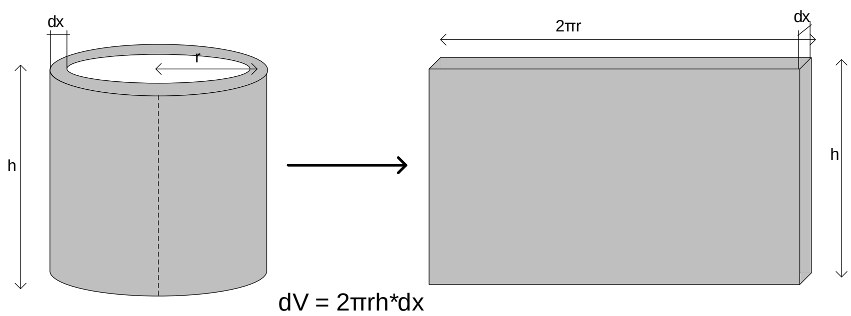
Integral Applications – Volumes of Revolution: Shell Method

In the previous section we computed the volume of solid objects that were constructed by rotating regions in the x-y plane around an axis using infinitesimal circular disks. A circular disk is a convenient shape since we know the formula for the area of a circle, i.e. πr^2. We now ask the question; "Is there another convenient shape that can be used to compute the volume of such objects?". The answer is yes, and the shape, which we will use in this section, is a cylindrical shell. To be sure of its usability let's derive a formula for the volume of a cylindrical shell with an infinitesimal thickness. We'll use the figure below to illustrate.



The figure on the left shows a cylindrical shell with height h, radius r, and infinitesimal thickness dx. To find the volume we cut the shell along the dotted line. When we do this, we have a rectangular surface as shown on the right. The height of the rectangle is h, and the length is equal to the circumference of the original shell, i.e. $2\pi r$. The area of the rectangle is then $2\pi rh$, and since the shell has a thickness of dx, the infinitesimal volume is represented as

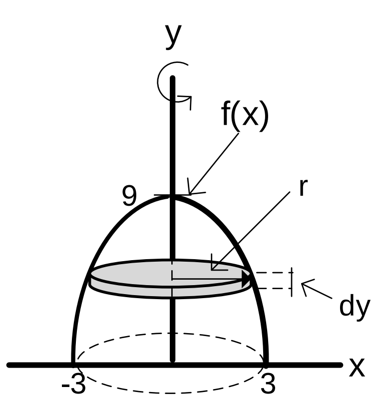
$$dV = 2\pi rhdx$$

Single Function Rotated About y-axis (Basic Shell Method)

We will introduce the basic shell method by redoing example 4 from the previous section.

Example 1 (4 from previous section): Find the volume of the solid generated by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the y-axis.

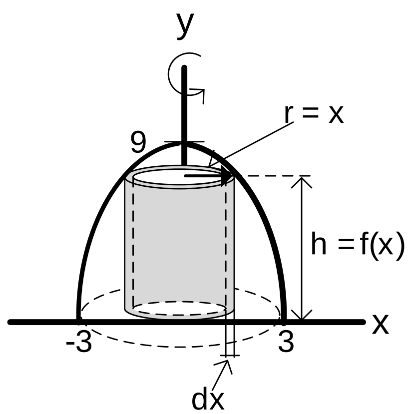
Solution: For the disk method we were required to find the inverse of $f(x)$, i.e. $x(y) = \sqrt{9-y}$, since the radius of the disk is equal to the x. For review, the solution, along with the figure from the previous section, is shown below.



$$V = \pi \int_0^9 (9-y)dy$$

$$V = \pi \left(9y - \frac{1}{2}y^2 \Big|_0^9 \right) = 40.5\pi$$

Let's see what happens when we instead build the solid from cylindrical shells.



