Integration – The Fundamental Theorems of Calculus

We started our previous lesson asking if there was a link between differentiation and integration. We subsequently showed this connection, albeit without a formal proof, when we defined the indefinite integral as the general antiderivative. In this lesson we introduce one of the most important theorems in mathematics, the Fundamental Theorem of Calculus (FTC), which formally connects differentiation and integration. The theorem is split into two parts. The first part, FTC I, allows for the computation of the definite integral using the antiderivative, (i.e. the indefinite integral). This part of the theorem has profound practical implications, as it allows us to compute definite integrals without having to rely on numerical techniques. The second part, FTC II, formally establishes the relationship between differentiation and integration. We begin with FTC I.

The Fundamental Theorem of Calculus Part 1

We started our study of integration by introducing the definite integral as the area under a curve. We then formally defined a method to compute the definite integral using the Riemann sum, where we were able to evaluate definite integrals for a few basic function types using a few well-known summation formulas. Next, we introduced the antiderivative and used so-called indefinite integral notation to representant this operation. We found that computing the indefinite integral is easier to do over a wider range of functions, compared to computing definite integrals, since it involves reversing derivative formulas that we have become quite familiar with. The first part of the fundamental theorem of calculus allows us to evaluate definite integrals using the indefinite integral, which as we mentioned is quite a bit easier than using the Riemann sum.

The Fundamental Theorem of Calculus Part 1
Assume that f is continuous on $[a, b]$. If F is an antiderivative of f on $[a, b]$, then $$\int_a^b f(x)dx = F(b) - F(a)$$ Note: We generally use the notation where $F(b) - F(a)$ is denoted as: $F(x)\|_a^b$

Rather than providing a formal proof of this theorem we will use an example from our earlier lesson on definite integrals and show how the same answer can be obtained using the above theorem.

Evaluate the following definite integral:

$$\int_{-3}^{3} (9x - 4x^2)\, dx$$

In the earlier lesson we solved the above definite integral as shown.

$$\begin{aligned}\int_{-3}^{3}(9x-4x^2)\,dx &= \int_{-3}^{0}(9x-4x^2)\,dx+\int_{0}^{3}(9x-4x^2)\,dx\\ &= -\left(9\int_{0}^{-3}x\,dx-4\int_{0}^{-3}x^2\,dx\right)+9\int_{0}^{3}x\,dx-4\int_{0}^{3}x^2\,dx\\ &= -\left(9\left(\frac{1}{2}(-3)^2\right)-4\left(\frac{1}{3}(-3)^3\right)\right)+9\left(\frac{1}{2}(3)^2\right)-4\left(\frac{1}{3}(3)^3\right)\\ &= \left(-\frac{81}{2}\right)-\left(\frac{108}{3}\right)+\left(\frac{81}{2}\right)-\left(\frac{108}{3}\right)\\ &= -\frac{216}{3} = -72\end{aligned}$$

Where we first needed to split the integral into two regions so that we could use the two formulas shown below.

$$\int_{0}^{b}x\,dx=\frac{1}{2}b^2 \qquad\qquad \int_{0}^{b}x^2\,dx=\frac{1}{3}b^3$$

Now let's solve this integral now using FTC I. Using the integration power rule, the antiderivative, with $C=0$, for the given integral is as shown.

$$F(x)=\frac{9}{2}x^2-\frac{4}{3}x^3$$

With this we can directly use FTC I evaluate the integral.

$$\begin{aligned}\int_{-3}^{3}(9x-4x^2)dx &= F(x)|_{-3}^{3}\\ &= \left(\frac{9}{2}3^2-\frac{4}{3}3^3\right)-\left(\frac{9}{2}(-3)^2-\frac{4}{3}(-3)^3\right)\\ &= \cancel{\frac{9}{2}3^2}-\frac{4}{3}3^3-\cancel{\frac{9}{2}(-3)^2}+\frac{4}{3}(-3)^3\\ &= \frac{4}{3}(-27-27)\\ &= -\frac{216}{3}=-72\end{aligned}$$

Although this is in no way a formal proof, we will accept the validity of FTC I in general and use it exclusively to evaluate definite integrals going forward. Let's do a few more examples to further illustrate the benefit of this theorem.

