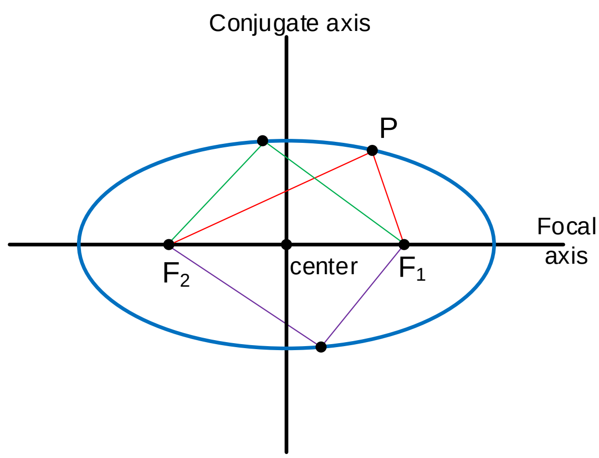
<u>Conic Sections</u>

Conic sections are curves which are obtained by intersecting the surface of a cone and a plane. The curves generated are the hyperbola, the parabola, the ellipse, and the circle (which can also be considered a special case of the ellipse). We are likely already familiar with the equations that represent these curves, however in this section we show how these curves can also be defined geometrically. For the ellipse case we will derive the algebraic expression from the geometric definition. For the remaining curves we will provide the geometric definition to help provide some intuition, however the algebraic equation will be given without an explicit derivation.

Ellipse

Using the figure below we can define the following geometric process to construct an ellipse.



1. Choose two separate points, $F_1$and F_2.
    - For convenience we choose the points to be on the x-axis and evenly spaced about the y-axis. In this case the ellipse is said to be in standard position.
2. Find all points, P, such that the distance from P to F_1 plus the distance from P to F_2 is equal to a constant, $K > 0$.

We can represent this as follows:

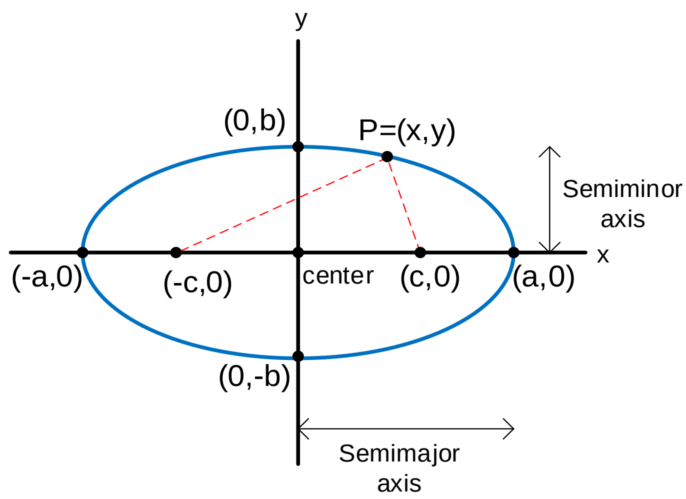
$$\overline{PF_1} + \overline{PF_2} = K$$

Note when $F_1 = F_2$ the equation above is $2PF_1 = K$ and we obtain a circle with radius F_1.

We use the following terminology when referring to an ellipse as shown in the figure.

- $F_1$and F_2 are called the foci, (plural of "focus"), of the ellipse.
- The midpoint of $\overline{F_1F_2}$ is center of the ellipse.
- The line through the foci is the focal axis of the ellipse.
- The line through the center perpendicular to the focal axis is the conjugate axis of the ellipse.

Now let's try to derive the algebraic equation for an ellipse based on the geometric description from above. We'll use the figure below which shows an ellipse in standard position.



The figure shows the foci with coordinates as $F_1 = (c, 0)$ and $F_2 = (-c, 0)$, the focal axis endpoints as $(a, 0)$ and $(-a, 0)$, and the conjugate axis endpoints as $(0, b)$ and $(0, -b)$. Next, we define a point, $P = (x, y)$. The left-hand side of the geometric relationship above represents the sum of the distances between the point P and the two focal points. We can use the distance formula to find these distances in terms of the coordinate points as follows.

$$\overline{PF_1} = \sqrt{(x-c)^2 + y^2}$$
$$\overline{PF_2} = \sqrt{(x+c)^2 + y^2}$$

To write an expression for the constant, K, in terms of the coordinate points we let $P = (a, 0)$, and find the sum of the two distances as shown.

$$\overline{PF_1} + \overline{PF_2} = K$$
$$\sqrt{(a-c)^2 + 0^2} + \sqrt{(a+c)^2 + 0^2} = K$$
$$a - c + a + c = K$$
$$2a = K$$

We can now rewrite the geometric expression in terms of the variables, x and y , to obtain an algebraic expression.

$$PF_1 + PF_2 = K$$
$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$
$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

This equation can be written in a form that we are more familiar with by performing some trivial, but not immediately obvious, algebraic manipulations.

