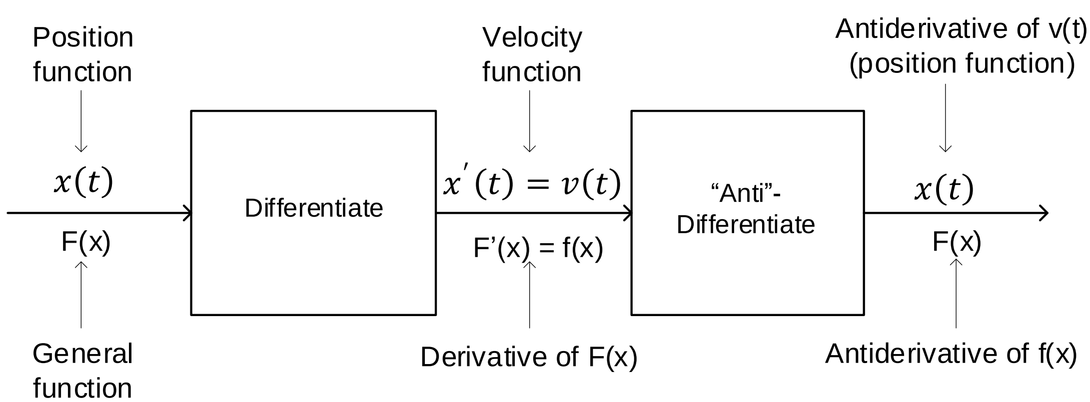
Integration – The Indefinite Integral

We started our study of calculus mentioning that there are two major areas of study; differential calculus and integral calculus. Having studied differential calculus and introducing integral calculus, we may now ask the question: "Are these two branches in any way connected?". We earlier described differential calculus as the study of how one quantity *changes* with respect to another, and integral calculus as the study of the *accumulation* of one quantity with respect to another. Considering a *change* as a difference, i.e. subtraction, and an *accumulation* as an addition, they at the least seem associated. As it turns out they are fundamentally linked, and we will begin to see that link in this lesson. We will have to wait until the next lesson on the *Fundamental Theorems of Calculus* to formally explore this link. We begin this lesson with the so-called *antiderivative*.

Indefinite Integral as the Antiderivative

We have seen the usefulness of finding the derivative of a function. For example, given a position function, $x(t)$, its often useful to find the velocity function, $v(t) = x'(t)$. The inverse problem is also common, i.e. given a velocity function, $v(t)$, we may want to find the position function, $x(t)$. Since $x'(t) = v(t)$, this amounts to finding a function, $x(t)$, whose derivative is $v(t)$. In particular since $v(t)$ is the derivative of $x(t)$, we may call $x(t)$ the *antiderivative* of $v(t)$. In the general sense, a function, $F(x)$, whose derivative is $f(x)$ is called the *antiderivative* of $f(x)$. The figure below illustrates this idea using both the position function and a generic function.



The above figure can be formalized with the definition below.

Antiderivative Definition
A function, F, is **an** antiderivative of f on an open interval (a, b) if: $$F'(x) = f(x) \text{ for all } x \text{ in } (a, b).$$

Note the bold '**an**' in the above definition. We emphasize this since it turns out there are an infinite number of possible antiderivatives of any particular function, f. Let's see why this is true. From the definition above, a function, $F(x)$, is an antiderivative of another function, $f(x)$, as long as $F'(x) = f(x)$.

Using the example $F_0(x) = x^2$, we have

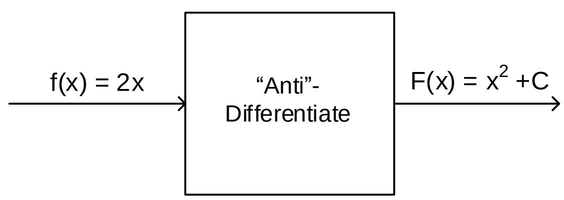
$$\frac{d}{dx}(x^2) = 2x$$

Therefore, $F_0(x) = x^2$, is an antiderivative of $f(x) = 2x$.

Now we may as the question: "Can we find another function, e.g. $F_1(x)$, that also has a derivative of $f(x) = 2x$?". How about $F_1(x) = x^2 + 1$?

$$\frac{d}{dx}(x^2 + 1) = \frac{d}{dx}(x^2) + \frac{d}{dx}(1) = 2x + 0 \ = \ 2x$$

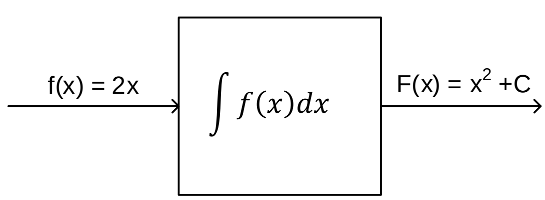
As we can see the function, $f(x) = 2x$, has at least two possible antiderivatives, $F_0(x) = x^2$ and $F_1(x) = x^2 + 1$. As a matter of fact, it should be clear that since the derivative of a constant is zero, we can create an infinite number of antiderivatives of $f(x) = 2x$, representing them as: $F(x) = x^2 + C$.



