# Integration – The Definite Integral

In the previous section we saw that the area under a graph can be approximated using properly constructed rectangles. Additionally, by letting the rectangles become infinitesimally small, i.e., letting the number of subintervals go to infinity, the approximation becomes exact.

| Area Under the Graph |
| --- |
| If $f$ is continuous on $[a, b]$, then the endpoint and midpoint approximations approach the same value, $A$, in the limit as $N \to \infty$. In other words: $$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = A$$ If $f(x) > 0$ on $[a, b]$, then $A$ represents the area under the graph of $f(x)$ on $[a, b]$ |

In this lesson we show this area is defined as the definite integral of $f$ over $[a, b]$. Before we do, however, we will introduce a more general area approximation called the *Riemann Sum*.

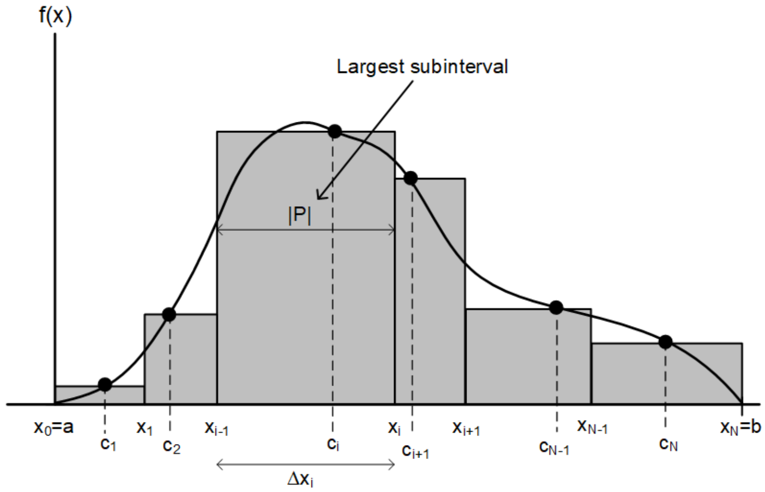
Similar to the endpoint and midpoint approximations, the Riemann sum also uses rectangles to approximate the area under a graph. Recall, however, with the endpoint and midpoint approximations, the width of the rectangles is fixed, $\Delta x = \frac{b-a}{N}$, and the heights are at a specified value, i.e. endpoints or midpoints. With the Riemann sum the width of each rectangle is not fixed, and the heights can be at an arbitrary point within each subinterval. We formally introduce the procedure below.

We start, as we have done in the past, by choosing an interval, $[a, b]$, and divide it into $N$ subintervals. We call this a partition, $P$, and in this case we choose the points, $x_0, x_1, \ldots, x_N$, that divide the subintervals as follows:

$$P: a = x_0 < x_1 < x_2 \cdots < x_N = b$$

Therefore, the length of each subinterval is: $\Delta x_i = x_i - x_{i-1}$

Next, we choose the sample points, $C = \{c_1, c_2, \cdots, c_N\}$, within each subinterval such that $c_i$ belongs to the subinterval, $[x_{i-1}, x_i]$. The figure below illustrates the above procedure.




Each rectangle has a variable width of $\Delta x_i$ and a height of $f(c_i)$. The Riemann sum can then be defined as a function of the partition, $P$, the set of sample points, $C$, and the function to be operated on, $f$, as shown below.

| **Riemann Sum** |
| --- |
| $R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i$ |

We refer to the largest subinterval as the norm of $P$, denoted as $\|P\|$, and you can imagine that as $\|P\| \to 0$, this summation tends to the exact area under the graph in the same manner as the endpoint and midpoint approximations did. Recall, from the previous section we mentioned that if the function, $f$, takes on negative values we cannot *strictly* interpret the summation as the area under the graph. In the sense we will call this limit, $L$, instead of $A$ as we have done previously.

$$\lim_{\|P\| \to 0} \{R(f, P, C)\} = \lim_{\|P\| \to 0} \left\{ \sum_{i=1}^{N} f(c_i)\Delta x_i \right\} = L$$