Example 1: Evaluate the following definite integrals using FTC I.

a. $\int_0^2 (12x^5 + 3x^2 - 4x)\, dx$

b. $\int_1^{27} \left(\frac{t+1}{\sqrt{t}}\right) dt$

c. $\int_{\pi/4}^{3\pi/4} sin(\theta)\, d\theta$

d. $\int_0^{\pi/2} cos\left(\frac{1}{3}\theta\right) d\theta$

Solutions:

a.

$$\int_0^2 (12x^5 + 3x^2 - 4x)\, dx = \frac{12}{6}x^6 + \frac{3}{3}x^3 - \frac{4}{2}x^2 \Big|_0^2$$

$$= (2 \cdot 2^6 + \cancel{2^3 - 2 \cdot 2^2}) - (\cancel{2 \cdot 0^6 + 0^3 - 2 \cdot 0^2})$$

$$= 2^7 = 128$$

b.

$$\int_1^{27} \left(\frac{t+1}{\sqrt{t}}\right) dt = \int_1^{27} \left(t^{1/2} + t^{-1/2}\right) dt$$

$$= \frac{2}{3}t^{3/2} + 2t^{1/2} \Big|_1^{27}$$

$$= \left(\frac{2}{3} \cdot 27^{3/2} + 2 \cdot 27^{1/2}\right) - \left(\frac{2}{3} \cdot 1^{3/2} + 2 \cdot 1^{1/2}\right) \cong 101.26$$

c.

$$\int_{\pi/4}^{3\pi/4} sin(\theta)\, d\theta = -\,cos(\theta)\, \Big|_{\pi/4}^{3\pi/4}$$

$$= -(cos(3\,\pi/4) - cos(\pi/4))$$

$$= -\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

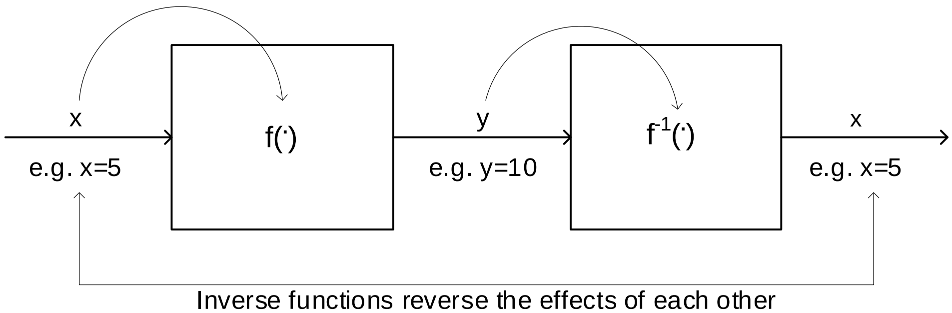
d.

$$\int_0^{\pi/2} cos\left(\frac{1}{3}\theta\right) d\theta = 3\, sin\left(\frac{1}{3}\theta\right) \Big|_0^{\pi/2}$$

$$= 3\left(\left(sin\left(\frac{\pi}{6}\right)\right) - (sin(0))\right) = 3\left(\frac{1}{2} - 0\right) = \frac{3}{2}$$

The Fundamental Theorem of Calculus Part 2

Recall in an earlier lesson we introduced the indefinite integral as notation for the general antiderivative. In that lesson we said that the indefinite integral is essentially the inverse process of differentiation. The second part of the fundamental theorem of calculus formalizes this notion. To start let's recall how inverse functions work using the diagram below.



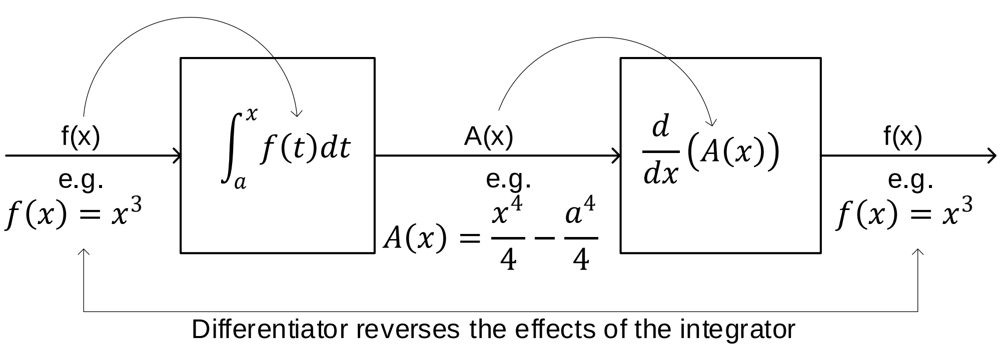
As we know, functions operate on numbers to produce new numbers. In the above diagram we input a value, x, e.g. $x = 5$, to the function, f, which produces an output y, e.g. $y = 10$. This output is then used as input to the inverse function, f^{-1}, which then outputs the original input value, x, e.g $x = 5$. In other words, the inverse function is like an "undo" function, i.e. it reverses the effects of the original function.