Extrapolating the above example, we state the more general antiderivative theorem below.

General Antiderivative Theorem
If $F(x)$ is an antiderivative of $f(x)$ on (a, b), then: Every other antiderivative on (a, b) is of the form $F(x) + C$. Where C is any constant.

We now introduce, what may seem curious, a notation for the operation of "anti-differentiation". We will use the same integral symbol, $\int$, that was used for the definite integral, but without writing any limits of integration, (hence we call it the *indefinite integral*). We will discover in the next lesson that the antiderivative is indeed connected to the integration process we have already learned so this notation is well justified. For now, we use the symbol as notation only. Below we redraw the figure from above as well as formally introduce the notation.



Indefinite Integral Notation
$$\int f(x)dx = F(x) + C$$ means that $F'(x) = f(x)$ We say that $F(x) + C$ is the **indefinite integral,** (the general antiderivative), of $f(x)$. Note: The function, $f(x)$, appearing in the integral is called the *integrand*, and dx is a differential.

As indicated by the term "antiderivative", computing the indefinite integral is essentially the inverse process of differentiation. Therefore, we can evaluate many integrals by reversing the differentiation rules developed earlier. We start with the power rule.

Power Rule for Integrals
$$\int x^n dx = \left(\frac{1}{n+1}\right)x^{n+1} + C$$ For $n \neq -1$

To prove this, we need to show that the derivative of $\left(\frac{1}{n+1}\right)x^{n+1} + C$ is equal to x^n.

$$\frac{d}{dx}\left(\left(\frac{1}{n+1}\right)x^{n+1} + C\right) = \left(\frac{1}{n+1}\right)\frac{d}{dx}(x^{n+1}) + \frac{d}{dx}(C)$$

$$= \left(\frac{1}{n+1}\right)(n+1)x^{(n+1)-1} + 0$$

$$= \left(\frac{\cancel{n+1}}{\cancel{n+1}}\right)x^n$$

$$\frac{d}{dx}\left(\left(\frac{1}{n+1}\right)x^{n+1} + C\right) = x^n$$

The fact that this rule does not apply for $n = -1$, results from the fact that:

$$\frac{d}{dx}(ln(x)) = \frac{1}{x} = x^{-1}$$

Reversing this we have

$$\int \frac{1}{x}dx = ln(x) + C$$

Note, however that the domain of $ln(x)$ is $x > 0$, but the domain of $\frac{1}{x}$ is $\{x: x \neq 0\}$. We would like to have the integral defined on the full domain of the integrand. To do this we can set $F(x) = ln|x|$ and define the following integration formula.

Integral Formula for x^{-1}
$$\int \frac{1}{x} dx = ln\|x\| + C$$ For $\{x: x \neq 0\}$

Using this same process, we can also define trigonometric integral formulas by reversing the derivative formulas we have previously learned.

Basic Trigonometric Integrals	
$\int sin(x)\, dx = -cos(x) + C$	$\int cos(x)\, dx = sin(x) + C$
$\int sec^2(x)\, dx = tan(x) + C$	$\int csc^2(x)\, dx = -cot(x) + C$
$\int csc(x)\, cot(x)\, dx = -csc(x) + C$	$\int sec(x)\, tan(x)\, dx = sec(x) + C$

Another formula that can be easily acquired is for the integral of e^x, since, as you will recall, the derivative of e^x is e^x!

Integral Formula for e^x
$$\int e^x dx = e^x + C$$

In general, it is not always easy, (or even possible), to find a closed formed solution to integrals. In later lessons we will introduce various techniques to help us evaluate more complicated integrals. For now, we can use our knowledge of the chain rule for differentiation to derive some common integrals, starting with sine and cosine functions

To find the integral of $cos(5x)$ we can start by differentiating $sin(5x)$.

$$\frac{d}{dx}(sin(5x)) = 5\,cos(5x)$$

Reversing this we have

$$\int 5\,cos(5x)\,dx = sin(5x) + C$$
$$\int cos(5x)\,dx = \frac{1}{5}sin(5x) + C$$

Based on this example we can define the following integration formulas.

Sine and Cosine Integral Formulas	
$\int sin(kx)\,dx = -\frac{1}{k}cos(kx) + C$	$\int cos(kx)\,dx = \frac{1}{k}sin(kx) + C$

We can apply the same technique to define the more general exponential integral formula below.

Integral Formula for e^{kx}
$\int e^{kx}dx = \frac{1}{k}e^{x} + C$

Let’s now do a few examples to practice using the above integral formulas.