This limit, $L$, is called the ***definite integral*** of $f$ over $[a, b]$, which is formally defined below.

| **Definite Integral Definition** |
| --- |
| The definite integral of $f$ over $[a, b]$, denoted by the integral sign, is the limit of the Riemann Sum.<br><br>$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \{R(f, P, C)\} = \lim_{\|P\| \to 0} \left\{ \sum_{i=1}^{N} f(c_i)\Delta x_i \right\}$<br><br>When this limit exists, we say $f$ is integrable over $[a, b]$. |

As the definite integral notation, $\int_a^b f(x)dx$, is new to us we provide comments on it below.

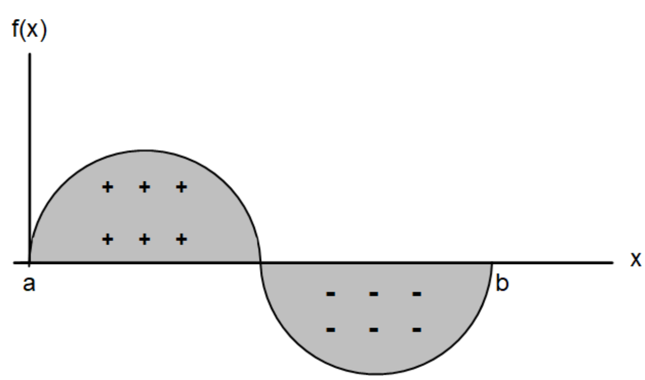
The definite integral is more commonly called the *integral* of $f$ over $[a, b]$. The process of computing the integral is called *integration*. The function, $f(x)$, is called the *integrand* and the points $a$ and $b$ are called the *limits of integration,* where $a$ is the *lower limit* and $b$ is the *upper limit*. The variable $x$ is called the *integration variable*. This variable is a "dummy variable" and can be replaced with any convenient variable. The symbol " $\int$ " is an elongated $S$ signifying that the integral is just the limit of the sums as the value to be summed becomes exceedingly small and the number of values to be summed becomes infinite. Although $dx$ has no formal mathematical meaning by itself it is heuristically considered an infinitesimal version of $\Delta x_i$. Interpreting as such proves useful in various applied problems and in some informal derivations that we will encounter in future lessons.

Lastly, we state the following theorem, which assures us that most functions are integrable. We usually rely on this theorem rather than try to directly prove various functions are integrable.

| **Integrable Theorem** |
|---|
| If $f$ is continuous on $[a, b]$, or if $f$ is continuous with at most finitely many jump discontinuities, then $f$ is *integrable* on $[a, b]$. |

**The Definite Integral as the Signed Area:**

When $f(x) \geq 0$ the definite integral defines the area under the graph. When $f(x)$ takes on both positive and negative values the definite integral results in positive area for portions of the graph that are above the $x$-axis and "*negative area"* for portions of the graph that are below the $x$-axis. Therefore, we can more generally say that the definite integral defines a so-called *signed area* of a graph. The figure below illustrates a graph with both positive and negative area on $[a, b]$.

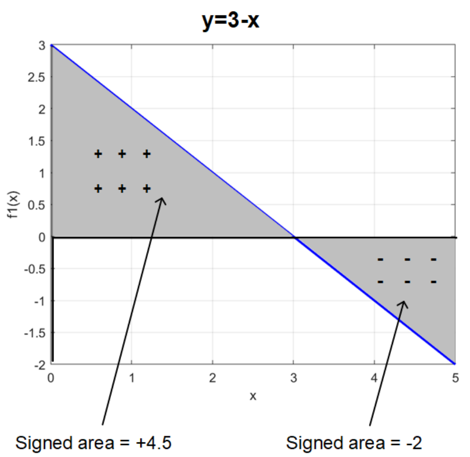


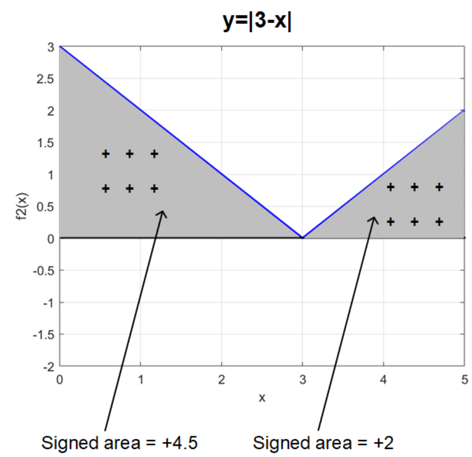

Let's look at a small example to illustrate the idea of signed area.

