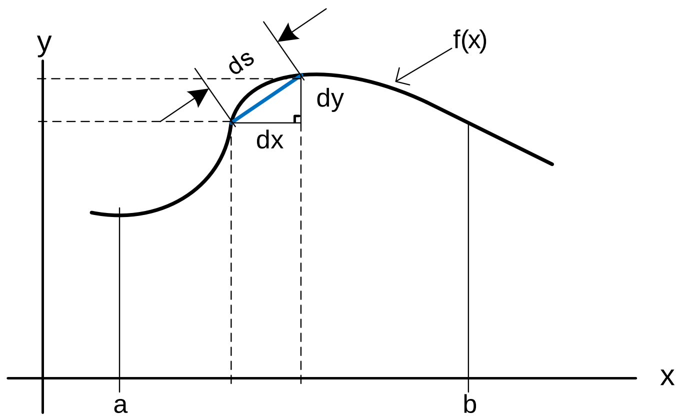
Integral Applications – Arc Length and Surface Area

In calculus 1 we learned that the integral can be used to find the "total amount" of something, e.g., area, volume, mass, etc. In this lesson we will learn how to use the integral to find the total length along a curve, which we refer to as *arc length*. We will also learn how the arc length can be used to find the surface area of a surface of revolution. Although the arc length integral can be set up, we'll see that in most cases it has a rather complicated integrand that does not have an elementary antiderivative. However, in most applications, we require the evaluation of a definite integral, for which we can use numerical methods.

Arc Length:

To derive an expression for the arc length we use an approach we learned in calculus 1. That is; derive an expression for an infinitesimal length, dL, and then integrate to find the total length.



As the figure above shows, the infinitesimal length, ds, is the hypotenuse of a right triangle with side lengths of dx and dy. Therefore, we can use the Pythagorean theorem to write

$$ds^2 = dx^2 + dy^2$$

The next step is not at all obvious, but nonetheless provides us with a desired form of ds. We start by multiplying the right-hand side by $1 = \frac{dx^2}{dx^2}$, while distributing the denominator term only. Next, we take the square root to find our desired expression ds.

$$ds^2 = (dx^2 + dy^2)\frac{dx^2}{dx^2}$$

$$ds^2 = \left(\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)dx^2$$

$$ds^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)dx^2$$

$$ds = \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right)dx$$

And if $y = f(x)$, we can write

$$ds = \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$

Note, we could have also initially multiplied by $\frac{dy^2}{dy^2}$, giving us an alternate formula for the arc length as shown.

$$ds = \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right) dy$$

$$ds = \left(\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$

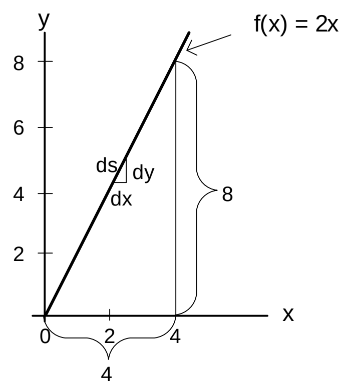
In either case we find the total arc length by integrating both sides of the above expressions.

$$s = \int_a^b \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx \quad \textbf{or} \quad s = \int_a^b \left(\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$

Formula for Arc Length
If f is continuous and differentiable on $[a, b]$, then the arc length, s, of $y = f(x)$ over $[a, b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$ Similarly, if g is continuous and differentiable on $[a, b]$, then the arc length, s, of $x = g(y)$ over $[a, b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$

Let's do some examples.

Example 1: Find the arc length for $y = 2x$, over $[0,4]$. Note, in this case our equation is linear and so we can check the results by directly applying the Pythagorean Theorem.



Solution: As mentioned, the length of the line segment can easily be computed as

$$s = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$$

Now let's use the arc length formula to verify its correctness.

$$\begin{aligned} s &= \int_0^4 \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx \\ &= \int_0^4 \left(\sqrt{1 + (2)^2}\right) dx \\ &= \int_0^4 \sqrt{5} dx = \sqrt{5}\Big|_0^4 = 4\sqrt{5} \end{aligned}$$

As expected, the arc length formula computes the same value!