The figure shows one particular shell with a radius equal to x, a height of $f(x)$, and an infinitesimal width of dx. The infinitesimal volume of this shell is therefore

$$\begin{aligned} dV &= 2\pi r h dx \\ &= 2\pi x f(x) dx \end{aligned}$$

Similar to the disk method, the volume of the entire object is obtained by integrating the shells as we vary x from 0 to 3.

$$\begin{aligned} V &= \int_0^3 2\pi x f(x) dx \\ &= 2\pi \int_0^3 x(9 - x^2) dx \\ &= 2\pi \int_0^3 (9x - x^3) dx \\ &= 2\pi \left(\frac{9}{2}x^2 - \frac{1}{4}x^4 \Big|_0^3 \right) = 2\pi \left(\frac{2 \cdot 81}{4} - \frac{81}{4} \right) = 40.5\pi \end{aligned}$$

Which, as expected, is the same answer we computed using the disk method! For the shell method we also recommend sketching the graph and deriving the expression for each case. Nonetheless, we provide a formula for the basic shell method.

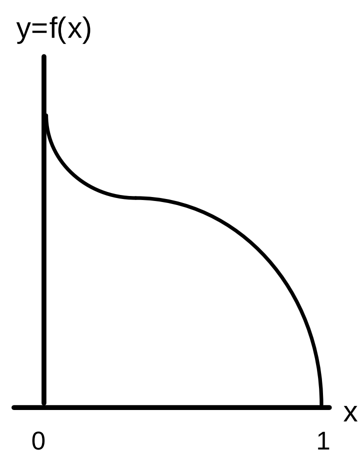
Volume of Revolution: Single Function Rotated About the y-axis (Basic Shell Method)
Consider a continuous function $f(x) \geq 0$ on $[a, b]$. Rotating this function about the y-axis creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $f(x)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $$V = \int_a^b (2\pi \cdot radius)(height) dx = 2\pi \int_a^b x f(x) dx$$

Since we now have two ways to compute volumes of revolutions, we might ask which method is simpler. The question is not simple to answer, however with some key observation we can provide some general guidelines.

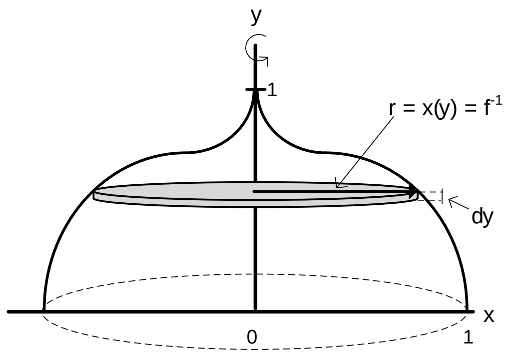
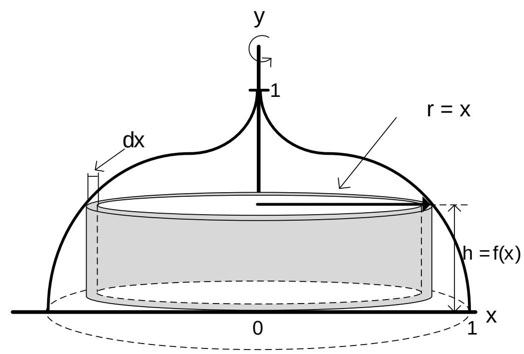
Disk vs. Shell Method General Guidelines
Observations: ***Disk Method:*** For horizontal axis rotation the disks are oriented such that the radius is directly related to the function(s) given. For vertical axis rotation the disks are oriented such that the radius is related to the inverse function(s) given. ***Shell Method:*** For vertical axis rotation the shells are oriented such that the height is directly related to the function(s) given. For horizontal axis rotation the shells are oriented such that the height is related to the inverse function(s) given. **General Guideline** Since finding the inverse function is not always simple, these observations suggest that we should choose the method which **does not** require the inverse to be computed. To avoid inverse functions, we should choose the disk method when the rotation is about a horizontal axis and the shell method when the rotation is about a vertical axis. Note: If the function(s) is easily inverted than either method should suffice.

