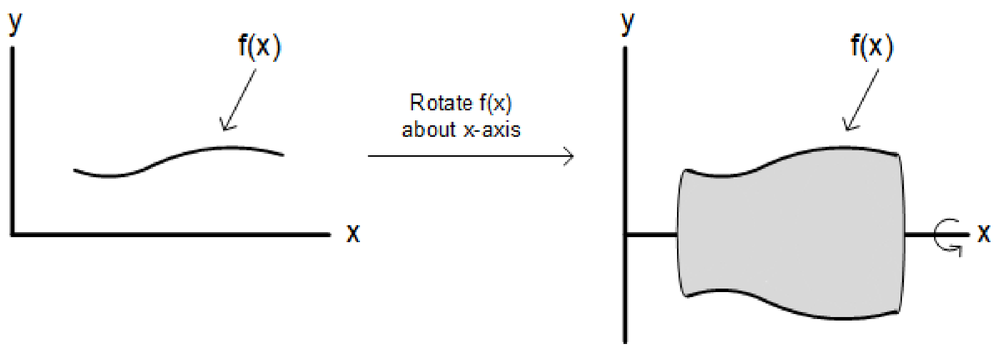
Integral Applications – Volumes of Revolution: Disk Method

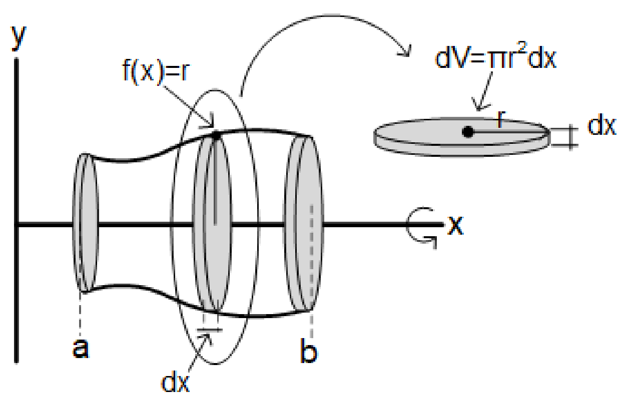
In the previous section we computed the volume of given objects with known cross section. In this section and the next we learn how solid objects can be constructed by rotating a region in the x-y plane around an axis. The objects we create will also have a fixed cross section and we will learn methods to compute the volume of such objects. In this section we use the so-called disk method, while in the next we use the shell method.

Single Function Rotated About x-axis (Basic Disk Method)

To illustrate the basic disk method, we consider a generic function, $f(x) \geq 0$, in the interval $[a, b]$. Next, we rotate this function about the x-axis to obtain a solid object as shown.



A special feature of such objects is that all vertical cross section are circles. Since the area of a circle has a well-known formula, we can add a width, dx, to create an infinitesimal *disk* with volume, dV, and integrate along the x-axis to find the total volume of the object.



Once we identify the disk we proceed as we did in the previous section, i.e. we write an expression for the volume of the infinitesimal disk and integrate to find the total volume.

$$dV = \pi r^2 dx$$
$$dV = \pi f^2(x) dx$$

Where, as shown in the figure we let $r = f(x)$. Integrating we have

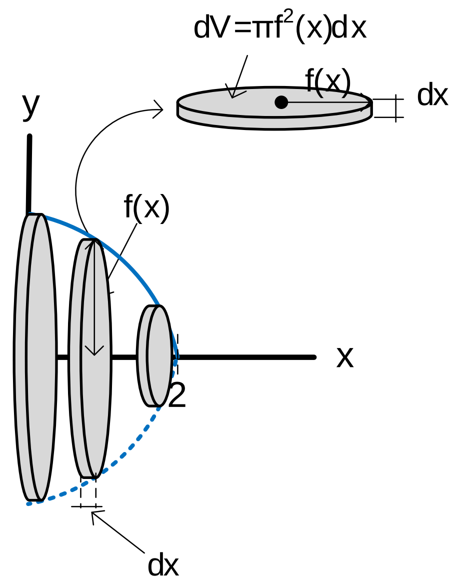
$$V = \pi \int_a^b f^2(x) dx$$

Volume of Revolution: Single Function Rotated About the x-axis (Basic Disk Method)
Consider a continuous function, $f(x) \geq 0$ on $[a, b]$. Rotating this function about the x-axis creates a solid object with a circular cross section and a radius, $r = f(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b f^2(x)dx$$

Let's do an example to illustrate.