Example 2: Find the arc length of the curve $f(x) = \frac{2}{3}x^{3/2}$, over the interval $[1,3]$.

Solution: Directly applying the arc length integral we have

$$\begin{aligned} s &= \int_1^3 \sqrt{1 + \left(f'(x)\right)^2} dx \\ &= \int_1^3 \sqrt{1 + \left(\frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} x^{1/2}\right)^2} dx \\ &= \int_1^3 \left(\sqrt{1 + x}\right) dx \\ &= \int_2^4 u^{1/2} du \\ &= \frac{2}{3} u^{3/2}\Big|_2^4 = \frac{2}{3}\left(4^{3/2} - 2^{3/2}\right) \cong 3.45 \end{aligned}$$

Where, we used the technique of substitution to evaluate the integral.

Example 3: Find the arc length of the curve $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$, over the interval $\left[\frac{1}{2}, 2\right]$.

Solution: We can start by computing $1 + \left(f'(x)\right)^2$

$$\begin{aligned} 1 + \left(f'(x)\right)^2 &= 1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2 \\ &= 1 + \frac{1}{4}(x^2 - x^{-2})^2 \end{aligned}$$

Although it's not immediately obvious, this function can be written as a perfect square as shown below.

$$\begin{aligned} 1 + \frac{1}{4}(x^2 - x^{-2})^2 &= 1 + \frac{1}{4}(x^4 + x^{-4} - 2) \\ &= 1 + \frac{x^4}{4} + \frac{x^{-4}}{4} - \frac{1}{2} \\ &= \frac{x^4}{4} + \frac{x^{-4}}{4} + \frac{1}{2} \\ &= \frac{1}{4}(x^4 + x^{-4} + 2) \\ &= \frac{1}{4}(x^2 + x^{-2})^2 \end{aligned}$$

Makes sure you verify for yourself from the last step that $(x^4 + x^{-4} + 2) = (x^2 + x^{-2})^2$. Finally, we evaluate the arc length integral.

$$\begin{aligned} s &= \int_{\frac{1}{2}}^{2} \left(\sqrt{\frac{1}{4}(x^2 + x^{-2})^2}\right) dx \\ &= \frac{1}{2}\int_{\frac{1}{2}}^{2} (x^2 + x^{-2}) dx \\ &= \frac{1}{2}\left(\frac{1}{3}x^3 - x^{-1}\right)\Big|_{\frac{1}{2}}^{2} \\ &= \frac{1}{2}\left(\left(\frac{1}{3}2^3 - \frac{1}{2}\right) - \left(\frac{1}{3}\left(\frac{1}{2}\right)^3 - 2\right)\right) = \frac{33}{16} \end{aligned}$$

Example 4: Find the arc length of the curve $y = sin(x)$, over the interval $[0, \pi]$.

Solution: In this case the function is simple enough to proceed directly to the arc length formula.

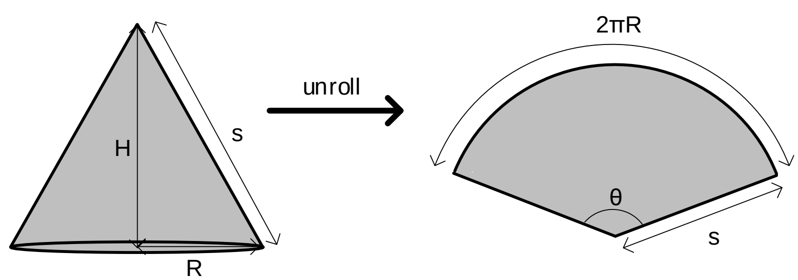
$$s = \int_0^{\pi} \left(\sqrt{1 + \left(\frac{d}{dx} sin(x) \right)^2} \right) dx$$

$$= \int_0^{\pi} \left(\sqrt{1 + cos^2(x)} \right) dx$$

Unfortunately, even though the original function was a simple sine wave this integral cannot be solved analytically. In this case we must use numerical techniques. We can use a computer system or one of the numerical techniques we learned in an earlier lesson, e.g., Trapezoid Rule. Using the TI-84 plus we find $s \cong 3.82$.