Example 1:

Find the indefinite integral of the following functions.

a. $\int(2x^4 - 24x^2 + 12x^{-1})dx$

b. $\int\left(\frac{x^3+4x^2+3x-4}{x^2}\right)dx$

c. $\int(x + sec^2(x))dx$

d. $\int\left(cos(3\theta) - \frac{1}{2}sin\left(\frac{\theta}{4}\right)\right)d\theta$

Solutions: Note: The linearity rules we used for definite integrals apply as well to indefinite integrals.

a.

$$\int (2x^4 - 24x^2 + 12x^{-1})dx = 2\int x^4\,dx - 24\int x^2\,dx + 12\int \frac{1}{x}dx$$
$$= \left[2\left(\frac{x^5}{5}\right) + C_1\right] - \left[24\left(\frac{x^3}{3}\right) + C_2\right] + [12(ln|x|) + C_3]$$
$$= \frac{2x^5}{5} - 8x^3 + 12\,ln|x| + C$$

Note: We combined the various constants into a single constant, $C = C_1 + C_2 + C_3$

b.

$$\int \left(\frac{x^3 + 4x^2 + 3x - 4}{x^2}\right) dx = \int x\,dx + 4\int x^0 dx + 3\int \frac{1}{x}dx - 4\int x^{-2}\,dx$$
$$= \left(\frac{x^2}{2}\right) + 4\left(\frac{x^1}{1}\right) + 3(ln|x|) - 4\left(\frac{x^{-1}}{-1}\right)$$
$$= \frac{x^2}{2} + 4x + 3\,ln|x| + \frac{4}{x} + C$$

c.

$$\int (x + sec^2(x))dx = \int x^1\,dx + \int sec^2(x)\,dx$$
$$= \frac{x^2}{2} + tan(x) + C$$

d.

$$\int \left(cos(3\theta) - \frac{1}{2}sin\left(\frac{\theta}{4}\right)\right) d\theta = \int cos(3\theta)\,d\theta + \frac{1}{2}\int sin\left(\frac{1}{4}\theta\right) d\theta$$
$$= \left(\frac{1}{3}sin(3\theta)\right) - \frac{1}{2}\left(-4\cos\left(\frac{1}{4}\theta\right)\right)$$
$$= \frac{sin(3\theta)}{3} + 2\cos\left(\frac{1}{4}\theta\right) + C$$

Initial Condition Integration

The general solution to an integration problem is as shown.

$$\int f(x)dx = F(x) + C$$

Which means that, since C can be any constant, there are an infinite number of solutions to any given integral problem. The question we can now ask is: "*Is there a way in which we can find C, and obtain a **particular solution**?*". The answer is yes and to illustrate we return to the example of position and velocity.

The velocity of a particle is given as $v(t) = t^2$ for $t \geq 0$. To find the position function, $x(t)$, we integrate the velocity function as follows.

$$x(t) = \int t^2 dt$$

$$x(t) = \frac{t^3}{3} + C$$

To find the constant, C, we need to be given the position of the particle at a given time, e.g. $x(t_a) = X_a$. Since the solution must hold for all t we can solve for C using this additional piece of information as follows

$$x(t_a) = \frac{{t_a}^3}{3} + C = X_a$$

$$C = \left(X_a - \frac{{t_a}^3}{3}\right)$$

Therefore, we have

$$x(t) = \frac{t^3}{3} + \left(X_a - \frac{{t_a}^3}{3}\right)$$

Although we may be given the value of $x(t)$ for any given time, in most cases we are given the initial position of the particle, i.e. $x(0) = X_0$. For this reason, we usually refer to this additional information as an *initial condition*.

Let's do a few examples to emphasize the point.

Example 2:

Find the particular solutions to the following integrals using the initial conditions provided.

a. $F(x) = \int x^3 dx\,,\ F(0) = 4$

b. $F(x) = \int 2x + 9x^2 dx\,,\ F(1) = 2$

c. $x(t) = \int t^2 dt\,,\ x(0) = 32$

d. $x(t) = \int sin(\pi t)\, dt\,,\ x(2) = 2$

Solutions:

a.

$$F(x) = \int x^3 dx = \frac{x^4}{4} + C$$

We now use the initial condition to find C.

$$F(0) = \frac{0^4}{4} + C = 4, \quad C = 4$$

Therefore, the particular solution is

$$\boxed{F(x) = \frac{x^4}{4} + 4}$$

b.

$$F(x) = \int 2x + 9x^2 dx = x^2 + 3x^3 + C$$

$$F(1) = 1^2 + 3 \cdot 1^3 + C = 2, \quad C = -2$$

$$\boxed{F(x) = x^2 + 3x^3 - 2}$$

c.

$$x(t) = \int t^2 dt = \frac{t^3}{3} + C$$

$$x(0) = \frac{0^3}{3} + C = 32, \quad C = 32$$

$$\boxed{x(t) = \frac{t^3}{3} + 32}$$

d.

$$x(t) = \int sin(\pi t)\, dt = -\frac{1}{\pi} cos(\pi t) + C$$

$$x(2) = -\frac{1}{\pi} \cancel{cos(2\pi)} + C = 2, \ \ C = 2$$

$$\boxed{x(t) = 2 - \frac{1}{\pi} cos(\pi t)}$$

Final Summary for Integration – The Indefinite Integral

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