Before moving on to other cases of the shell method let's take a look at a very similar problem to the one above in order to illustrate the guidelines.

Example 2: Find the volume of the solid generated by rotating the region under the graph of $f(x) = 1 - 2x + 3x^2 - 2x^3$, shown below, over $[0,1]$ about the y-axis.



Solution: The figures below illustrate the generated solid using both the disk method and the shell method.

<table>
<tr><th>Disk Method</th><th>Shell Method</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$V = \pi \int_0^1 \left(f^{-1}(x)\right)^2 dy$</td><td>$V = 2\pi \int_0^1 x f(x) dx$</td></tr>
</table>

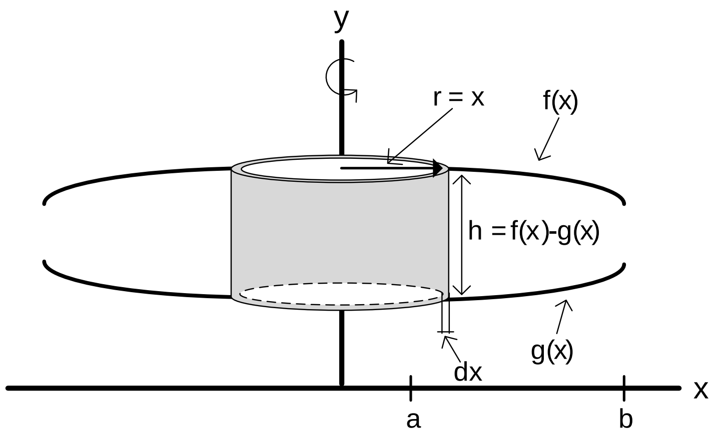
Since we are rotating about a vertical axis the disk method, as in the last problem, requires the inverse function. However, unlike the previous case the inverse of the current function is not easily computed. For the shell method the inverse function is not required, and therefore in this case, as the guidelines suggest, it is much easier to use the shell method. We compute the volume using this method below.

$$
\begin{aligned}
V &= 2\pi \int_0^1 x(1 - 2x + 3x^2 - 2x^3)dx \\
&= 2\pi \int_0^1 (x - 2x^2 + 3x^3 - 2x^4)dx \\
&= 2\pi \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \Big|_0^1\right) \\
&= 2\pi \left(\frac{30}{60} - \frac{40}{60} + \frac{45}{60} - \frac{24}{60}\right) = \pi \frac{11}{30}
\end{aligned}
$$

Similar to the disk method there are many variations to the basic shell method. Also similar is the strong suggestion that a sketch of the graph should be made for each problem and the integral should be derived from it. However, we did identify three general cases for the disk method, and we will generalize those same three types for the shell method for completeness.

Volume for a Region Between Two Curves

Suppose we have two functions, $f(x)$ and $g(x)$, $f(x) > g(x)$ on $[a, b]$. Next, we rotate these functions about the y-axis and consider the solid generated by only the region between the two functions.



The radius of the shell is still equal to x, however the height is shown to be $f(x) - g(x)$. The infinitesimal volume of the shell is therefore

$$\begin{aligned} dV &= 2\pi r h dx \\ &= 2\pi x\big(f(x) - g(x)\big)dx \end{aligned}$$

And the total volume is

$$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$

Volume of Revolution: Region Between Two Curves: (Shell Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a, b]$. Rotating these functions about the y-axis while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $\big(f(x) - g(x)\big)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$