Surface Area of a Surface of Revolution:

To find the surface area for a surface of revolution we will again start by deriving an expression for an infinitesimal surface area, dA, and then integrate to find the surface area of the entire object. The infinitesimal object we will use is called a conical frustum. To find an expression for the surface area of a frustum, a truncated cone, we begin with finding the surface area of a cone.



The figure above shows a cone and the circular sector it creates when we "unroll" it. Finding the surface area of the cone, therefore, is equivalent to finding the area of the circular sector shown above. The area of the sector, A_s, can be found using a ratio as follows:

$$\frac{Area\ of\ Sector}{Area\ of\ Circle} = \frac{sector\ radians}{circle\ radians}$$

$$\frac{A_s}{\cancel{\pi} s^2} = \frac{\theta}{2\cancel{\pi}}$$

$$A_s = \frac{s^2}{2}\theta$$

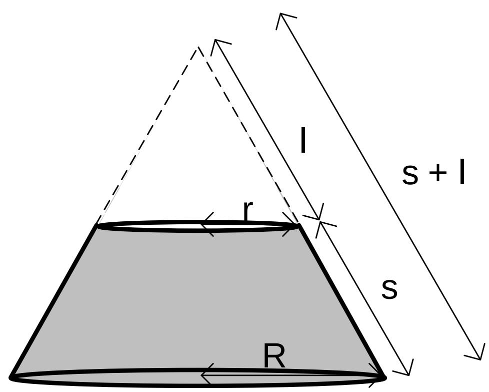
Furthermore, we can find a relationship for θ using another ratio.

$$\frac{sector\ radians}{circle\ radians} = \frac{Arc\ Length}{Circumference}$$
$$\frac{\theta}{2\pi} = \frac{\cancel{2\pi}R}{\cancel{2\pi}s}$$
$$\theta = \frac{2\pi R}{s}$$

Substituting θ into the first equation, we have a formula for the surface area of a cone in terms of the radius and the slanted height.

$$A_s = \frac{s^{\cancel{2}}}{\cancel{2}} \cdot \frac{\cancel{2}\pi R}{\cancel{s}}$$
$$A_s = \pi R s$$

We can now use this formula to find the surface area of the frustum shown below.



The area of the frustum is given by the surface are of the larger cone minus smaller top cone.

$$\begin{aligned} A_F &= A_C - A_T \\ &= \pi R(s + l) - \pi r l \\ &= \pi R s + \pi R l - \pi r l \end{aligned}$$

Next, we remove l using another ratio as follows

$$\frac{l}{r} = \frac{s + l}{R}$$
$$rs + rl = Rl$$
$$Rl - rl = rs$$
$$l = \frac{rs}{R - r}$$

Substituting for l we have

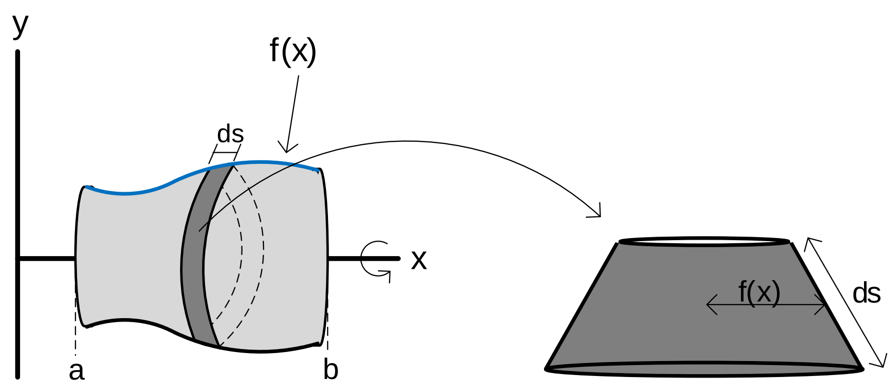
$$\begin{aligned}
A_F &= \pi R s + \pi R l - \pi r l \\
&= \pi R s + \pi R \cdot \frac{rs}{R-r} - \pi r \cdot \frac{rs}{R-r} \\
&= \frac{\pi R s(R-r) + \pi R r s - \pi r^2 s}{R-r} \\
&= \frac{\pi R^2 s \cancel{- \pi R r s + \pi R r s} - \pi r^2 s}{R-r} \\
&= \frac{\pi R^2 s - \pi r^2 s}{R-r} \\
&= \frac{\pi s(R^2 - r^2)}{R-r} \\
&= \frac{\pi s\cancel{(R-r)}(R+r)}{\cancel{R-r}} \\
&= \pi s(R+r)
\end{aligned}$$