We start by squaring both sides, rearranging the terms, and simplifying.

$$\left(\sqrt{(x+c)^2+y^2}\right)^2 = \left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+\cancel{y^2} = 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+\cancel{y^2}$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2+(x-c)^2-(x+c)^2$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2+\cancel{x^2}-2xc\cancel{+c^2-x^2}-2xc\cancel{-c^2}$$

$$4a\sqrt{(x-c)^2+y^2} = 4a^2-4xc$$

$$a\sqrt{(x-c)^2+y^2} = a^2-xc$$

Next, we square both sides again and further simplify.

$$\left(a\sqrt{(x-c)^2+y^2}\right)^2 = (a^2-xc)^2$$

$$a^2(x-c)^2+a^2y^2 = (a^2-xc)^2$$

$$a^2x^2\cancel{-2xca^2}+a^2c^2+a^2y^2 = a^4\cancel{-2xca^2}+x^2c^2$$

$$a^2x^2-x^2c^2+a^2y^2 = a^4-a^2c^2$$

$$x^2(a^2-c^2)+a^2y^2 = a^2(a^2-c^2)$$

Dividing through by $a^2(a^2-c^2)$ we have

$$\frac{x^2}{a^2}+\frac{y^2}{(a^2-c^2)} = 1$$

Next, if we let $P=(0,b)$ we can rewrite the numerator of the second term in terms of b as follows:

$$\overline{PF_1}+\overline{PF_2} = K$$

$$\sqrt{c^2+b^2}+\sqrt{c^2+b^2} = 2a$$

$$\cancel{2}\sqrt{c^2+b^2} = \cancel{2}a$$

$$c^2+b^2 = a^2$$

$$b^2 = a^2-c^2$$

Finally, we can write the algebraic expression for the equation of an ellipse in a form that we are likely familiar with.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

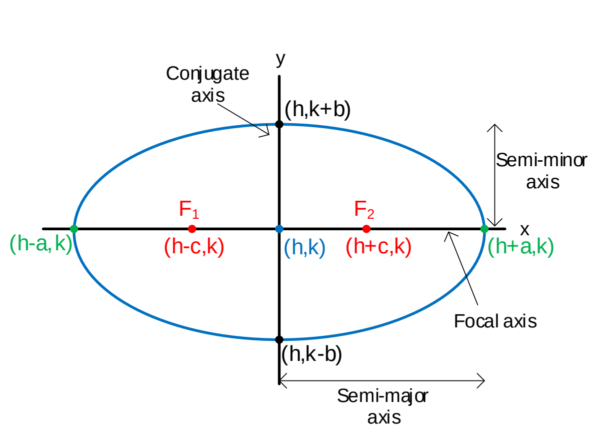
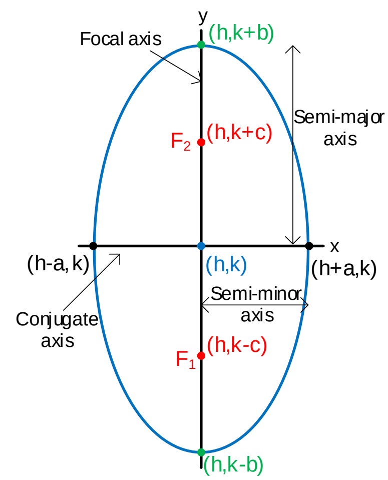
The equation above describes an ellipse that is centered at the origin, with an x-axis length of $2a$ and a y-axis length of $2b$.

Although the figure above shows the ellipse oriented so that it's wider in the horizontal axis, the opposite is also possible. Furthermore, the entire ellipse can be shifted away from the origin, e.g., center at (h, k).

The equation for an ellipse centered at (h, k) is given as

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some texts associate the variable a, with the major axis and therefore they provide two equations depending on the orientation of the ellipse. I prefer to associate a with the horizontal axis and b with the vertical axis. A summary of the two cases is shown below.

<table>
<tr><th colspan="2">Basic Ellipse Equation</th></tr>
<tr><td colspan="2">$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>

- $c^2 = a^2 - b^2$
- Center: (h, k)
- Foci: $F_1 = (h - c, k)$, $F_2 = (h + c, k)$
- Horizontal Vertices: $(h - a, k), (h + a, k)$
- Vertical Vertices: $(h, k - b), (h, k + b)$
- Semi-major Axis length: a
- Semi-minor Axis length: b

</td><td>

- $c^2 = b^2 - a^2$
- Center: (h, k)
- Foci: $F_1 = (h, k - c)$, $F_2 = (h, k + c)$
- Horizontal Vertices: $(h - a, k), (h + a, k)$
- Vertical Vertices: $(h, k - b), (h, k + b)$
- Semi-major Axis length: b
- Semi-minor Axis length: a

</td></tr>
</table>

Example 1: Find the equation of an ellipse with foci $(4,3)$, $(4,7)$ and a semimajor axis of 3, then sketch the graph.

Solution: Since the foci have fixed locations on the x-axis, the major axis is vertically aligned.

The center of the ellipse is on the conjugate axis at $x = 4$, and is centered on the focal axis between the two foci. Therefore $(h, k) = \left(4, \frac{3+7}{2}\right) = (4,5)$ and $c = 5 - 3 = 2$.

The vertical vertices are located $b = 3$ vertical units above and below the center, i.e., $(h, k - b), (h, k + b) = (4,2), (4,8)$.

To locate the horizontal vertices, we need to find a as shown below.