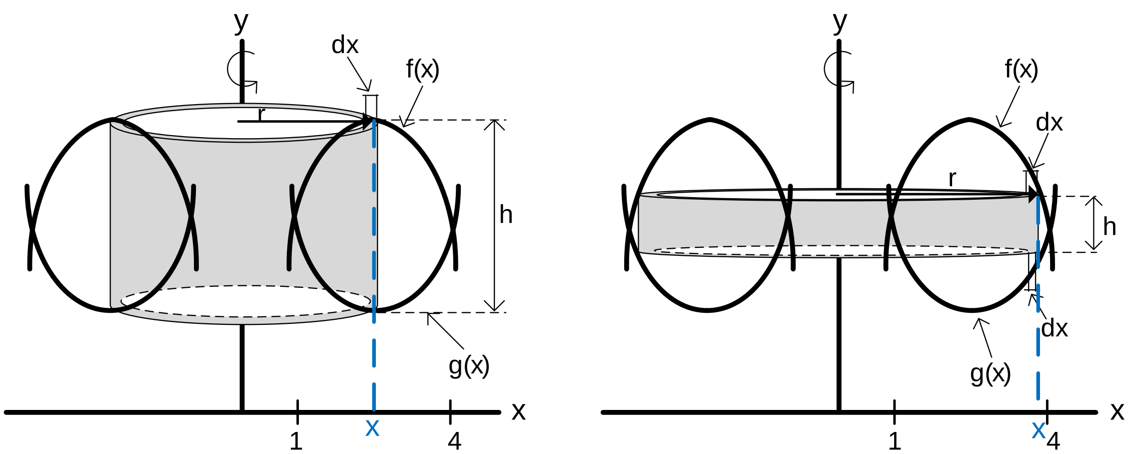
Let's do an example to illustrate.

Example 3: Find the volume of the solid generated using the region between $f(x) = x(5-x)$ and $g(x) = 8 - x(5-x)$, when they are rotated about the y-axis.

Solution: We start by finding the intersection of the curves by setting the functions equal to each other.

$$
\begin{aligned}
x(5-x) &= 8 - x(5-x) \\
5x - x^2 &= 8 - 5x + x^2 \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0 \\
x &= 1,4
\end{aligned}
$$

Next, we sketch the graph and derive the integral. To illustrate we show two different shells for two values of the integration variable x.



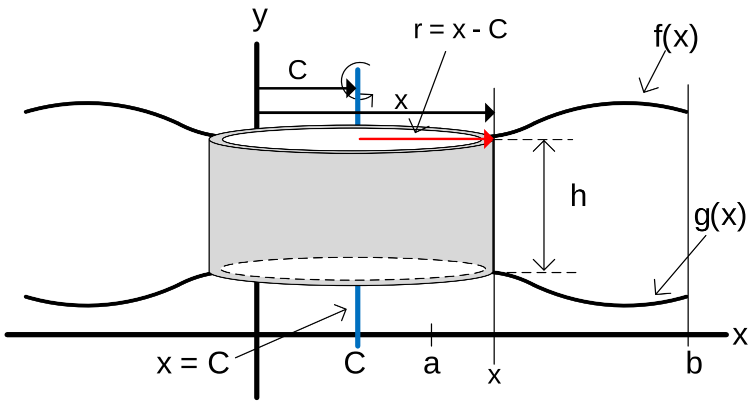
Based on the figures, the volume is computed as follows.

$$
\begin{aligned}
V &= 2\pi \int_1^4 x\left(f(x) - (g(x))\right) dx \\
&= 2\pi \int_1^4 x\left(x(5-x) - (8 - x(5-x))\right) dx \\
&= 2\pi \int_1^4 x(5x - x^2 - 8 + 5x - x^2) dx \\
&= 2\pi \int_1^4 (-2x^3 + 10x^2 - 8x) dx \\
&= 2\pi \left(\frac{-1}{2}x^4 + \frac{10}{3}x^3 - 4x^2 \Big|_1^4\right) \\
&= 2\pi \left(\frac{64}{3} - \left(\frac{-7}{6}\right)\right) = 45\pi
\end{aligned}
$$

Rotation Around an Arbitrary Vertical Line

The shell method can also be used when the rotation axis is shifted from the y-axis. Just as with the disk method there are different variations for the volume expression based on the placement of the rotation axis. With the disk method we provided an expression for a specific variation, while reminding the reader that it is always best to sketch the graphs and derive the expression from basic principles. We will follow a similar path for the shell method.