Calculate the definite integral, *signed area*, over $[0,5]$ for the following two functions.

$$f_1(x) = 3 - x \qquad\qquad f_2(x) = |3 - x|$$

The graphs of these equations are both linear function and therefore, we can compute the definite integral using basic geometry. The graphs are shown below.




Based on the figures above we can write the definite integrals as follows:

$$\int_0^5 (3-x)\,dx = (+4.5) + (-2) = 2.5 \qquad \int_0^5 |3-x|\,dx = (+4.5) + (+2) = 6.5$$

**Definite Integral Formulas:**

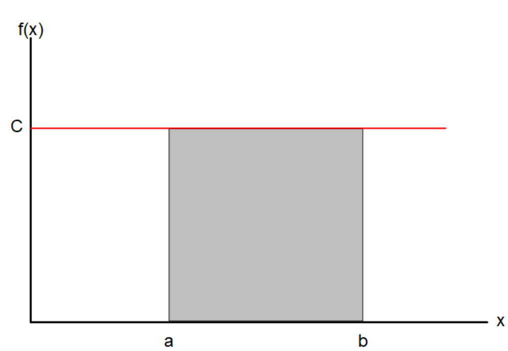
In the previous lesson we encountered some formulas for computing the area. We can now restate these formulas into the language of the definite integral.

1. *The Definite Integral of a Constant,* $f(x) = C$:

We first encountered this formula in the previous section with regard to straight line constant velocity, where we defined the following:

| **Straight Line Constant Velocity Distance Traveled** |
|---|
| $D = v\Delta t$ <br> Where, $\Delta t = t_2 - t_1$ |

We can now interpret this using the language of the definite integral. We do this below using a generic function, $f(x) = C$.




$$\int_a^b C\,dx = C(b-a)$$

2. *The Definite Integral of* $f(x) = x$:

In the previous section we derived the following formula:

| The area under the graph of $f(x) = x$ on $[0, b]$, is given as: $$A = \frac{1}{2}b^2$$ |
|---|

Which we can now restate using the definite integral as:

$$\boxed{\int_0^b x\,dx = \frac{1}{2}b^2}$$

3. *The Definite Integral of* $f(x) = x^2$:

In the previous section we derived the following formula:

| The area under the graph of $f(x) = x^2$ on $[0, b]$, is given as: $$A = \frac{1}{3}b^3$$ |
|---|

Which we can also now restate using the definite integral as:

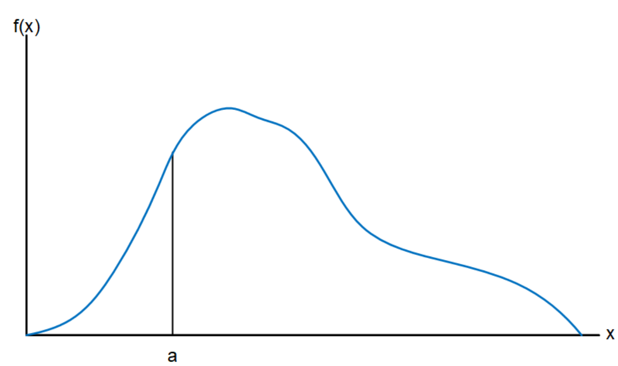
$$\boxed{\int_0^b x^2\,dx = \frac{1}{3}b^3}$$

**Properties of Definite Integrals:**

We now introduce some basic properties of definite integrals. These properties will greatly aid us in the evaluation of various, sometimes complicated looking, definite integrals that we will encounter as we progress.