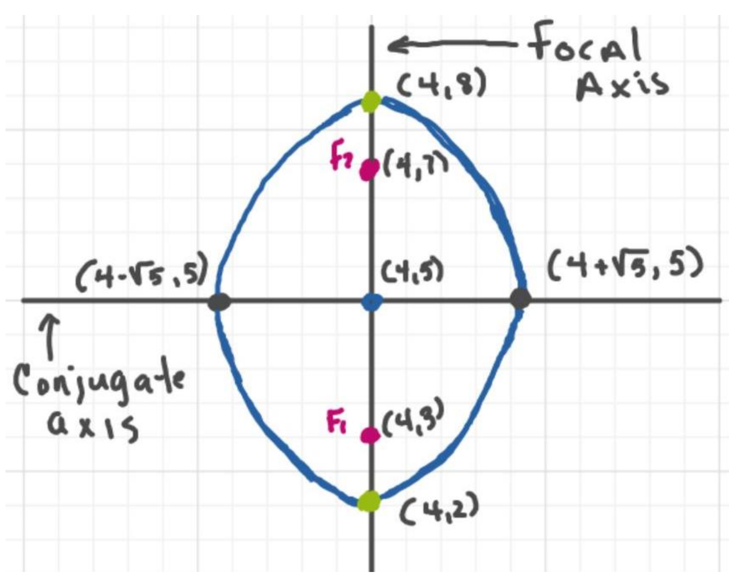
$$c^2 = b^2 - a^2$$
$$a = \sqrt{b^2 - c^2}$$
$$a = \sqrt{3^2 - 2^2}$$
$$a = \sqrt{5}$$

Therefore the $(h - a, k), (h + a, k) = \left(4 - \sqrt{5}, 5\right), \left(4 + \sqrt{5}, 5\right)$

Finally, the equation for the ellipse is given below with a sketch for illustration.

$$\frac{(x-4)^2}{\left(\sqrt{5}\right)^2} + \frac{(y-5)^2}{3^2} = 1$$

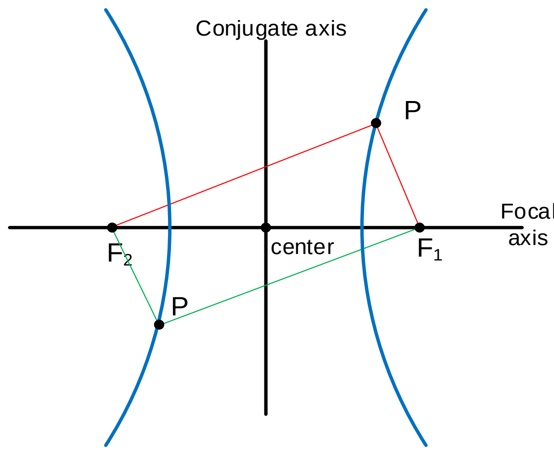
$$\frac{(x-4)^2}{5} + \frac{(y-5)^2}{9} = 1$$



Hyperbola

A hyperbola is defined geometrically as the set of all points, P, such that the difference of the distances between P and the foci is equal to $\pm K$.

$$\overline{PF_1} - \overline{PF_2} = \pm K$$

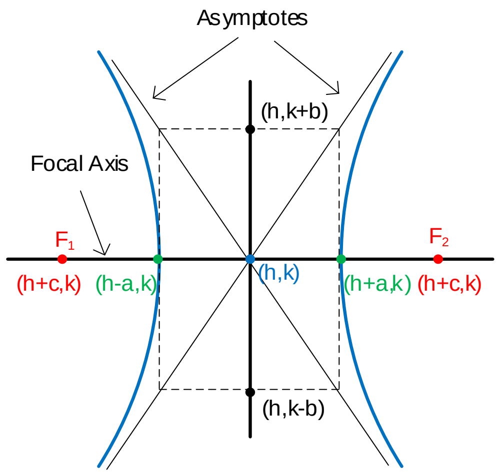
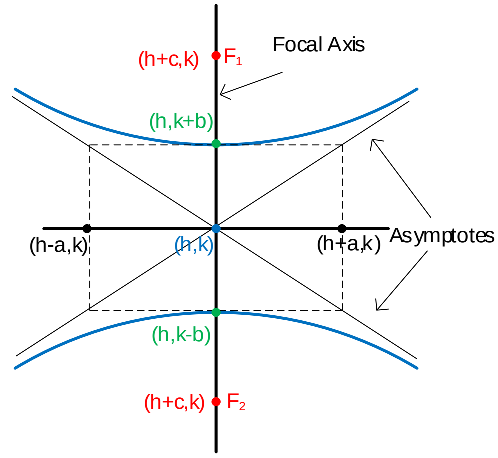


The points on the curve from the positive x-axis correspond to $+K$ and the points on the left correspond to $-K$.

The algebraic equation of a hyperbola can be derived similar to the way the ellipse was, however, we simply states the results below.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Again, this equation describes a hyperbola centered at the origin and oriented with a horizontal focal axis. Shifting the hyperbola away from the origin is done the same way it was done for the ellipse. Rotating the hyperbola so that the focal axis is oriented vertically is accomplished by multiplying the left hand side by -1. Similar to the ellipse we summarize the two cases below. In this case there are two equations depending on the orientation.

