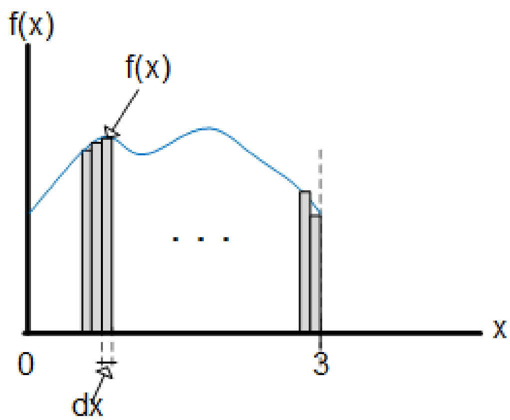
Integral Applications – Setting up Integrals

In this lesson we will learn how one may go about setting up integrals in practice. In general terms, the integral can be used to represent the "total amount" of something, e.g., area, volume, mass. We'll start this section by describing a procedure using a quantity we are already familiar with; area. With this knowledge we'll apply the technique to other quantities.

Setting up an Integral to Compute Area:

We are already familiar with the fact that a definite integral can represent the area under a function, e.g. $f(x)$. Using the generic function shown below let's try to develop an intuitive approach to setting up an integral to compute the area.



We start by drawing a rectangle within the area region with an infinitesimal width, dx, and a height, $f(x)$, that depends on the location of the rectangle. The choice of a rectangle is made for convenience as it is a simple geometric shape whose area is easily computed. Furthermore, since our rectangle has an infinitesimal width it will also have an infinitesimal area, which we denote as dA.

$$\underbrace{dA}_{area} = \underbrace{f(x)}_{height} \underbrace{dx}_{base}$$

As we learned earlier, the integral symbol, $\int$, is used to represent a summation symbol, $\sum$, when we are summing infinitely many infinitesimal objects, e.g., dA's. Furthermore, the limits of integration represent the interval over which these infinitesimal objects exist, in our case for $0 \leq x \leq 3$. Finally, we integrate both sides of the equation, using x-axis limits on the right-hand side. For the left-hand side, summing all of dA's results in the total area, A.

$$\int dA = \int_0^3 f(x)dx$$

$$A = \int_0^3 f(x)dx$$

Note that by treating the differentials, e.g. dx and dA, as actual variables, we were able to easily set up an integral to compute the area shown in the figure.

Volume:

Let's see if we can extend the ideas from above to compute the volume of shapes with a known cross-sectional area. For this we first need to be able to identify an appropriate 2D cross section of the 3D object. Secondly, we need to be able to represent this area as a function of the 3rd dimension for which it extends into. As an illustration the object shown below has a cross section that lies in the x-z plane and extends up in the y direction. Assuming we can represent the area of this cross section as a function of y, we can use a procedure very similar to the one we used for the area integral. In that example we used rectangles as our infinitesimal objects, with area

$$\underbrace{dA}_{area} = \underbrace{f(x)}_{height} \underbrace{dx}_{base}$$

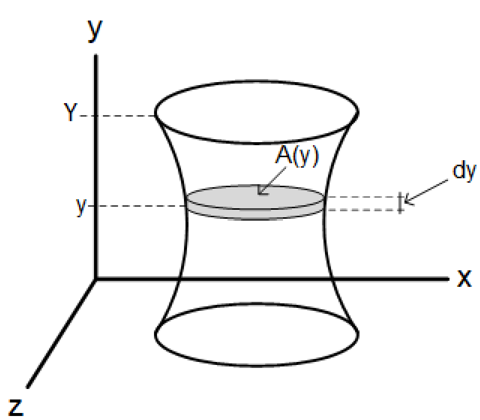
Here we use the cross-section shape with an infinitesimal width, dy, and area $A(y)$, giving is an infinitesimal volume, dV.

$$\underbrace{dV}_{volume} = \underbrace{A(y)}_{area} \underbrace{dy}_{width}$$

Finally, we again integrate both sides of the equation to find the total volume of the object.