We can use the above as an analogy for integrators and differentiators that operate in a similar manner. The difference being that these entities operate on *functions* to produce new *functions*. The claim of FTC II is that differentiation will reverse the effects of integration. In other words, a differentiator is an inverse operator with respect to an integrator. For this purpose, we define an integral with a *variable* bound of integration, which will output a function. We define this integral below and use it in the statement of FTC II.

$$\int_a^x f(t)dt = A(x)$$

Where, $A(x)$, referred to as the area function, is the signed area from a to x.

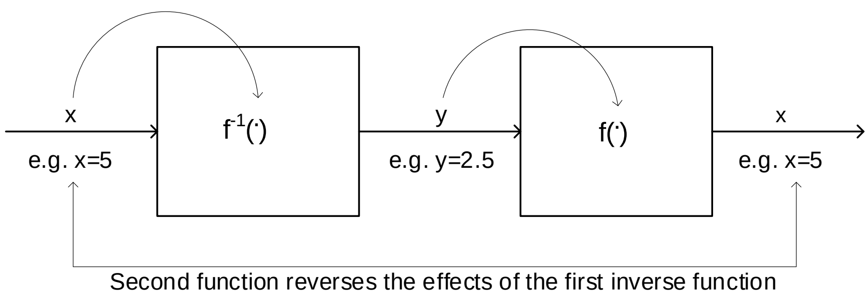
With this definition we can now formally state FTC II. The figure below is meant to provide a visual representation of the stated theorem. Please refer to it as you read the theorem.



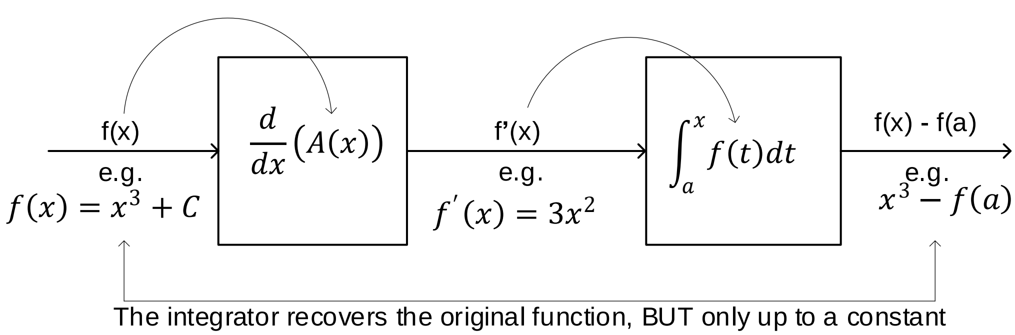
The Fundamental Theorem of Calculus Part 2
Assume that f is continuous on an open interval, I, and let a be a point in I. The area function $$A(x) = \int_a^x f(t)dt$$ is an antiderivative of f on I. In other words $$\frac{d}{dx}\big(A(x)\big) = f(x)$$ $$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Similar to the first part of FTC we will not provide a formal proof of this theorem, however instead of just showing an example we will attempt to provide a physical intuition. Before we do that however, we will explore the theorem as presented one step further.

We start by returning back to our example of the inverse function. Note that if we switch the order of operations the result is the same; i.e. the second function still reverse the effect of the first function.



Now let's do the same with our calculus operators and see what happens. The diagram below shows that if you differentiate first you can still recover $f(x)$, ***but*** only up to an unknown constant, $f(a)$.



We summarize the inverse relationship between differentiation and integration below.

Integration and Differentiation Inverse Relationship

If we integrate first and then differentiate, we get the original function back.

$$f(x) \overset{Integrate}{\longrightarrow} \int_a^x f(t)dt \overset{Differntiate}{\longrightarrow} \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

If we differentiate first and then integrate, we get the original function back, BUT only up to a constant, $f(a)$.

$$f(x) \overset{Differntiate}{\longrightarrow} f'(x) \overset{Integrate}{\longrightarrow} \int_a^x f'(t)dt = f(x) - \boldsymbol{f(a)}$$

Physical Intuition for FTC II

In this section we provide a heuristic explanation of the inverse relationship between integration and differentiation in order to provide physical intuition. For this purpose, we use the example of position, $x(t)$, and velocity, $v(t)$ functions.