We begin with two functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a, b]$, that we rotate about the vertical line $x = C$, where $C \leq a$. The figure below shows the solid being generated with shells.



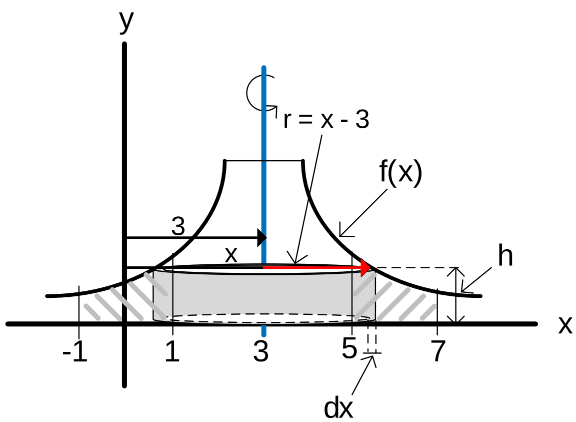
As you can see the radius is no longer the x value but is a shifted version, i.e. $r = x - C$. The volume can then be computed as

$$V = 2\pi \int_a^b (x - C)\big(f(x) - g(x)\big)dx$$

Volume of Revolution: Rotation Around an Arbitrary Vertical Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a, b]$. Rotating these functions about the horizontal line $x = C$, where $C \leq a$, while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius $x - C$, height $\big(f(x) - g(x)\big)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $$V = 2\pi \int_a^b (x - C)\big(f(x) - g(x)\big)dx$$

Example 4: Find the volume of the solid generated using the region between $f(x) = \frac{1}{x-4}$ and the x-axis over $[5,7]$ when we rotate this region about the axis $x = 3$.

Solution: Since $C \leq a$, the formula above may be directly used, however we sketch the graph below for illustration.



With the sketch we can more easily verify the volume expression given below.

$$V = 2\pi \int_5^7 rhdx$$

$$= 2\pi \int_5^7 (x-3)\left(\frac{1}{x-4}\right) dx = 2\pi \int_5^7 \left(\frac{x-3}{x-4}\right) dx$$

For this integral we use the substitution $u = x - 4$. Therefore $du = dx$ and $x = u + 4$. Applying the substitutions and updating the limits of integration appropriately we have

$$V = 2\pi \int_1^3 \left(\frac{u+1}{u}\right) du$$

$$= 2\pi \left(\int_1^3 (1) du + \int_1^3 \left(\frac{1}{u}\right) du \right)$$

$$= 2\pi\big((3-1) + (ln|3| - ln|1|)\big)$$

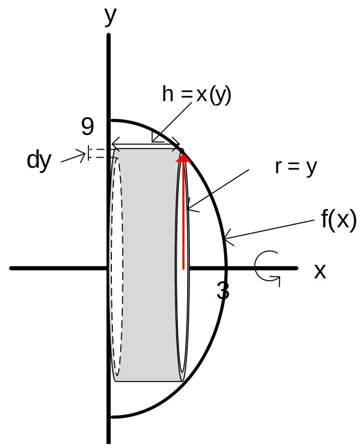
$$= 2\pi(2 + ln(2)) \cong 19.5$$

Volume of a Region Rotated About a Horizontal Axis

Finally, we consider the case when the rotation is about a horizontal axis. There are similar variations as there were in the vertical axis case, however, we will illustrate the idea with a specific example rather than trying to develop general expressions as we did above. Remember, it is always best to sketch the graphs to derive the volume integral rather than rely on a particular formula.

Example 5: Find the volume of the solid generated by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \leq x \leq 3$ about the x-axis.

Solution: We begin as we should for all volume of revolution problems - we sketch the graph.