Next, we multiply by $1 = \frac{2}{2}$ so that we can rewrite the formula in a more intuitive form.

$$\begin{aligned}
A_F &= 2\pi s\left(\frac{R+r}{2}\right) \\
A_F &= 2\pi s\bar{r}
\end{aligned}$$

Where, $\bar{r}$ is the average radius.

Now, let's see how this formula is used to find the surface area for surfaces of revolution. The figure below shows a surface of revolution created by rotating the curve, $f(x)$, around the x-axis. It also shows the infinitesimal frustum that will be used to compute the entire surface area.



For this infinitesimal frustum, the average radius is $f(x)$ and the side length is the infinitesimal, ds. With this we can write the infinitesimal area as

$$dA = 2\pi f(x) ds$$

Furthermore, ds is an infinitesimal arc length given as

$$ds = \sqrt{1 + \left(f'(x)\right)^2} dx$$

Substituting we have,

$$dA = 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

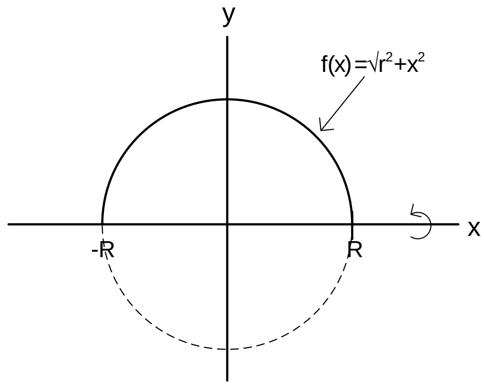
Finally, the surface area of the surface of revolution in the interval $a \leq x \leq b$, is found by integrating.

$$A_S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Formula for Surface Area of a Surface of Revolution
If $f(x) \geq 0$ and if f is continuous and differentiable on $[a, b]$, then the surface area, A_S, of the surface obtained by rotating the graph of f about the x-axis for $a \leq x \leq b$ is equal to $$A_S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example 5: Derive the formula for the surface area of a sphere of radius R, by rotating the function the following function about the x-axis.

$$f(x) = \sqrt{R^2 - x^2}$$



Solution:

We directly apply the formula from above with the limits of integration from $-R$ to R.

$$\begin{aligned}
A_S &= 2\pi \int_{-R}^{R} f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx \\
&= 2\pi \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right) \sqrt{1 + \left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{R^2 - x^2}}\right)^2}\, dx \\
&= 2\pi \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right) \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx \\
&= 2\pi \int_{-R}^{R} \left(\sqrt{R^2 - x^2}\right) \sqrt{\frac{R^2 - \cancel{x^2} + \cancel{x^2}}{R^2 - x^2}}\, dx \\
&= 2\pi \int_{-R}^{R} \left(\cancel{\sqrt{R^2 - x^2}}\right) \left(\frac{R}{\cancel{\sqrt{R^2 - x^2}}}\right) dx \\
&= 2\pi R \int_{-R}^{R} dx \\
&= 2\pi R\left(R - (-R)\right) \\
&= 4\pi R^2
\end{aligned}$$

Example 6: Find the surface area of a paraboloid, which is a surface that is obtained by rotating the graph of $f(x) = \sqrt{x}$ about the x-axis, for $0 \leq x \leq 1$.