1. *The Zero Rule*:

$$\int_a^a f(x)\,dx = 0$$




The vertical line region contains no area.

2. *The Order of Integration*:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

If the endpoints of integration are reversed, the value of the definite integration changes sign!

3. *Linearity Properties of Integration*:

**Sum and Difference Property**

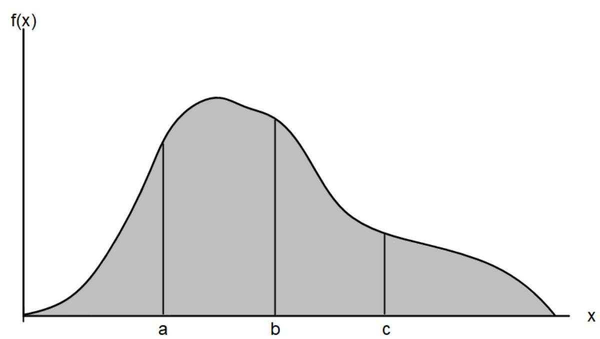
$$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

**Constant Multiplier Property**

$$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$$

4. *Additivity of Adjacent Regions*:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$




Let's finish this lesson by evaluating the following definite integrals using the formulas and properties from above.

*a.* $\int_0^4 (6t - 3)\,dt$

*b.* $\int_0^9 x^2\,dx$

*c.* $\int_0^1 (x^2 - 2x)\,dx$

*d.* $\int_{-3}^1 (7u^2 + u + 1)\,du$

*e.* $\int_{-a}^1 (x^2 + x)\,dx$

*f.* $\int_{-3}^3 (9x - 4x^2)\,dx$

Solutions:

*a.*

$$\begin{aligned}\int_0^4 (6t-3)\,dt &= \int_0^4 6t\,dt - \int_0^4 3\,dt\\ &= 6\int_0^4 t\,dt - 3(4-0)\\ &= 6\left(\frac{1}{2}4^2\right) - 12\\ &= 36\end{aligned}$$

*b.*

$$\int_0^9 x^2\,dx = \frac{1}{3}9^3 = 243$$

*c.*

$$\begin{aligned}\int_0^1 (x^2-2x)\,dx &= \int_0^1 x^2\,dx - 2\int_0^1 x\,dx\\ &= \left(\frac{1}{3}1^3\right) - 2\left(\frac{1}{2}1^2\right)\\ &= -\frac{2}{3}\end{aligned}$$

*d.*

$$\begin{aligned}\int_{-3}^1 (7u^2+u+1)\,du &= \int_{-3}^0 (7u^2+u+1)\,du + \int_0^1 (7u^2+u+1)\,du\\ &= -\int_0^{-3} (7u^2+u+1)\,du + 7\int_0^1 u^2\,du + \int_0^1 u\,du + \int_0^1 1\,du\\ &= -\left(7\int_0^{-3} u^2\,du + \int_0^{-3} u\,du + \int_0^{-3} 1\,du\right) + 7\left(\frac{1^3}{3}\right) + \left(\frac{1^2}{2}\right) + \big(1(1)\big)\\ &= -\left(7\left(\frac{1}{3}(-3)^3\right) + \left(\frac{1}{2}(-3)^2\right) + \big(1(-3)\big)\right) + \left(\frac{7}{3}\right) + \left(\frac{1}{2}\right) + (1)\\ &= \left(\frac{189}{3}\right) - \left(\frac{9}{2}\right) + (3) + \left(\frac{7}{3}\right) + \left(\frac{1}{2}\right) + (1)\\ &= \left(\frac{378}{6}\right) - \left(\frac{27}{6}\right) + \left(\frac{18}{6}\right) + \left(\frac{14}{6}\right) + \left(\frac{3}{6}\right) + \left(\frac{6}{6}\right)\\ &= \frac{392}{6} = \frac{196}{3}\end{aligned}$$