<table>
<tr><th colspan="2">Basic Hyperbola Equations</th></tr>
<tr><td>$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$</td><td>$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>• $c^2 = a^2 + b^2$
• Center: (h,k)
• Foci: $F_1 = (h-c,k)$, $F_2 = (h+c,k)$
• Vertices: $(h-a,k),(h+a,k)$
• Center Rectangle Width and Height:
○ $2a$ and $2b$.
• Asymptotes:
○ $y = \frac{b}{a}(x-h)+k, y = -\frac{b}{a}(x-h)+k$</td><td>• $c^2 = a^2 + b^2$
• Center: (h,k)
• Foci: $F_1 = (h,k-c)$, $F_2 = (h,k+c)$
• Vertices: $(h,k-b),(h,k+b)$
• Center Rectangle Width and Height:
○ $2a$ and $2b$.
• Asymptotes:
○ $y = \frac{b}{a}(x-h)+k, y = -\frac{b}{a}(x-h)+k$</td></tr>
</table>

Example 2: Find the equation of a hyperbola with foci $(0,0)$, $(8,0)$ and vertices at $(2,0)$, $(6,0)$, then sketch the graph.

Solution: Since the foci, (and vertices), have fixed locations on the y-axis, the hyperbola has a vertical focal axis.

The center of the hyperbola is midway between that foci on the x-axis with a y coordinate of 0. Therefore $(h,k) = \left(\frac{0+8}{2},0\right) = (4,0)$, and $c = 4 - 0 = 4$.

The vertices, which are on the x-axis are given as $(2,0)$, $(6,0)$. Therefore, $a = 4 - 2 = 2$.

Next, we find b as follows

$$c^2 = a^2 + b^2$$
$$b = \sqrt{c^2 - a^2}$$
$$b = \sqrt{4^2 - 2^2}$$
$$b = \sqrt{12}$$

Finally, the equation for the hyperbola is given below with a sketch for illustration.

$$\frac{(y-0)^2}{\left(\sqrt{12}\right)^2} - \frac{(x-4)^2}{2^2} = 1$$

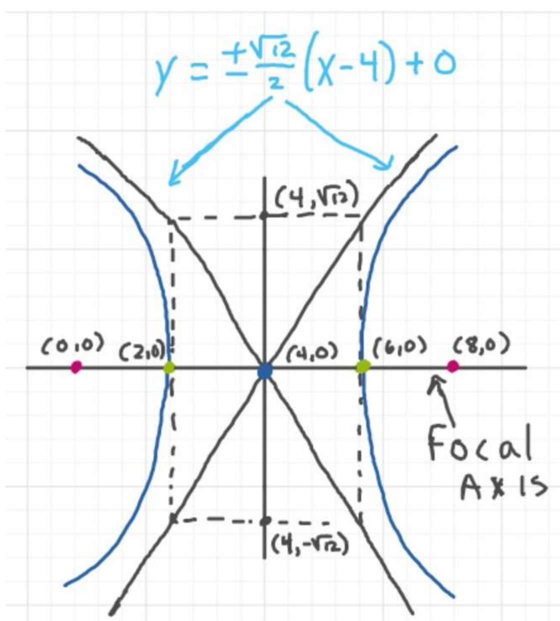
$$\frac{y^2}{12} - \frac{(x-4)^2}{4} = 1$$

The best way to sketch the hyperbola is to proceed as follows:

- Identify the center, vertices, and the foci.
  - Center: $(4,0)$, Vertices: $(2,0), (6,0)$, Foci: $(0,0), (8,0)$.
- Identify the rectangle, which is used to sketch the asymptotes. In this case we use b to find the upper and lower coordinates of the rectangle as shown in the sketch
  - $\left(4, \sqrt{12}\right), \left(4, -\sqrt{12}\right)$
- The asymptotes pass through the center and the edges of the rectangle. The equations can be identified using the slope, $\pm\frac{b}{a}$, and the center point (h, k). With this we can use the point-slope formula to write the asymptotes as follows

$$y = \pm\frac{\sqrt{12}}{2}(x-4)$$

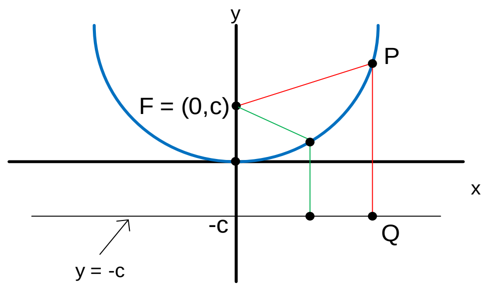
Finally, the sketch is shown below.



Parabola

A parabola is defined geometrically as the set of all points, P, equidistant from a focus, F, and a line D, called the directrix.

$$\overline{PF} = \overline{PD}$$



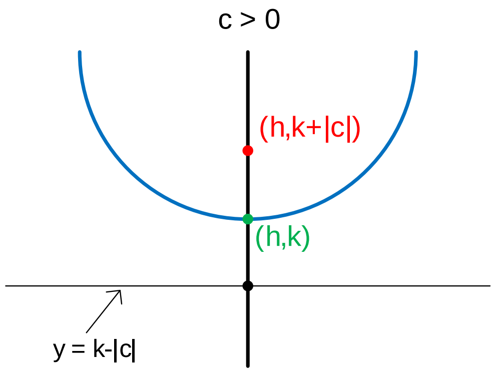
In this case the distance $\overline{PD}$ is the distance from the point P to a point Q on D, obtained by dropping a perpendicular line from P to D. The line through the focus, F, perpendicular to D is called the axis of the parabola. The vertex is the point where the parabola intersects its axis. The parabola is in standard position when the focus is $F = (0, c)$ and the directrix is $y = -c$. The algebraic equation of a parabola can again be derived similar to the way the ellipse was, however, we again simply states the results below.