$$\int dV = \int_0^Y A(y)dy$$

$$V = \int_0^Y A(y)dy$$



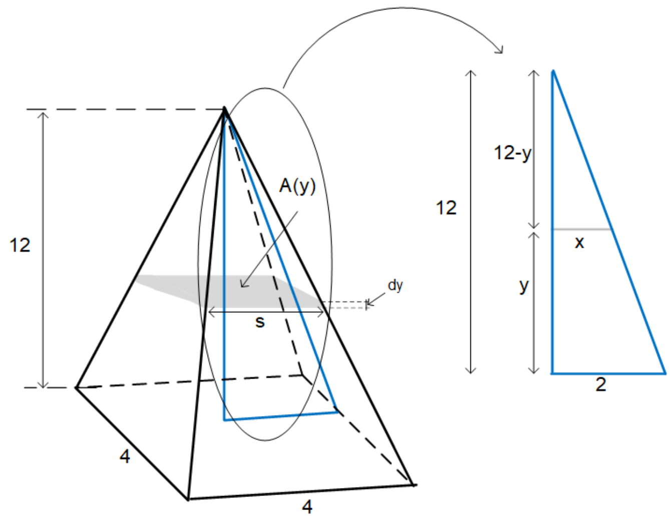
We can generalize the volume integral as follows.

Volume as the Integral of the Cross-Sectional Area
Let $A(y)$ be the area of a x-z plane cross section at height y of a solid body that extends from $y = a$ to $y = b$. The volume of the object, V, can then be computed as $$V = \int_a^b A(y)dy$$

Let's do an example to illustrate.

Example 1: Calculate the volume of a pyramid of height 12 m whose base is a 4 m side square.

Solution: Let's start by drawing a figure as shown below.



Identifying the cross section as a square in the x-z plane, we need to find an expression for the area of this square as a function of the height of the pyramid, $A(y)$. One way to do this is to draw similar triangles as shown above and set up the following proportion.

$$\frac{2}{12} = \frac{x}{12-y}$$
$$x = \frac{1}{6}(12-y)$$

Since the side length, s, is $2x$ and the area is s^2 we have the following expression for the area of the cross section as a function of the height of the pyramid.

$$A(y) = s^2 = (2x)^2 = \left(\frac{1}{3}(12-y)\right)^2$$
$$A(y) = \frac{1}{9}(12-y)^2$$

The infinitesimal volume is given as $dV = A(y)dy$, and the total volume is the integral as shown below.

$$V = \frac{1}{9}\int_0^{12} (12-y)^2\, dy$$

Which we can solve with the substitution method by letting $u = 12 - y$ and $du = -dy$.

$$V = -\frac{1}{9}\int_{u=12}^{u=0} u^2\, du = \frac{1}{9}\int_0^{12} u^2\, du$$
$$= \frac{1}{9}\left(\frac{u^3}{3}\bigg|_0^{12}\right) = \frac{1728}{27} = 64\ m^3$$

Total Mass:

We can use a similar procedure to find the total mass of an object whose mass is not uniformly distributed throughout the object. To illustrate we consider a rod with a linear density, $\rho(l)$, with units of mass per unit length. We know if the density is constant, e.g., ρ, then the total mass is given as

$$M = \rho \cdot L$$

Where, ρ has units of kg/m, L is the length of the rod in meters, and M is the total mass in kg.

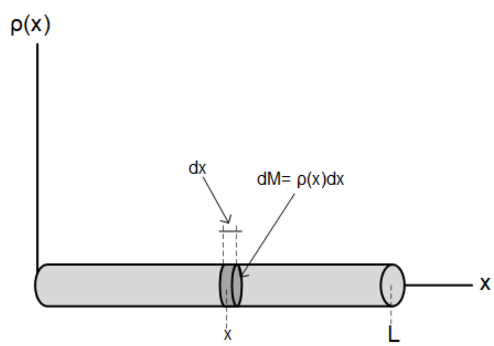
However, when the density is not constant, we can use the integral technique as in the previous applications. As an example, the figure below shows a rod placed along the x-axis with a density that varies with the location along the rod, $\rho(x)$. To find the total mass we again start by defining an infinitesimal mass element, dM, which is equal to the density at the given location multiplied by an infinitesimal width, dx.

$$\underbrace{dM}_{mass} = \underbrace{\rho(x)}_{density} \underbrace{dx}_{length}$$

The total mass, as usual, is found by integrating along the length of the rod.

$$\int dM = \int_0^L \rho(x)dx$$

$$M = \int_0^L \rho(x)dx$$



Let's do an example to illustrate.