In this case the shell is oriented such that the thickness is dy. Therefore, we need to integrate along the y-axis. The radius of our solid is equal to y and the height is equal to the inverse of the function given, i.e., $h = x(y) = \sqrt{9 - y}$.

$$V = 2\pi \int_0^9 (y)\left(\sqrt{9 - y}\right)dy$$

We can use a similar substitution for this integral as we did in example 4.

$u = (9 - y) \rightarrow y = (9 - u)$ and $du = -dy$

$$V = -2\pi \int_9^0 (9 - u)\left(\sqrt{u}\right)du$$

$$= 2\pi \int_0^9 \left(9u^{1/2} - u^{3/2}\right)du$$

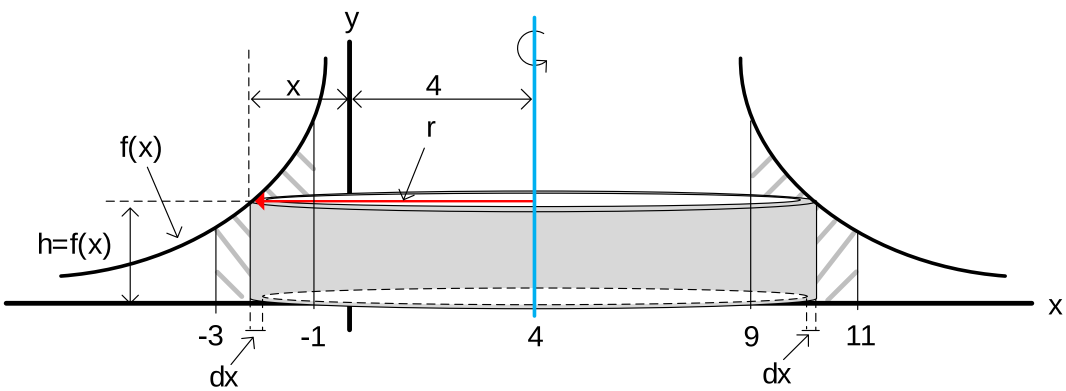
$$= 2\pi \left(6u^{3/2} - \frac{2}{5}u^{5/2}\Big|_0^9\right)$$

$$= 2\pi(162 - 97.2) = 129.6\pi$$

With all the major variations illustrated above let's continue with a few more examples to strengthen our skills at solving such problems.

Example 6: Find the volume of the solid generated by rotating the region under the graph of $f(x) = x^{-4}$ on $[-3, -1]$ about the vertical axis $x = 4$.

Solution: The graph is sketched below.



In this case let's identify the infinitesimal volume of the shell first.

$$dV = 2\pi r h dx$$

As the figure indicates the height is equal to the function value, $f(x)$. The radius, however, needs to be considered carefully. The figure shows that $r = 4 + x$, however, since the radius is a positive quantity and $x < 0$, we need to negate x, i.e. $r = 4 + (-x) = 4 - x$. Therefore

$$dV = 2\pi(4 - x)x^{-4}dx$$

The volume is then computed as

$$V = 2\pi \int_{-3}^{-1} (4 - x)x^{-4}dx$$

$$= 2\pi \int_{-3}^{-1} (4x^{-4} - x^{-3})dx$$

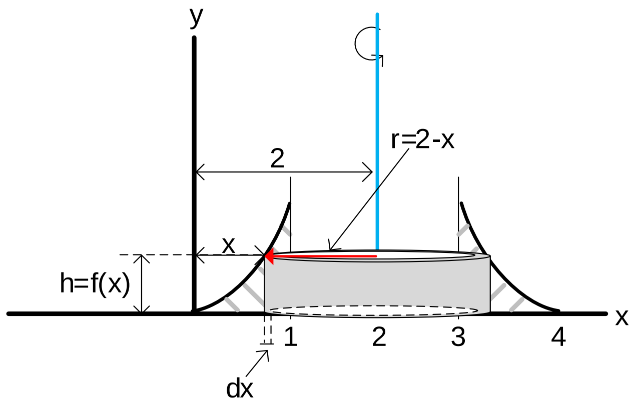
$$= 2\pi \left(-\frac{4}{3x^3} + \frac{1}{2x^2} \Big|_{-3}^{-1} \right)$$