Example 1: Find the volume of the solid generated when the function, $f(x) = 4 - x^2$, is rotated about the x-axis in the interval $[0,2]$.

Solution: We start by graphing the function on the given interval and sketching the solid by rotating about the x-axis.



Based on the figure we compute the volume as follows:

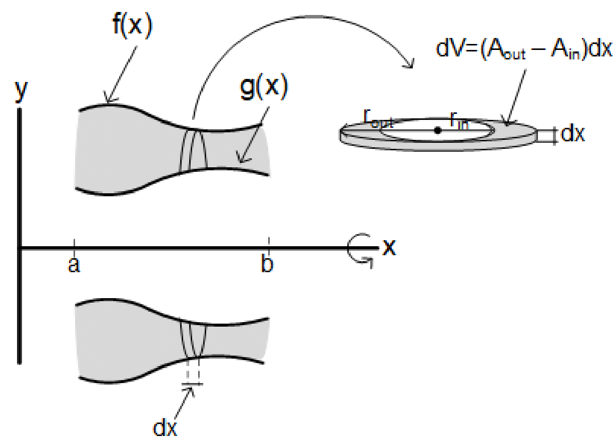
$$\begin{aligned} V &= \pi \int_0^2 (4 - x^2)^2 dx \\ &= \pi \int_0^2 16 - 8x^2 + x^4 dx \\ &= \pi \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \Big|_0^2 \right) \\ &= \pi \left(\frac{256}{15} \right) \end{aligned}$$

The formula derived and used in the above example is the basic disk method formula for volumes of revolution, however there are many variations. Three such examples are:

1. Rotate two curves, e.g. $f(x)$ and $g(x)$, and consider only the area between the two functions for the objects volume.
2. Rotate a region around an arbitrary horizontal line, e.g. $y = 3$.
3. Rotate a region around a vertical axis.

Volume for a Region Between Two Curves

Suppose we have two functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Next, we consider rotating these functions about the x-axis and consider the solid generated by only the region between the two functions.



The infinitesimal object is a disk with the center removed, i.e. a washer, with a volume given as

$$dV = (A_{out} - A_{in})dx$$
$$dV = (\pi f^2(x) - \pi g^2(x))dx$$
$$dV = \pi(f^2(x) - g^2(x))dx$$

The total volume is found by integrating.

$$V = \pi \int_a^b (f^2(x) - g^2(x))dx$$

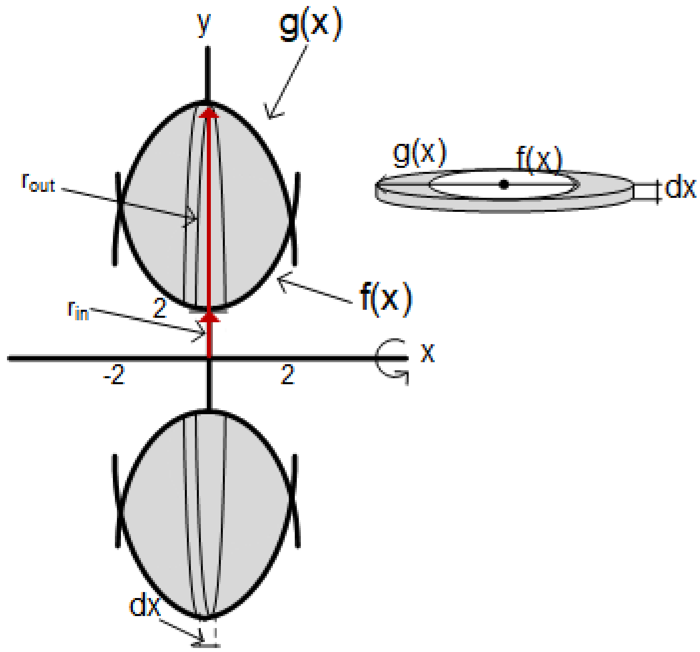
Volume of Revolution: Region Between Two Curves: (Basic Washer Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the x-axis while considering only the region between the functions creates a solid object with a cross section in the shape of a circular washer. The outer radius of the washer is $f(x)$ and the inner radius is $g(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b (f^2(x) - g^2(x))dx$$

Example 2: Find the volume of the solid generated using the region between $f(x) = x^2 + 2$ and $g(x) = 10 - x^2$, when they are rotated about the x-axis.

Solution: Since we are not given an interval, we assume the two curves intersect to form a region. We can find the intersection points by setting the functions equal to each other.

$$x^2 + 2 = 10 - x^2$$
$$2x^2 = 8$$
$$x = \pm\sqrt{4} = \pm 2$$

Next, we sketch the regions below to aid us in setting up the integral.