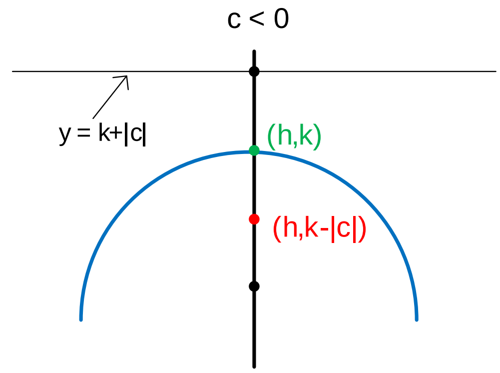
$$(4c)y = x^2$$

Once again, this equation describes a parabola centered at the origin, oriented upright and vertical. Shifting the parabola away from the origin is done the same way it was done for the ellipse and hyperbola. However, in this case there are 4 possible orientations described below.

Basic Parabola Equations

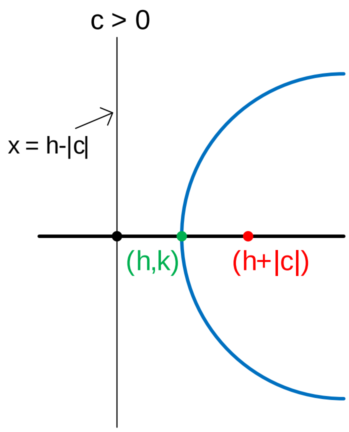
$$(x-h)^2 = 4c(y-k)$$

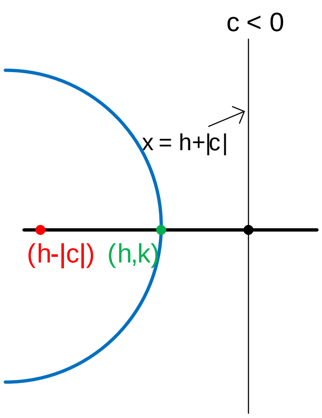




- Vertex: (h, k)
- Focus: $(h, k + c)$
- Directrix: $y = k - c$

$$(y-k)^2 = 4c(x-h)$$





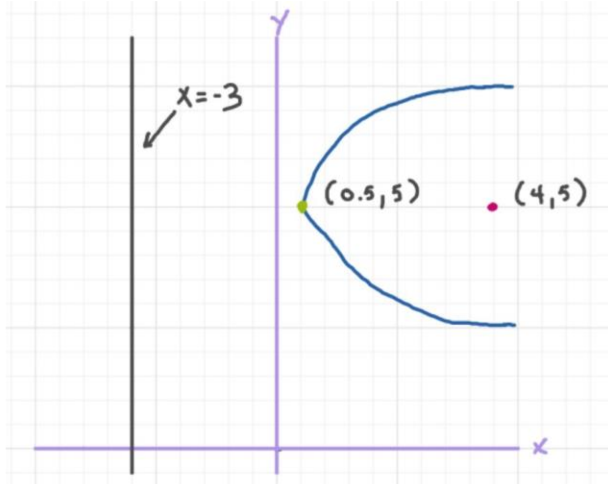
- Vertex: (h, k)
- Focus: $(h + c, k)$
- Directrix: $x = h - c$

Example 3: Find the equation of a parabola with focus $(4,5)$ and directrix $x = -3$, then sketch the graph.

Solution: Based on the directrix being a vertical line we know the parabola opens in the horizontal direction. Furthermore, since the focus is the right of the directrix the parabola opens to the right and c is positive. The vertex is located midway between the directrix and focus, i.e., $\left(\frac{4+-3}{2}, 5\right) = (0.5, 5) = (h, k)$. Finally, c is the positive distance between the focus and the vertex on the horizontal axis, i.e., $c = 4 - 0.5 = 3.5$. The equation is then given as

$$(x-0.5)^2 = 4 \cdot 3.5(y-5)$$
$$(x-0.5)^2 = 14(y-5)$$

Finally, the sketch is shown below.



General Equation of Degree 2

The algebraic equations that describe the conic sections described above are actually special cases of a general equation of degree 2 in x and y shown here.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Apart from certain "degenerate" conditions, which we will not cover here, the equation above describes a conic section that is not necessarily in standard position, i.e., not aligned with one of the coordinate axes. The term bxy is called the cross term. When $b \neq 0$ the conic section is rotated with respect to the coordinate axes. For our purposes we will assume $b = 0$. In this case the conic sections are shifted and scaled horizontally and vertically depending on the value of the remaining constants. Given an equation in the form above, with $b = 0$, we need to complete the squares to determine the type, scale, and position of the conic section. We illustrate this with the remaining examples.

Example 4: Determine the type of conic section given by the equation below. Describe its properties and sketch a graph.

