$

$$= \left(\frac{x^2}{4}(2\,ln(x) - 1)\right)\Bigg|_1^2$$

$$= \left(\frac{4}{4}(2\,ln(2) - 1)\right) - \left(\frac{1}{4}(2\,ln(1) - 1)\right) = 2\,ln(2) - \frac{3}{4}$$

Example 5: Evaluate $\int ln(x)\,dx$

Solution: In this case it seems there is only one function and doesn't seem like we can use IBP. However, we use the following "trick", by rewriting the integral as: $\int(1)\,ln(x)\,dx$.

Then we choose the following.

$u = ln(x)$	$dv = 1dx$
$du = \frac{1}{x}dx$	$v = x$

Applying IBP, we have

$$\int(1)\,ln(x)\,dx = x\,ln(x) - \int(\cancel{x})\frac{1}{\cancel{x}}dx$$

$$= x\,ln(x) - x = x(ln(x) - 1) + C$$

Example 6: Evaluate $\int x5^x dx$

Solution: According to LIATE we let $u = x$, and $dv = 5^x dx$. Therefore, we need to integrate an exponential function of the form $\int a^x dx$. As a refresher we show derive the integral of a general exponential function. Using properties of exponential and logarithms we can write any number, a, as $e^{ln(a)}$. Therefore, we can rewrite the integral as

$$\int \left(e^{ln(a)}\right)^x dx = \int e^{ln(a)x} dx$$

For which we can use substitution to evaluate.

$$u = ln(a)\,x \quad \rightarrow \quad du = ln(a)\,dx \quad \rightarrow \quad dx = \frac{1}{ln(a)} du$$

$$\int e^{ln(a)x} dx = \frac{1}{ln(a)} \int e^u du = \frac{e^{ln(a)x}}{ln(a)} = \frac{1}{ln(a)} a^x$$

Giving us the general formula

$$\boxed{\int a^x dx = \frac{1}{ln(a)} a^x + C}$$

Returning to the original problem we make the following choices for IBP.

$u = x$	$dv = 5^x dx$
$du = dx$	$v = \frac{1}{ln(5)} 5^x$

$$\begin{aligned}\int x5^x dx &= \frac{x}{ln(5)} 5^x - \frac{1}{ln(5)} \int 5^x dx \\ &= \frac{x5^x}{ln(5)} - \frac{1}{ln(5)} \cdot \frac{1}{ln(5)} 5^x \\ &= \frac{5^x}{ln(5)} \left(x - \frac{1}{ln(5)}\right) + C\end{aligned}$$

Example 7: Evaluate $\int (x^2 + 3x + 1) \ln(x)\, dx$

Solution: We choice u and dv based on LIATE.

$u = \ln(x)$	$dv = (x^2 + 3x + 1)dx$
$du = \frac{1}{x} dx$	$v = \frac{1}{3} x^3 + \frac{3}{2} x^2 + x$

Then evaluate using IBP.

$$\int (x^2+3x+1)\,ln(x)\,dx = \left(\frac{1}{3}x^3+\frac{3}{2}x^2+x\right)ln(x) - \int \left(\frac{1}{3}x^3+\frac{3}{2}x^2+x\right)\frac{1}{x}dx$$

$$= \left(\frac{1}{3}x^3+\frac{3}{2}x^2+x\right)ln(x) - \left(\frac{1}{3}\int x^2dx+\frac{3}{2}\int xdx+\int dx\right)$$

$$= \left(\frac{1}{3}x^3+\frac{3}{2}x^2+x\right)ln(x) - \left(\frac{1}{9}x^3+\frac{3}{4}x^2+x\right)$$

$$= \left(\frac{3\,ln(x)-1}{9}\right)x^3+\left(\frac{6\,ln(x)-3}{4}\right)x^2+(ln(x)-1)x+C$$

Example 8: Evaluate $\int \frac{x}{\sqrt{x+1}}dx$

Solution: In this case it's not clear, based on LIATE, how to choose u and dv. However, the general rule was to choose u so that $\frac{du}{dx}$ is simpler. Therefore, we try the following:

$u = x$	$dv = (x+1)^{-1/2}dx$
$du = dx$	$v = \frac{1}{2}\sqrt{x+1}$

Where we used substitution to evaluate the integral of $(x+1)^{-1/2}$ as follows:

$$u = x+1 \quad \rightarrow \quad du = dx \quad \rightarrow \quad \int (u)^{-1/2}du = \frac{1}{2}\sqrt{u} = \frac{1}{2}\sqrt{x+1}$$

Evaluating with IBP we have

$$\int \frac{x}{\sqrt{x+1}}dx = \frac{x\sqrt{x+1}}{2} - \frac{1}{2}\int \frac{1}{\sqrt{x+1}}dx$$

$$= \frac{x\sqrt{x+1}}{2} - \frac{\sqrt{x+1}}{4}$$

$$= \frac{\sqrt{x+1}}{4}(2x-1)+C$$

Example 9: Evaluate $\int cos(x)\, ln(sin(x))\, dx$

Solution: In this case the integral of $ln(sin(x))$ is not easily computed and the derivative does not seem to give us a simpler function. Instead, let's try to change the integrand by using substitution first.

$$u = sin(x) \quad \rightarrow \quad du = cos(x)\, dx$$

The integral can now be written as

$$\int cos(x)\, ln(sin(x))\, dx = \int ln(u)\, du$$

Which can be solved with IBP according to example 5 above.

$$\int ln(u)\, du = u(ln(u) - 1) + C$$

The final step is to resubstitute for u, which gives us the following result.

$$\int cos(x)\, ln(sin(x))\, dx = sin(x)\,(ln(sin(x)) - 1) + C$$

Let's check this result with differentiation.

$$\begin{aligned}
\frac{d}{dx}\big(sin(x)\,(ln(sin(x)) - 1)\big) &= \left(\frac{d}{dx}(sin(x)\, ln(sin(x))) - \frac{d}{dx}(sin(x))\right) \\
&= \left(\left(cos(x)\, ln(sin(x)) + \cancel{sin(x)} \cdot \frac{1}{\cancel{sin(x)}} cos(x)\right) - (cos(x))\right) \\
&= (cos(x)\, ln(sin(x)) + \cancel{cos(x) - cos(x)}) \\
&= cos(x)\, ln(sin(x))
\end{aligned}$$

Example 11: Evaluate the following definite integral:

$$\int_1^5 x^2\, ln(x^2)\, dx$$

Solution: Using LIATE to choose u we have

$u = ln(x^2)$	$dv = x^2 dx$
$du = \frac{1}{x^2} 2x dx = \frac{2}{x} dx$	$v = \frac{1}{3} x^3$

$$\int_1^5 x^2\, ln(x^2)\, dx = \left(\frac{x^3\, ln(x^2)}{3} - \int_1^5 \frac{1}{3} x^3 \frac{2}{x} dx\right)\Bigg|_1^5$$

$$= \left(\frac{x^3\, ln(x^2)}{3} - \frac{2}{3}\int_1^5 x^2 dx\right)\Bigg|_1^5$$

$$= \left(\frac{3x^3\, ln(x^2)}{9} - \frac{2}{9} x^3\right)\Bigg|_1^5$$

$$= \left(\frac{x^3}{9}(3\, ln(x^2) - 2)\right)\Bigg|_1^5$$

$$= \left(\frac{125}{9}(3\, ln(25) - 2)\right) - \left(\frac{1}{9}(0 - 2)\right)$$

$$= \frac{1}{9}(375\, ln(25) - 248) \cong 106.6$$

Final Summary for Integration Techniques – Integration by Parts

Integration by Parts Formula
$$\int udv = uv - \int vdu$$
Integration by Parts Formula for Definite Integrals
$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu = \left(uv - \int vdu\right)\Big|_a^b$$
Guidelines for Choosing u and dv

There are no hard and fast rules for how to choose u and dv for IBP, however the following guidelines can sometimes be useful.

1. Choose dv so that $v = \int dv$ can be evaluated.
2. Choose u so that $\frac{du}{dx}$ is "simpler" that u itself.

Another guideline is referred to as "LIATE", and is stated as follows:

Choose u to be the function that comes first in the list:

L:	Logarithmic Function
I:	Inverse Trigonometric Function
A:	Algebraic Function
T:	Trigonometric Function
E:	Exponential Function

Note: The last two, (T and E), can actually be in either order.

By: ferrantetutoring