Using the illustration above the volume is computed as shown. (Note in this case the outer function is $g(x)$).

$$V = \pi \int_{-2}^{2} ((10 - x^2)^2 - (x^2 + 2)^2) dx$$

$$= \pi \int_{-2}^{2} (100 - 20x^2 + x^4 - x^4 - 4x^2 - 4) dx$$

$$= \pi \int_{-2}^{2} (96 - 24x^2) dx$$

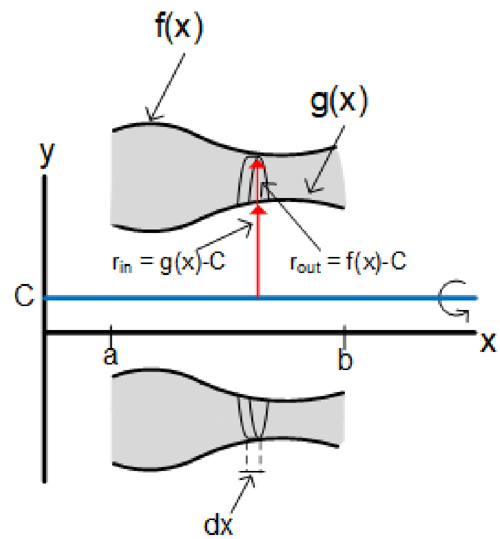
$$= \pi \left(96x - \frac{24}{3} x^3 \Big|_{-2}^{2} \right)$$

$$= \pi\big((128) - (-128)\big) = 256\pi$$

Rotation Around an Arbitrary Horizontal Line

For the two previous revolution types we generalized the procedure to the point where we derived a general expression for the total volume. It is always best, however, to sketch the graph for each problem and "re-derive" the expression. This is especially true for rotations around an arbitrary horizontal line since there are different variations within this type of rotation. Nonetheless, we will illustrate this type of rotation with a specific variation and provide a formula to use as a guide.

We begin with two functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$, that we rotate about the horizonal line $y = C$, where C is a positive constant. The figure below shows the solid generated.



The key thing to notice is that the radii are measured from the rotation line, in this case $y = C$, whereas the function values are specified from the horizontal axis, $y = 0$. For the specific case shown here where $C < g(x) < f(x)$, the radii are as follows:

$$r_{out} = f(x) - C \qquad\qquad r_{in} = g(x) - C$$

The volume is then computed as

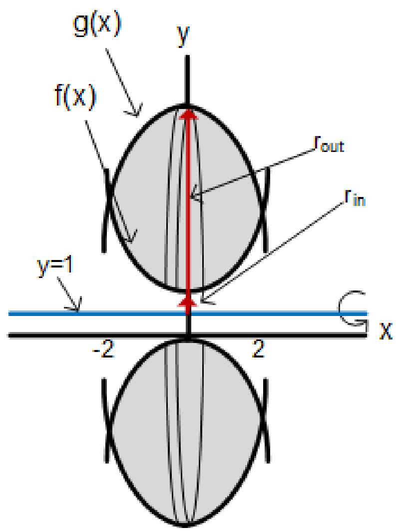
$$V = \pi \int_a^b ((f(x) - C)^2 - (g(x) - C)^2)dx$$

Volume of Revolution: Rotation Around a Horizontal Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the horizontal line $y = C$, where C is a positive constant such that $C < g(x) < f(x)$ on $[a, b]$ creates a solid object with a cross section in the shape of a circular washer. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b ((f(x) - C)^2 - (g(x) - C)^2)dx$$

Example 3: Find the volume of the solid generated using the area between $f(x) = x^2 + 2$ and $g(x) = 10 - x^2$, when they are rotated about the line $y = 1$.

Solution: The two functions are the same as those used in example 2, except in this case they are being rotated around the line $y = 1$. The inner and outer radii are therefore given as

$$r_{out} = g(x) - 1 \qquad\qquad r_{in} = f(x) - 1$$



The infinitesimal washer in this case is expressed as shown

$$dV = \pi({r_{out}}^2 - {r_{in}}^2)dx$$
$$dV = \pi((g(x) - 1)^2 - (f(x) - 1)^2)dx$$
$$dV = \pi((9 - x^2)^2 - (x^2 - 1)^2)dx$$
$$dV = \pi(81 - 18x^2 + x^4 - x^4 + 2x^2 - 1)dx$$
$$dV = \pi(80 - 16x^2)dx$$

The volume is then

$$V = \pi \int_{-2}^{2} (80 - 16x^2)dx$$
$$= \pi \left(80x - \frac{16}{3}x^3 \Big|_{-2}^{2} \right)$$
$$= \pi \left(\frac{704}{3}\right) \cong 234.7\pi$$

Note the volume is less than in example 2 since the radii were reduced by the rotation about $y = 1$.