The average velocity of an object over $[t_1, t_N]$, whose position function is given by $x(t)$, is

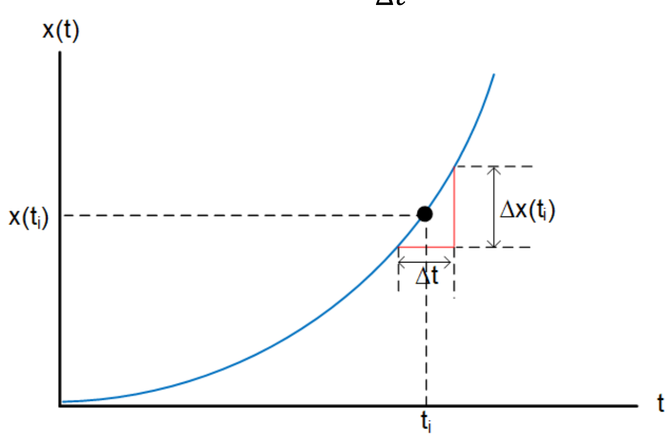
$$v_{avg} = \frac{x(t_N) - x(t_1)}{t_N - t_1} = \frac{\Delta x}{\Delta t}$$

The instantaneous velocity at t_1 is obtained by allowing $t_N \to t_1$, (i.e. $\Delta t \to 0$). We refer to this as the derivative of $x(t)$ evaluated at $t = t_1$.

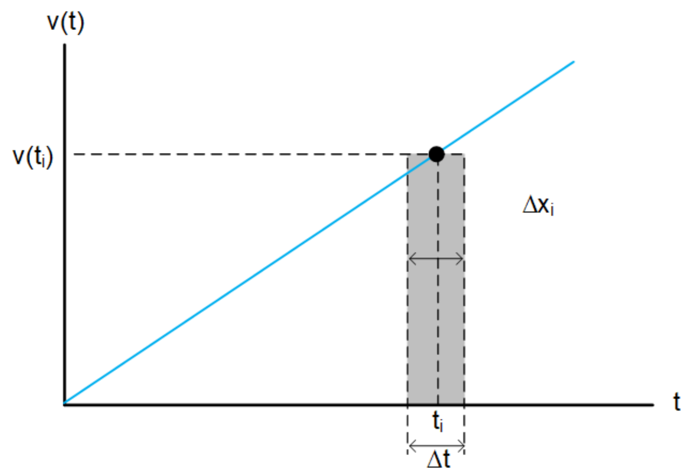
$$v_{inst}(t_1) = \frac{d}{dt}\left(x(t)\right)\Big|_{t=t_1} = \lim_{t_N \to t_1}\left\{\frac{x(t_N) - x(t_1)}{t_N - t_1}\right\} = \lim_{\Delta t \to 0}\left\{\frac{\Delta x}{\Delta t}\right\}$$

Let's remove the strictness of the limit to gain additional insight. Using the figure below for illustration, we create a time interval, Δt, centered at t_i, and a corresponding position interval, $\Delta x(t_i)$. We can then approximate the velocity at t_i, (i.e. the derivative of $x(t)$ at t_i), as

$$v(t_i) \cong \frac{\Delta x(t_i)}{\Delta t}$$



If we continue this process using, for example $i = 1 \ldots N$, we can graph the velocity, $v(t_i)$ of the object over, $[t_1, t_N]$. The figure below shows an example velocity function in blue, $v(t_i)$, where we may assume the sample points are so close together that the curve looks smooth.



Since $v(t_i)$ is essentially the slope at time t_i, we see that the slope starts at zero and increases as we increment i, as we would expect from looking at $x(t)$. Next, we take the equation we used to approximate the $v(t_i)$ and solve it for $\Delta x(t_i)$, i.e., compute its inverse.

$$v(t_i) \cong \frac{\Delta x(t_i)}{\Delta t}$$
$$\Delta x(t_i) \cong v(t_i)\Delta t$$

This function allows us to approximate the change in distance for each subinterval. Looking at the figure again we can see that the right-hand side of this equation is the area of a rectangle with height $v(t_i)$ and width Δt. Of course, to find the total change in distance over the interval, $[t_1, t_N]$, we add each of the subintervals.

$$\Delta x \cong \sum_{i=1}^{N} v(t_i)\Delta t$$

Remarkedly, this is essentially how we defined the integral of a function, less the strictness of the limit of course. Note that Δx is a change in distance and does not tell us the actual position of the object, $x(t_i)$. We can obtain the position by substituting $\Delta x = x(t_N) - x(t_1)$ as follows:

$$x(t_N) - x(t_1) \cong \sum_{i=1}^{N} v(t_i)\Delta t$$
$$x(t_N) \cong \sum_{i=1}^{N} v(t_i)\Delta t + x(t_1)$$

Finally, if we imagine continuing this process for $N + 1, N + 2, etc.$, we are able to construct a graph of the position from the given velocity graph.