$$= 2\pi \left(\left(\frac{4}{3} + \frac{1}{2} \right) - \left(\frac{4}{81} + \frac{1}{18} \right) \right)$$

$$= 2\pi \left(\left(\frac{11}{6} \right) - \left(\frac{17}{162} \right) \right) = \frac{280}{81}\pi$$

Example 7: Find the volume of the solid generated by rotating the region under the graph of $f(x) = x^3$ on $[0, 1]$ about the vertical axis $x = 2$.

Solution: We first sketch the graph.



Based on the figure shown the infinitesimal volume is given as below

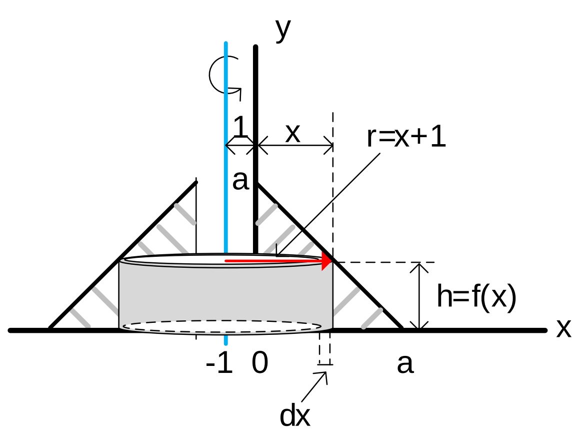
$$dV = 2\pi r h dx = 2\pi(2 - x)x^3 dx$$

The volume is then computed as

$$\begin{aligned} V &= 2\pi \int_0^1 (2 - x)x^3 dx \\ &= 2\pi \int_0^1 (2x^3 - x^4) dx \\ &= 2\pi \left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right) \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{5}\pi \end{aligned}$$

Example 8: Find the volume of the solid generated by rotating the region under the graph of $f(x) = (a - x)$, $a > 0$, on $[0, a]$ about the vertical axis $x = -1$.

Solution: We first sketch the graph.



Based on the figure shown the infinitesimal volume is given as below

$$dV = 2\pi rhdx$$

$$dV = 2\pi(x+1)(a-x)dx$$

The volume is then computed as

$$V = 2\pi\int_0^a (x+1)(a-x)dx$$

$$= 2\pi\int_0^a (-x^2+(a-1)x+a)dx$$

$$= 2\pi\left(-\frac{1}{3}x^3+\frac{a-1}{2}x^2+ax\Big|_0^a\right)$$

$$= 2\pi\left(-\frac{1}{3}a^3+\frac{a-1}{2}a^2+a^2\right)$$

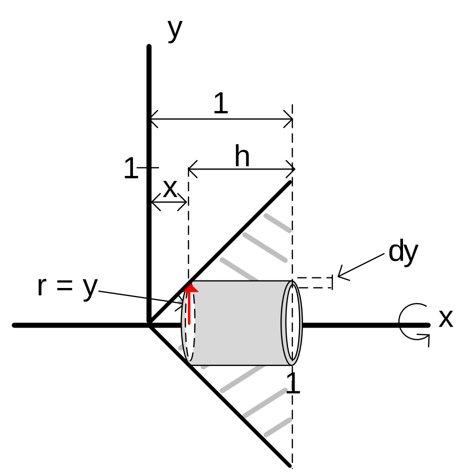
$$= \pi\left(-\frac{2}{3}a^3+(a+1)a^2\right)$$

$$= \pi\left(\frac{1}{3}a^3+a^2\right)$$

Example 9: Find the volume of the solid generated by rotating the region enclosed by the given functions about the x-axis.