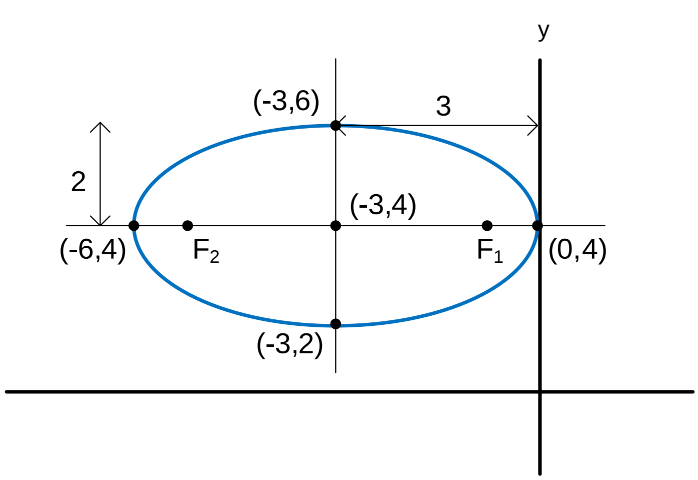
$$4x^2 + 9y^2 + 24x - 72y + 144 = 0$$

Solution: To get the equation into a form we can recognize we must complete the square of the terms involving x and y separately and simplify as shown.

$$(4x^2 + 24x) + (9y^2 - 72y) = -144$$
$$4(x^2 + 6x) + 9(y^2 - 8y) = -144$$
$$4[(x+3)^2 - 9] + 9[(y-4)^2 - 16] = -144$$
$$4(x+3)^2 + 9(y-4)^2 = -144 + 36 + 144$$
$$\frac{4(x+3)^2}{36} + \frac{9(y-4)^2}{36} = \frac{36}{36}$$
$$\frac{(x+3)^2}{9} + \frac{(y-4)^2}{4} = 1$$
$$\frac{(x+3)^2}{3^2} + \frac{(y-4)^2}{2^2} = 1$$

The equation represents an ellipse centered at $(-3,4)$, with a horizontal focal axis since $a > b$. Below will list various properties that are then used to sketch the ellipse.

- $c = \sqrt{3^2 - 2^2} = \sqrt{5} \cong 2.2$
- Center: $(-3,4)$
- Foci: $F_1 = (-3 - \sqrt{5}, 4) \cong (-5.2,4)$, $F_2 = (-3 + \sqrt{5}, 4) \cong (-0.76,4)$
- Horizontal Vertices: $(-3-3,4) = (-6,4)$, $(-3+3,4) = (0,4)$
- Vertical Vertices: $(-3,4-2) = (-3,2)$, $(-3,4+2) = (-3,6)$
- Semi-major Axis length: 3
- Semi-minor Axis length: 2



Example 5: Determine the type of conic section given by the equation below. Describe its properties and sketch a graph.

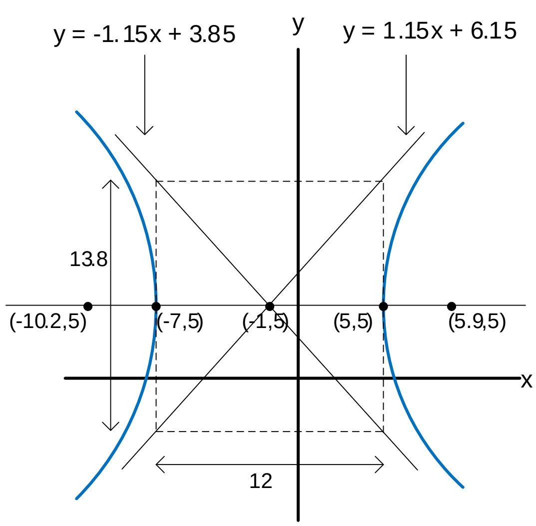
$$4x^2 - 3y^2 + 8x + 30y - 215 = 0$$

Solution: Similar to the previous question we start by completing the squares.

$$(4x^2 + 8x) - (3y^2 - 30y) = 215$$
$$4(x^2 + 2x) - 3(y^2 - 10y) = 215$$
$$4[(x+1)^2 - 1] - 3[(y-5)^2 - 25] = 215$$
$$4(x+1)^2 - 3(y-5)^2 = 144$$
$$\left(\frac{x+1}{6}\right)^2 - \left(\frac{y-5}{\sqrt{48}}\right)^2 = 1$$

The equation represents a hyperbola centered at $\left(6, \sqrt{48}\right)$, with a horizontal focal axis since the with the variable y is negative. Below will list various properties that are then used to sketch the ellipse.

- $c = \sqrt{36 + 48} = \sqrt{84} \cong 9.2$
- Center: $(-1, 5)$
- Foci: $F_1 = \left(-1 - \sqrt{84}, 5\right) \cong (-10.2, 5)$, $F_2 = \left(-1 + \sqrt{84}, 5\right) \cong (8.2, 5)$
- Vertices: $(-1 - 6, 5) = (-7, 5)$, $(-1 + 6, 5) = (5, 5)$
- Center Rectangle Width and Height:
  - $2 \cdot 6 = 12$ and $2 \cdot \sqrt{48} \cong 13.8$
- Asymptotes:
  - $y = \pm\frac{\sqrt{48}}{6}(x + 1) + 5$



Example 6: Determine the type of conic section given by the equation below. Describe its properties and sketch a graph.