Volume of a Region Rotated About a Vertical Axis

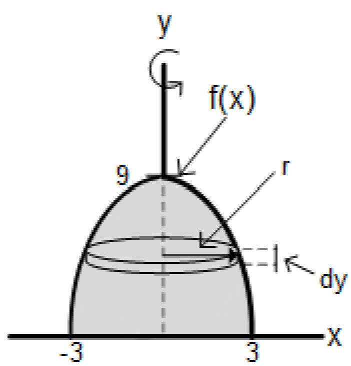
Finally, we consider the case when the rotation is about a vertical axis. Of course, there are similar variations as there were in the horizontal axis case, however, we will illustrate the idea with a specific example rather than trying to develop general expressions as we did above. Remember, it is always best to sketch the graphs to derive the volume integral rather than rely on a particular formula anyway.

Example 4: Find the volume of the solid generated by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the y-axis.

Solution: The figure below shows the function for $0 \le x \le 3$. The rotation about the y-axis results in horizontal cross sections instead of vertical ones. Since the radius of the disk is equal to the x value of our function we need find its inverse.

$$r = x(y) = \sqrt{9 - y}$$

Where, we take the positive square root since $x \ge 0$.



The infinitesimal volume of our disk is then given as

$$dV = \pi\left(\sqrt{9 - y}\right)^2 dy$$
$$dV = \pi(9 - y)dy$$

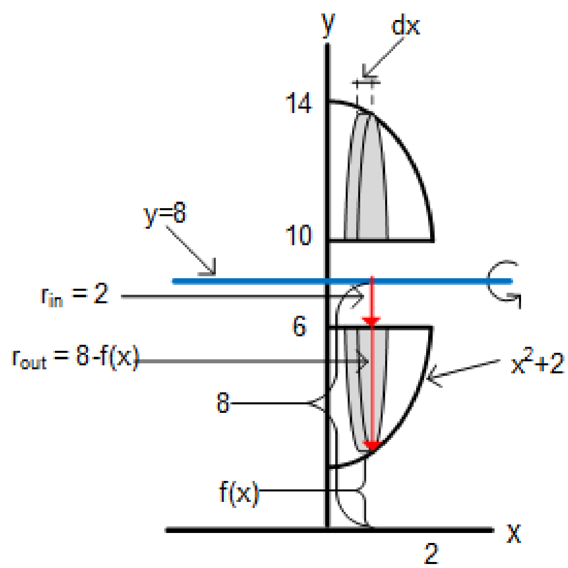
Integrating over $0 \le y \le 9$ we can find the total volume.

$$V = \pi \int_0^9 (9 - y)d$$
$$V = \pi\left(9y - \frac{1}{2}y^2\Big|_0^9\right) = 40.5\pi$$

With all the major variations illustrated above let's continue with a few more examples to strengthen our skills at solving such problems.

Example 5: Find the volume of the solid generated by rotating the region between the functions $f(x) = x^2 + 2$, $x = 0$, and the line $y = 6$ about the line $y = 8$.

Solution: We start by sketching the region below.



As the figure illustrates the cross section is a washer with radii shown in red and given as

$$r_{out} = 8 - f(x) \qquad\qquad r_{in} = 2$$

The infinitesimal volume of the washer is then given as

$$dV = \pi\left(\left(8 - (x^2 + 2)\right)^2 - (2)^2\right)dx$$
$$dV = \pi(x^4 - 12x^2 + 32)dx$$