*e.*

$$\begin{aligned}\int_{-a}^{1}(x^2+x)\,dx &= \int_{-a}^{0}(x^2+x)\,dx+\int_{0}^{1}(x^2+x)\,dx\\ &= -\left(\int_{0}^{-a}x^2\,dx+\int_{0}^{-a}x\,dx\right)+\int_{0}^{1}x^2\,dx+\int_{0}^{1}x\,dx\\ &= -\left(\left(\frac{1}{3}(-a)^3\right)+\left(\frac{1}{2}(-a)^2\right)\right)+\left(\frac{1}{3}(1)^3\right)+\left(\frac{1}{2}(1)^2\right)\\ &= \left(\frac{a^3}{3}\right)-\left(\frac{a^2}{2}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\\ &= \left(\frac{2a^3}{6}\right)-\left(\frac{3a^2}{6}\right)+\left(\frac{2}{6}\right)+\left(\frac{3}{6}\right)\\ &= \frac{1}{6}(2a^3-3a^2+5)\end{aligned}$$

*f.*

$$\begin{aligned}\int_{-3}^{3}(9x-4x^2)\,dx &= \int_{-3}^{0}(9x-4x^2)\,dx+\int_{0}^{3}(9x-4x^2)\,dx\\ &= -\left(9\int_{0}^{-3}x\,dx-4\int_{0}^{-3}x^2\,dx\right)+9\int_{0}^{3}x\,dx-4\int_{0}^{3}x^2\,dx\\ &= -\left(9\left(\frac{1}{2}(-3)^2\right)-4\left(\frac{1}{3}(-3)^3\right)\right)+9\left(\frac{1}{2}(3)^2\right)-4\left(\frac{1}{3}(3)^3\right)\\ &= \left(-\frac{81}{2}\right)-\left(\frac{108}{3}\right)+\left(\frac{81}{2}\right)-\left(\frac{108}{3}\right)\\ &= -\frac{216}{3} = -72\end{aligned}$$

## ***Final Summary for Integration – The Definite Integral***

### Riemann Sum

The Riemann Sum is a general method for computing the *signed area* of a graph over a given interval, and is defined as

$$R(f,P,C)=\sum_{i=1}^{N} f(c_i)\Delta x_i$$

Where,

- $f$ represents the function whose graph we are computing the *signed area* for.
- $P$ defines the partition of the interval, $[a,b]$, for which the sum is computed over. The points, $x_i$, divide the interval into $N$ subintervals.
  $$P\colon a=x_0<x_1<x_2\cdots<x_N=b$$
- $C=\{c_1,c_2,\cdots c_N\}$ are the sample points within each subinterval where the function is evaluated.
- $\Delta x_i$ is the width of each subinterval.
  $$\Delta x_i=x_i-x_{i-1}$$

### Definite Integral Definition

The definite integral of $f$ over $[a,b]$, denoted by the integral sign, is the limit of the Riemann Sum.

$$\int_a^b f(x)dx=\lim_{\|P\|\to 0}\{R(f,P,C)\}=\lim_{\|P\|\to 0}\left\{\sum_{i=1}^{N} f(c_i)\Delta x_i\right\}$$

When this limit exists, we say $f$ is integrable over $[a,b]$.
$\|P\|$ is defined as the largest of the lengths, $\Delta x_i$.

**$\int_a^b f(x)dx$ represents the signed area of $f(x)$ over $[a,b]$.**

### Integrable Theorem

If $f$ is continuous on $[a,b]$, or if $f$ is continuous with at most finitely many jump discontinuities, then $f$ is *integrable* on $[a,b]$.

### Definite Integral Formulas

$$\int_a^b C\,dx=C(b-a) \qquad \int_0^b x\,dx=\frac{1}{2}b^2 \qquad \int_0^b x^2\,dx=\frac{1}{3}b^3$$

| Definite Integral Properties |
| --- |
| *The Zero Rule:* $$\int_a^a f(x)dx = 0$$ |
| *Order of Integration:* $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$ |
| *Linearity Properties:* **Sum and Difference Property** $$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$ **Constant Multiplier Property** $$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$$ |
| *Additivity of Adjacent Regions:* $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$ |

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