Solution: In this case we can start by simplifying the integrand from the surface area formula.

$$\begin{aligned} f(x)\sqrt{1+\left(f'(x)\right)^2} &= \left(\sqrt{x}\right)\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2} \\ &= \left(\sqrt{x}\right)\sqrt{1+\frac{1}{4x}} \\ &= \sqrt{\cancel{x}\left(\frac{4x+1}{4\cancel{x}}\right)} \\ &= \frac{1}{2}\sqrt{4x+1} \end{aligned}$$

Plugging into the integral we have

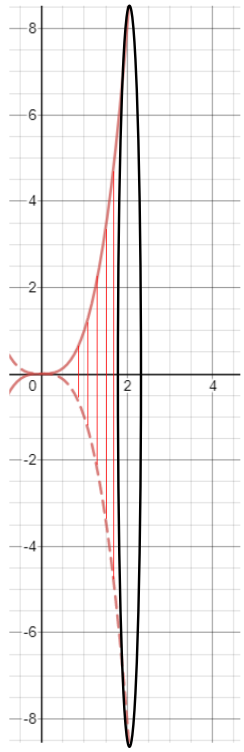
$$A_S = 2\pi\frac{1}{2}\int_0^1 \sqrt{4x+1}\,dx = \pi\int_0^1 \sqrt{4x+1}\,dx$$

Which can be evaluated using the following substitution.

$$u = 4x + 1 \qquad\qquad du = 4dx$$

$$\begin{aligned} A_S &= \frac{\pi}{4}\int_1^5 u^{1/2}\,du \\ &= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\Big|_1^5\right) \\ &= \frac{\pi}{6}\left(5^{3/2}-1\right) \cong 5.33 \end{aligned}$$

Example 7: Find the surface area of the surface obtained by rotating the graph of $f(x) = x^3$ about the x-axis, for $0 \le x \le 2$.



Solution:

$$A_S = 2\pi \int_0^2 f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx$$

$$= 2\pi \int_0^2 (x^3)\sqrt{1 + (3x^2)^2}\,dx$$

$$= 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\,dx$$

This integral can be solved with following substitution.

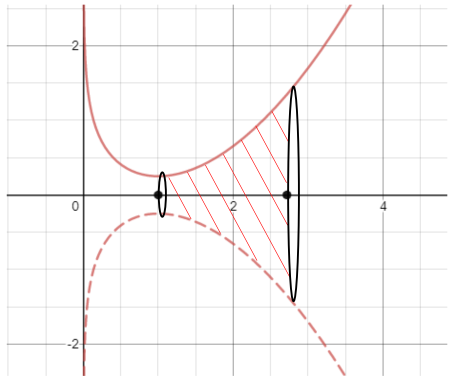
$$u = 1 + 9x^4 \qquad du = 36x^3dx \rightarrow \frac{1}{36}du = x^3dx$$

$$2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\,dx = \frac{2\pi}{36}\int_1^{145} u^{1/2}\,du$$

$$= \frac{\pi}{18}\left(\frac{2}{3}u^{3/2}\Big|_1^{145}\right)$$

$$= \frac{\pi}{27}\left(145^{3/2} - 1\right) \cong 203$$

Example 8: Find the surface area of the surface obtained by rotating the graph of $f(x) = \frac{1}{4}x^2 - \frac{1}{2}ln(x)$ about the x-axis, for $1 \le x \le e$.