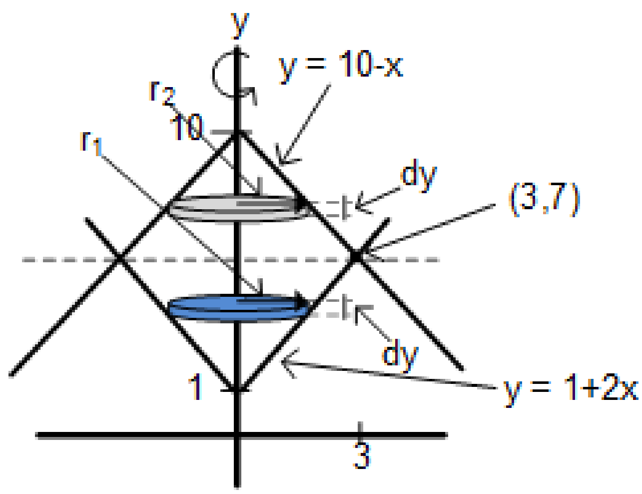
Integrating for $0 \le x \le 2$, the total volume is then

$$V = \pi \int_0^2 (x^4 - 12x^2 + 32)dx$$
$$= \pi\left(\frac{1}{5}x^5 - \frac{12}{3}x^3 + 32x\Big|_0^2\right)$$
$$= \pi\left(\frac{1}{5}\cdot 2^5 - \frac{12}{3}\cdot 2^3 + 32\cdot 2\right) \cong 102.4\pi$$

Example 6: Find the volume of the solid generated by rotating the region enclosed by the below functions about the axis given.

$f(x) = 10 - x$ $\qquad g(x) = 1 + 2x$ $\qquad x = 0$ $\qquad y$-axis

Solution: We start by sketching the region below.

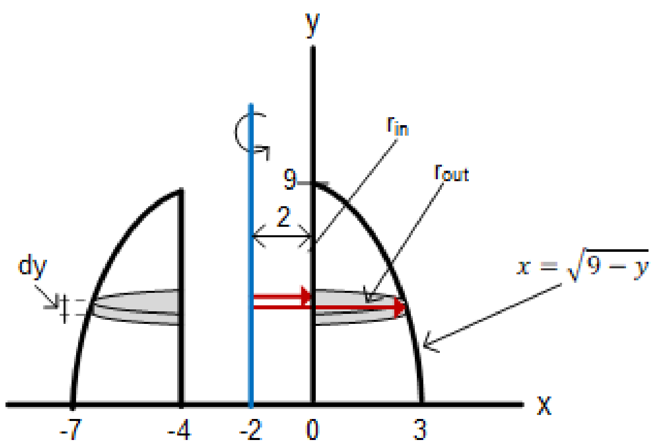


This solid is created using two different cross sections. The first disk, (in blue), has a radius that is equal to the x value from $g(x)$, therefore $r_1 = x(y) = (y-1)/2$. It is used to create the solid when $1 \leq y \leq 7$. The second disk, (in gray), has a radius that is equal to the x value from $f(x)$, therefore $r_2 = x(y) = 10 - y$. This disk is used to create the remaining part of the solid, i.e., when $7 \leq y \leq 10$. The volume of the solid is therefore computed as the sum of the two integrals as shown below.

$$
\begin{aligned}
V &= \pi \int_1^7 \left(\frac{1}{2}y - \frac{1}{2}\right)^2 dy + \pi \int_7^{10} (10-y)^2 dy \\
&= \pi \left(\int_1^7 \left(\frac{1}{4}y^2 - \frac{1}{2}y + \frac{1}{4}\right) dy + \int_7^{10} (100 - 20y + y^2) dy \right) \\
&= \pi \left(\left(\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{1}{4}y \Big|_1^7 \right) + \left(100y - 10y^2 + \frac{1}{3}y^3 \Big|_7^{10} \right) \right) \\
&= \pi\big((18) + (9)\big) = 27\pi
\end{aligned}
$$

Example 7: Find the volume of the solid generated by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \le x \le 3$ about the vertical axis $x = -2$.

Solution: We start by sketching the region as shown below.



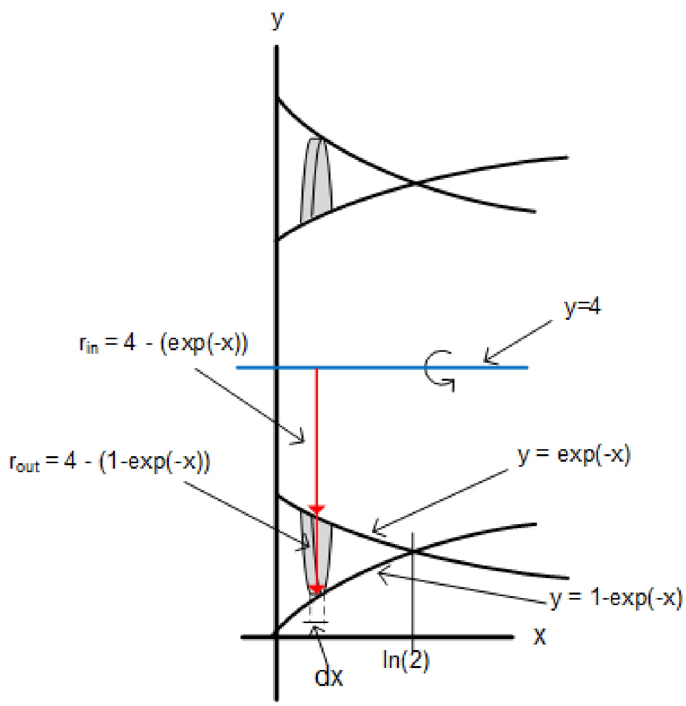
The cross section is a washer with inner radius, $r_{in} = 2$, and outer radius, $r_{out} = 2 + \sqrt{9 - y}$. The volume of the solid can now be computed as shown below.