$y = x$ $\quad\quad$ $y = 0$ $\quad\quad$ $x = 1$

Solution: Sketching the graph we note that since we are rotating around the x-axis we will integrate along the y-axis.



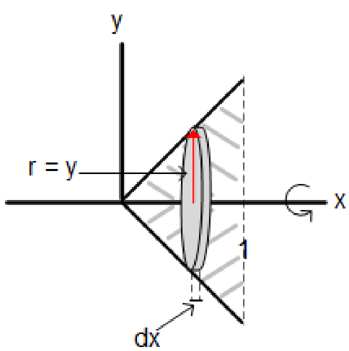
Since we are integrating along y, we need the radius and height to be functions of y. The radius is directly the y value. The height, however, is shown to be $1 - x$. Replacing x with the inverse function (simply $x = y$ in this case), we write the infinitesimal volume as

$$dV = 2\pi r h dy$$
$$dV = 2\pi(y)(1 - y)dy$$
$$dV = 2\pi(-y^2 + y)dy$$

The volume is then computed as

$$V = 2\pi \int_0^1 (y - y^2)dy$$
$$= 2\pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\Big|_0^1\right)$$
$$= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{3}$$

Extra: Since this problem is rotation about the x-axis it is usually more straightforward to use the disk method. Let's do this to verify the volume we computed using the shell method. Below we resketch the graph and solve the resulting integral.



$$V = \pi \int_0^1 r^2 dx$$
$$= \pi \int_0^1 x^2 dx$$
$$= \pi \left(\frac{1}{3}x^3\Big|_0^1\right) = \frac{\pi}{3}$$

Final Summary for Integral Applications – Volumes of Revolution: Shell Method

Volumes of Revolution: Shell Method
Rotating a region in the x-y plane around an axis results in a solid which can be divided into infinitely many cylindrical shells. The general procedure to find the volume of the solid can be summarized as: 1. Find the radius and height of the shell in terms of the functions that specify the region in the x-y plane. 2. Give the shell an infinitesimal width, dx or dy, and write an expression for the infinitesimal volume. $$dV = 2\pi \cdot (radius)(height)$$ 3. Integrate over the appropriate interval to find the total volume of the solid. There are many variations for these types of solids and some general expressions were shown, however, it is always best to follow the guideline above and derive the expressions for each specific case. With that we show the expressions for some of the specific cases we reviewed.
Volume of Revolution: Single Function Rotated About y-axis (Basic Shell Method)
Consider a continuous function $f(x) \geq 0$ on $[a,b]$. Rotating this function about the y-axis creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $f(x)$, and width dx. The volume of such objects may be found by evaluating the following definite integral. $$V = \int_a^b (2\pi \cdot radius)(height)dx = 2\pi \int_a^b xf(x)dx$$
Volume of Revolution: Region Between Two Curves: (Shell Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a,b]$. Rotating these functions about the y-axis while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $f(x) - g(x)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$$
Volume of Revolution: Rotation Around a Vertical Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a,b]$. Rotating these functions about the horizontal line $x = C$, where $C \leq a$, while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius $x - C$, height $f(x) - g(x)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $$V = 2\pi \int_a^b (x - C)\big(f(x) - g(x)\big)dx$$

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Observations: ***Disk Method:*** For horizontal axis rotation the disks are oriented such that the radius is directly related to the function(s) given. For vertical axis rotation the disks are oriented such that the radius is related to the inverse function(s) given. ***Shell Method:*** For vertical axis rotation the shells are oriented such that the height is directly related to the function(s) given. For horizontal axis rotation the shells are oriented such that the height is related to the inverse function(s) given. **General Guideline** Since finding the inverse function is not always simple, these observations suggest that we should choose the method which **does not** require the inverse to compute. To avoid inverse functions, we should choose the disk method when the rotation is about a horizontal axis and the shell method when the rotation is about a vertical axis. Note: If the function(s) is easily inverted than either method should suffice.

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