Solution: Let's begin by trying to simplify the integrand.

$$
\begin{aligned}
f(x)\sqrt{1+\left(f'(x)\right)^2} &= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2} \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\sqrt{1+\left(\frac{x^2-1}{2x}\right)^2} \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\sqrt{\frac{4x^2+(x^4-2x^2+1)}{4x^2}} \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\sqrt{\frac{x^4+2x^2+1}{4x^2}} \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\sqrt{\frac{(x^2+1)^2}{4x^2}} \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\left(\frac{x^2+1}{2x}\right) \\
&= \left(\frac{1}{4}x^2 - \frac{1}{2}ln(x)\right)\left(\frac{x}{2}+\frac{1}{2x}\right) \\
&= \left(\frac{1}{8}x^3 + \frac{1}{8}x - \frac{1}{4}x\,ln(x) - \frac{1}{4}\frac{ln(x)}{x}\right)
\end{aligned}
$$

Substituting into the integral formula we have

$$
\begin{aligned}
A_S &= 2\pi\int_1^e \left(\frac{1}{8}x^3 + \frac{1}{8}x - \frac{1}{4}x\,ln(x) - \frac{1}{4}\frac{ln(x)}{x}\right)dx \\
&= 2\pi\left(\frac{1}{8}\int_1^e x^3\,dx + \frac{1}{8}\int_1^e x\,dx - \frac{1}{4}\int_1^e x\,ln(x)\,dx - \frac{1}{4}\int_1^e \frac{ln(x)}{x}\,dx\right)
\end{aligned}
$$

The first two integrals are straightforward to evaluate.

$$\frac{1}{8}\int_1^e x^3\,dx = \frac{1}{8}\left(\frac{x^4}{4}\bigg|_1^e\right) = \frac{(e^4-1)}{32} \qquad \frac{1}{8}\int_1^e x\,dx = \frac{1}{8}\left(\frac{x^2}{2}\bigg|_1^e\right) = \frac{(e^2-1)}{16}$$

The third integral is evaluated using integration by parts.

$$u = ln(x) \qquad dv = xdx$$

$$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$$

$$\begin{aligned}\frac{1}{4}\int_1^e x\,ln(x)\;dx &= \frac{1}{4}\left(\frac{1}{2}x^2\,ln(x) - \int_1^e \frac{1}{2}x^2\cdot\frac{1}{x}dx\right)\\ &= \frac{1}{8}\left(x^2\,ln(x) - \int_1^e xdx\right)\\ &= \frac{1}{8}\left(x^2\,ln(x) - \frac{1}{2}x^2\bigg|_1^e\right)\\ &= \frac{1}{8}\left(\left(e^2 - \frac{1}{2}e^2\right) - \left(0 - \frac{1}{2}\right)\right) = \frac{e^2+1}{16}\end{aligned}$$

Finally, the last integral is evaluated using substitution as follows.

$$u = ln(x) \qquad du = \frac{1}{x}dx$$

$$\frac{1}{4}\int_1^e \frac{ln(x)}{x}\;dx = \frac{1}{4}\int_0^1 udu = \frac{1}{4}\left(\frac{1}{2}u^2\bigg|_0^1\right) = \frac{1}{8}$$

Substituting we have

$$\begin{aligned}A_S &= 2\pi\left(\frac{1}{8}\int_0^e x^3\,dx + \frac{1}{8}\int_0^e x\,dx - \frac{1}{4}\int_0^e x\,ln(x)\;dx - \frac{1}{4}\int_0^e \frac{ln(x)}{x}\;dx\right)\\ &= 2\pi\left(\frac{(e^4-1)}{32} + \frac{(e^2-1)}{16} - \frac{e^2+1}{16} - \frac{1}{8}\right)\\ &= \frac{\pi}{16}\left(e^4 - 1 + \cancel{2e^2} - 2 - \cancel{2e^2} - 2 - 4\right)\\ &= \frac{\pi}{16}\left(e^4 - 9\right) \cong 8.95\end{aligned}$$

Final Summary for Integral Applications – Arc Length and Surface Area

Formula for Arc Length
If f is continuous and differentiable on $[a,b]$, then the arc length, s, of $y = f(x)$ over $[a,b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$ Similarly, if g is continuous and differentiable on $[a,b]$, then the arc length, s, of $x = g(y)$ over $[a,b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$
Formula for Surface Area of a Surface of Revolution
If $f(x) \geq 0$ and if f is continuous and differentiable on $[a,b]$, then the surface area, A_S, of the surface obtained by rotating the graph of f about the x-axis for $a \leq x \leq b$ is equal to $$A_S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

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