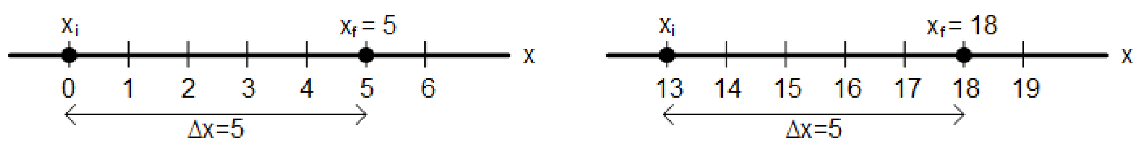
Let's review what we have done. We started with a position function, $x(t)$, and found a velocity function, $v(t)$, by approximating the differentiation process.

$$v(t_i) \cong \frac{\Delta x(t_i)}{\Delta t}$$

We then performed an approximation to the integration process to get back the original position function.

$$x(t_N) \cong \sum_{i=1}^{N} v(t_i)\Delta t + x(t_1)$$

Not only does this example provide physical intuition for the fact that differentiation and integration are inverse operations, but also gives a physical meaning as to why, when we perform differentiation first, we get the position function back only up to a constant. As we have seen the summation, i.e., integration, provides the change in position only. The figure below illustrates, for two specific examples, that even with the same change in distance there are an infinite number of final positions depending on the initial position of the object.



Let's do a few examples to become more familiar with this part of the theorem.

Example 2: Integrate the velocity function, $v(t) = 3t^2 + 2t - 4, t \geq 0$, to get a position function. Then differentiate the resulting position function to show that you can fully recover the original velocity function.

Integrating for the position function we have:

$$x(t) = \int (3t^2 + 2t - 4)dt$$

$$x(t) = t^3 + t^2 - 4t + C$$

Differentiating to verify FTC II.

$$v(t) = \frac{d}{dt}(t^3 + t^2 - 4t)$$

$$v(t) = 3t^2 + 2t - 4$$

Example 3: Differentiate the position function, $x(t) = 8t^3 + 2t^2 + 4, t \geq 0$, to get a velocity function. Then integrate the resulting velocity function to show that you can only recover the original position function up to a constant.

Differentiating for the position function we have:

$$v(t) = \frac{d}{dt}(8t^3 + 2t^2 + 4)$$

$$v(t) = 24t^2 + 4t$$

Integrating we have:

$$x(t) = \int (24t^2 + 4t)dt$$

$$x(t) = (8t^3 + 2t^2) + C$$

Assuming we started with $v(t)$ we have no way of computing the unknown constant, C. In order to find the full position function, we need to be given an "initial condition". In this case, and for illustration we can use $x(t) = 4$. Therefore, the position function is found as follows

$$x(0) = (8 \cdot 0^3 + 2 \cdot 0^2) + C = 4, \rightarrow C = 4 \ \rightarrow x(t) = 8t^3 + 2t^2 + 4$$

Example 4: Differentiate $F(x)$, as defined below. Then find $F'(2)$ and $F(2)$.

$$F(x) = \int_2^x \left(\sqrt{1 + t^3}\right) dt$$

Using FTC II, we can easily differentiate this integral function.

$$F'(x) = \frac{d}{dx}\left(\int_2^x \left(\sqrt{1 + t^3}\right) dt\right) = \sqrt{1 + x^3}$$

Furthermore

$$F'(2) = \sqrt{1 + 2^3} = \sqrt{9} = 3$$

On the other hand, we have:

$$F(2) = \int_2^2 \left(\sqrt{1 + t^3}\right) dt = 0$$

FTC combined with the Chain Rule:

Note the area function defined for FTC II uses an integral with a constant lower limit and an upper limit of x. In this case the derivative of the area function is equal to the integrand, as we have shown. In some cases, however, the lower and/or upper limit may be functions of x. When this is the case, we need to utilize the chain rule to differentiate the integral. The general case is shown below, where we start by using FTC I.