$$x^2 + y^2 - 6x - 8y + 21 = 0$$

Solution: Again, we start by completing the squares

$$(x^2 - 6x) + (y^2 - 8y) = -21$$

$$(x-3)^2 - 9 + (y-4)^2 - 16 = -21$$

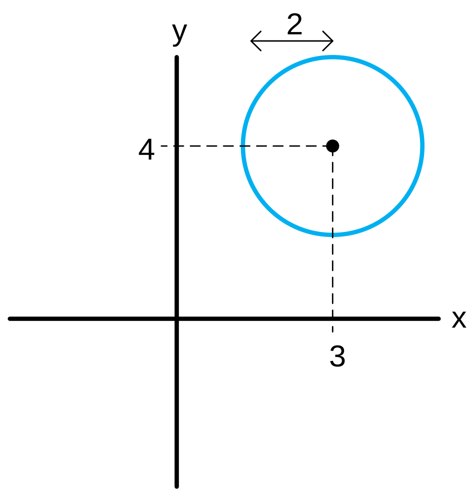
$$(x-3)^2 + (y-4)^2 = 4$$

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y-4}{2}\right)^2 = 1$$

Since a and b are the same this equation represents the equation of a circle, more commonly written as

$$(x-3)^2 + (y-4)^2 = 2^2$$

The circle is centered at $(3,4)$ with a radius of 2.



Example 7: Determine the type of conic section given by the equation below. Describe its properties and sketch a graph.

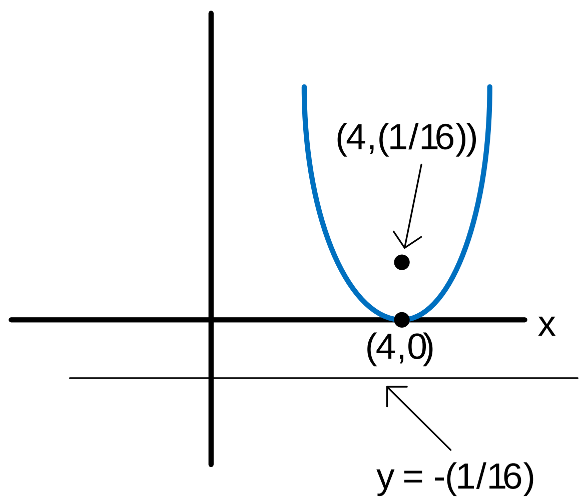
$$4x^2 - 32x - y + 64 = 0$$

Solution: In this case since the y term is linear the equation is a parabola. Completing the square with the y term we have

$$\begin{aligned} 4x^2 - 32x - y + 64 &= 0 \\ 4(x^2 - 8x) &= y - 64 \\ 4((x-4)^2 - 16) &= y - 64 \\ 4(x-4)^2 - 64 &= y - 64 \\ (x-4)^2 &= \frac{1}{4}y \\ (x-4)^2 &= 4\left(\frac{1}{16}\right)y \end{aligned}$$

The equation represents a parabola with the following properties

- Vertex: $(4,0)$
- Focus: $\left(4,0+\frac{1}{16}\right) = \left(4,\frac{1}{16}\right)$
- Directrix: $y = 0 - \frac{1}{16},\ \ y = -\frac{1}{16}$



Final Summary for Conic Sections

Conic Sections

Conic sections are curves which are obtained by intersecting the surface of a cone and a plane. The curves are the hyperbola, the parabola, the ellipse, and the circle (which can also be considered a special case of the ellipse).

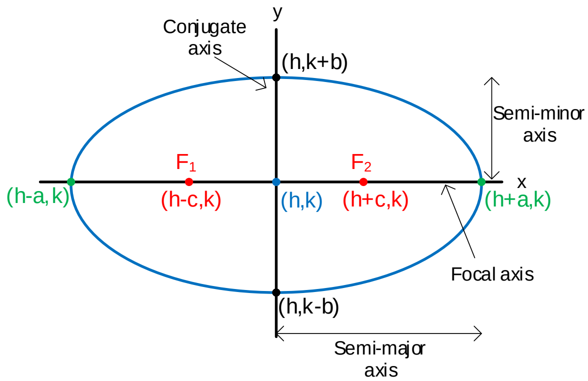
The algebraic equations that describe the conic sections are special cases of a general equation of degree 2 in x and y shown here.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

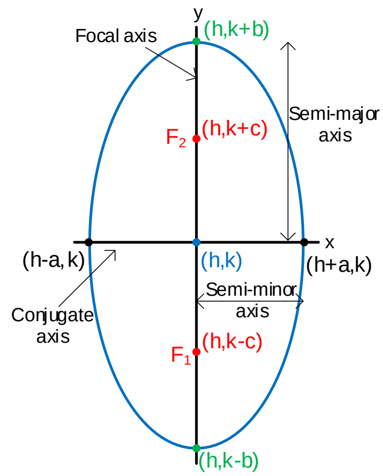
Apart from certain "degenerate" conditions, the equation above describes a conic section that is not necessarily aligned with the coordinate axes. The term bxy is called the cross term. When $b \neq 0$ the conic section is rotated with respect to the coordinate axes.