$$
\begin{aligned}
V &= \pi \int_0^9 ({r_{out}}^2 - {r_{in}}^2) dy \\
&= \pi \int_0^9 \left(\left(2 + \sqrt{9-y}\right)^2 - 2^2 \right) dy \\
&= \pi \int_0^9 \left(4 + 4\sqrt{9-y} + 9 - y - 4\right) dy \\
&= \pi \int_0^9 \left(4\sqrt{9-y} + 9 - y\right) dy \\
&= \pi \left(-\frac{8}{3}(9-y)^{3/2} + 9y - \frac{1}{2}y^2 \Big|_0^9 \right) \\
&= \pi\big((0 + 81 - 40.5) - (-72)\big) = 112.5\pi
\end{aligned}
$$

Example 8: Find the volume of the solid generated by rotating the region enclosed by the functions below about the axis given.

$y = e^{-x}$ $\qquad$ $y = 1 - e^{-x}$ $\qquad$ $x = 0$ $\qquad$ Rotation axis: $y = 4$

Solution: We start by sketching the region below.



The cross section has inner radius, $r_{in} = (4 - e^{-x})$, and outer radius, $r_{out} = (3 + e^{-x})$. The volume of the solid can be computed as shown below.

$$V = \pi \int_0^{ln(2)} ({r_{out}}^2 - {r_{in}}^2) dx$$

$$= \pi \int_0^{ln(2)} ((3 + e^{-x})^2 - (4 - e^{-x})^2) dx$$

$$= \pi \int_0^{ln(2)} \left((9 + 6e^{-x} + e^{-2x}) - (16 - 8e^{-x} + e^{-2x})\right) dx$$

$$= \pi \int_0^{ln(2)} (14e^{-x} - 7) dx$$

$$= \pi\left(-14e^{-x} - 7x\Big|_0^{ln(2)}\right)$$

$$= \pi\left(\left(-14e^{-ln(2)} - 7\,ln(2)\right) - (-14)\right)$$

$$= \pi(-7 - 7\,ln(2) + 14) = 7\pi(1 - ln(2))$$

Final Summary for Integral Applications – Volumes of Revolution: Disk Method

Volumes of Revolution: Disk Method
Rotating a region in the x-y plane around an axis results in a solid with cross sections that are either disks or washers. The general procedure to find the volume of the solid can be summarized as: 1. Find the radius of the disk or the outer and inner radius of the washer in terms of the functions that specify the region in the x-y plane. 2. Give the disk or washer an infinitesimal width, dx or dy, and write an expression for the infinitesimal volume. 3. Integrate over the appropriate interval to find the total volume of the solid. There are many variations for these types of solids and some general expressions were shown, however, it is always best to follow the guideline above and derive the expressions for each specific case. With that we show the expressions for some of the specific cases we reviewed.
Volume of Revolution: Single Function Rotated About the x-axis (Basic Disk Method)
Consider a continuous function $f(x) \geq 0$ on $[a, b]$. Rotating this function about the x-axis creates a solid object with a circular cross section and a radius, $r = f(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b f^2(x)dx$$
Volume of Revolution: Region Between Two Curves: (Basic Washer Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the x-axis while considering only the region between the functions creates a solid object with a cross section in the shape of a circular washer. The outer radius of the washer is $f(x)$ and the inner radius is $g(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b \left(f^2(x) - g^2(x)\right)dx$$
Volume of Revolution: Rotation Around a Horizontal Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the horizontal line $y = C$, where C is a positive constant such that $C < g(x) < f(x)$ on $[a, b]$ creates a solid object with a cross section in the shape of a circular washer. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b ((f(x) - C)^2 - (g(x) - C)^2)dx$$

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