$$\int_{l(x)}^{u(x)} f(t)dt = F\big(u(x)\big) - F\big(l(x)\big)$$

Next, we differentiate using the chain rule on the right side and the fact that $F'(x) = f(x)$.

$$\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = \frac{d}{dx}\Big(F\big(u(x)\big)\Big) - \frac{d}{dx}\Big(F\big(l(x)\big)\Big)$$

$$\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = F'\big(u(x)\big)u'(x) - F'\big(l(x)\big)l'(x)$$

$$\boldsymbol{\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = f\big(u(x)\big)u'(x) - f\big(l(x)\big)l'(x)}$$

We can show that this expression evaluates to satisfy FTC II when the integral is the area function as defined previously, i.e. $l(x) = a$ and $u(x) = x$.

$$\begin{aligned} \frac{d}{dx}\left(\int_{l(x)=a}^{u(x)=x} f(t)dt\right) &= f\big(u(x)\big)u'(x) - f\big(l(x)\big)l'(x) \\ &= f(x)\cdot 1 - f(a)\cdot 0 \\ \frac{d}{dx}\left(\int_{a}^{x} f(t)dt\right) &= f(x) \end{aligned}$$

The generalized version of the derivative of an integral is given below.

Generalized Derivative of an Integral
$\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = f\big(u(x)\big)u'(x) - f\big(l(x)\big)l'(x)$

Let's see how this formula is used with a few examples.

Example 5: Find the derivative of

$$G(x) = \int_{-2}^{x^2} sin(t)\,dt$$

Using the generalized formula, we have

$$\begin{aligned}\frac{d}{dx}\big(G(x)\big) &= \frac{d}{dx}\left(\int_{-2}^{x^2} sin(t)\,dt\right)\\ &= \left(sin(x^2)\frac{d}{dx}(x^2)\right) - \left(sin(-2)\frac{d}{dx}(-2)\right)\\ &= (sin(x^2)2x) - (sin(-2)\cdot 0)\\ &= 2xsin(x^2)\end{aligned}$$

Example 6: Find the derivative of

$$G(x) = \int_{1/x}^{1} cos^3(t)\,dt$$

Solution:

$$\begin{aligned}\frac{d}{dx}\big(G(x)\big) &= \frac{d}{dx}\left(\int_{1/x}^{1} cos^3(t)\,dt\right)\\ &= (cos^3(1)\cdot 0) - \left(cos^3(1/x)\left(-\frac{1}{x^2}\right)\right)\\ &= \frac{cos^3(1/x)}{x^2}\end{aligned}$$

Example 7: Find the derivative of

$$G(x) = \int_{x^2}^{x^4} tan(t)\,dt$$

Solution:

$$\begin{aligned}\frac{d}{dx}\big(G(x)\big) &= \frac{d}{dx}\left(\int_{x^2}^{x^4} tan(t)\,dt\right)\\ &= \big(tan(x^4)\,(4x^3)\big) - \big(tan(x^2)\,(2x)\big)\end{aligned}$$

Final Summary for Integration – The Fundamental Theorems of Calculus

The Fundamental Theorems of Calculus
The fundamental theorems of calculus reveal the deep connection between the two main branches of calculus, i.e. differentiation and integration. It is one of the most important theorems in mathematics. The theorem is split into two parts. Part 1 allows for the computation of definite integrals without having to use the Riemann sum. Part 2 formally defines the inverse relationship between integration and differentiation.
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Assume that f is continuous on an open interval, I, and let a be a point in I. The area function $$A(x) = \int_a^x f(t)dt$$ is an antiderivative of f on I. In other words $$\frac{d}{dx}\big(A(x)\big) = f(x)$$ $$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$
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If we integrate first and then differentiate, we get the original function back. $$f(x) \overset{Integrate}{\longrightarrow} \int_a^x f(t)dt \overset{Differntiate}{\longrightarrow} \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$
If we differentiate first and then integrate, we get the original function back, BUT only up to a constant $f(a)$. $$f(x) \overset{Differntiate}{\longrightarrow} f'(x) \overset{Integrate}{\longrightarrow} \int_a^x f'(t)dt = f(x) - \boldsymbol{f(a)}$$
Generalized Derivative of an Integral
$$\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = f\big(u(x)\big)u'(x) - f\big(l(x)\big)l'(x)$$

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