Basic Ellipse Equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



- $c^2 = a^2 - b^2$
- Center: (h, k)
- Foci: $F_1 = (h-c, k)$, $F_2 = (h+c, k)$
- Horizontal Vertices: $(h-a, k), (h+a, k)$
- Vertical Vertices: $(h, k-b), (h, k+b)$
- Semi-major Axis length: a
- Semi-minor Axis length: b



- $c^2 = b^2 - a^2$
- Center: (h, k)
- Foci: $F_1 = (h, k-c)$, $F_2 = (h, k+c)$
- Horizontal Vertices: $(h-a, k), (h+a, k)$
- Vertical Vertices: $(h, k-b), (h, k+b)$
- Semi-major Axis length: b
- Semi-minor Axis length: a

<table>
<tr><th colspan="2">Basic Hyperbola Equations</th></tr>
<tr><td>$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$</td><td>$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$</td></tr>
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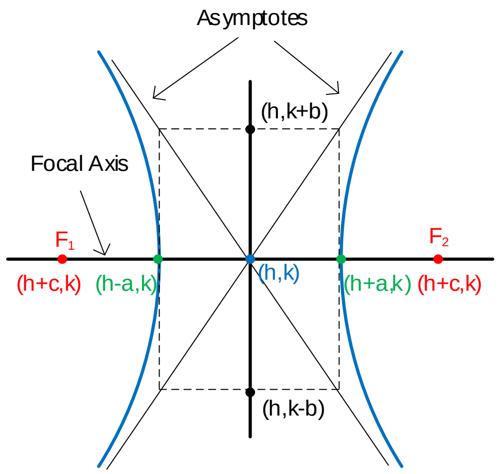

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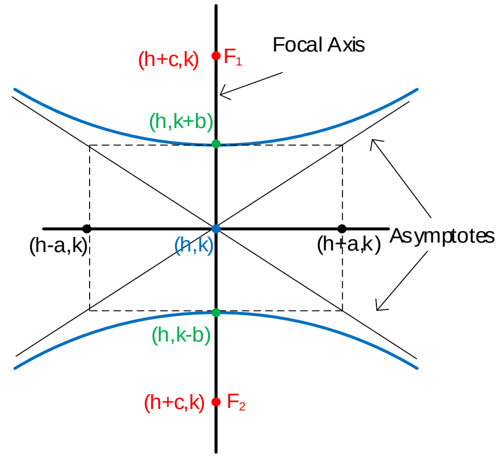

</td></tr>
<tr><td>

- $c^2=a^2+b^2$
- Center: (h,k)
- Foci: $F_1=(h-c,k),\ \ F_2=(h+c,k)$
- Vertices: $(h-a,k),(h+a,k)$
- Center Rectangle Width and Height:
  - $2a$ and $2b$.
- Asymptotes:
  - $y=\frac{b}{a}(x-h)+k, y=-\frac{b}{a}(x-h)+k$

</td><td>

- $c^2=a^2+b^2$
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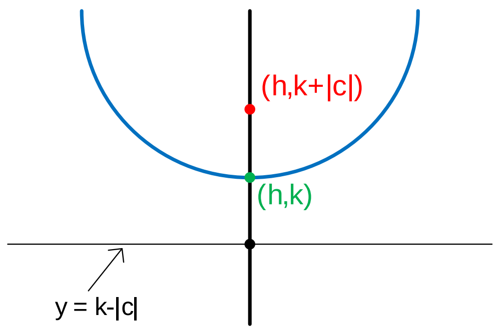
</td></tr>
</table>

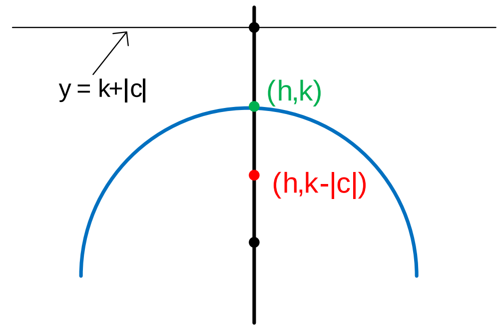
Basic Parabola Equations

$$(x-h)^2 = 4c(y-k)$$

c > 0



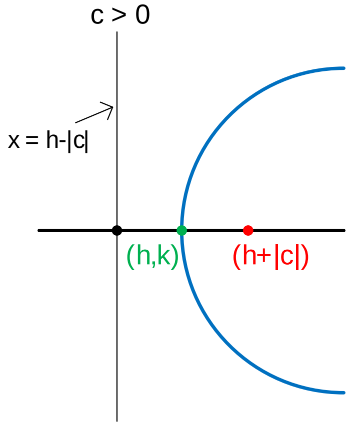
c < 0



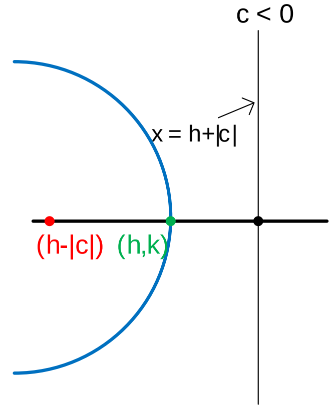
- Vertex: (h, k)
- Focus: $(h, k+c)$
- Directrix: $y = k - c$

$$(y-k)^2 = 4c(x-h)$$

c > 0



c < 0



- Vertex: (h, k)
- Focus: $(h+c, k)$
- Directrix: $x = h - c$

By: ferrantetutoring