Example 2: Calculate the mass of a 2 m rod of linear density $\rho(x) = 1 + x(2 - x)\ kg/m$, where x is the distance from one end of the rod.

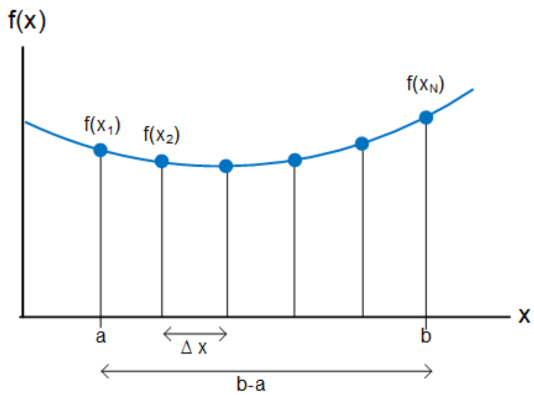
$$M = \int_0^2 \big(1 + x(2 - x)\big)dx$$

$$= \int_0^2 (-x^2 + 2x + 1)dx$$

$$= \left(-\frac{x^3}{3} + x^2 + x\Big|_0^2\right) = -\frac{8}{3} + \frac{12}{3} + \frac{6}{3} = \frac{10}{3}\ kg$$

Average Value:

The last application we'll look at is the average value of a continuous function. Rather than a rigorous proof we'll instead intuitively derive an expression for the average value. We begin with something we are all likely familiar with, i.e. the average value of N discrete values. The figure below shows these N values being sampled from a continuous function over an interval $a \leq x \leq b$.



The average value of this array of numbers sampled from the function is given as

$$f_{avg} = \frac{f(x_1) + f(x_2) + \ldots + f(x_N)}{N} = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

Next, let's express N in terms of the length of the interval, $(b-a)$, and the sample spacing, Δx.

$$N = \frac{(b-a)}{\Delta x}$$

The average value can now be expressed as

$$f_{avg} = \frac{\Delta x}{(b-a)}\sum_{i=1}^{N} f(x_i) = \frac{1}{(b-a)}\sum_{i=1}^{N} f(x_i)\,\Delta x$$

Where we moved Δx inside the summation for reasons that will soon be obvious.

Finally, we let Δx approach zero, which of course means that N will approach infinity. We should instantly recognize that this is exactly how we defined the definite integral in a previous lesson. Therefore, letting $\Delta x \to dx$, and $\sum_{i=1}^{N} \to \int_a^b$, we have derived our desires expression for the average value of a continuous function.

Average Value
The average value of a continuous function, $f(x)$, over $[a, b]$ is $$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

Let's do an example to illustrate.

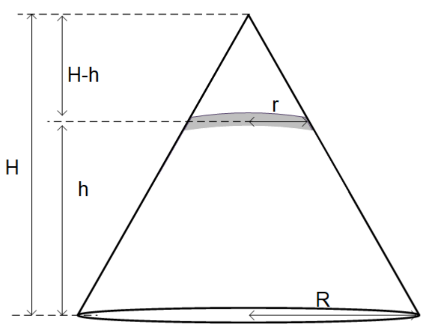
Example 3: Find the average value of $f(x) = sin(x)$ over $[0, \pi]$.

Solution: Directly applying the formula above we find the average value as

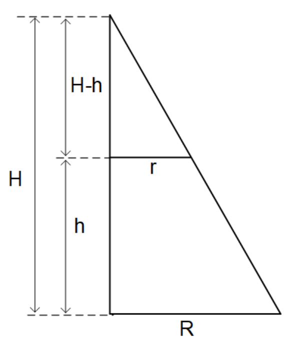
$$\begin{aligned} f_{avg} &= \frac{1}{\pi}\int_0^{\pi} sin(x)\,dx \\ &= \frac{1}{\pi}\left(-cos(x)|_0^{\pi}\right) \\ &= -\frac{1}{\pi}(-1\ -\ 1) = \frac{2}{\pi} \end{aligned}$$

Now that we have become more familiar with the idea of setting up integrals for various applications let's do a few more examples.

Example 4: Use the method of example 1 to find the formula for the volume the right circular cone shown below.



Solution: The horizontal cross section is a circle which varies as a function of the height. Since the area of the cross section is a function of the radius, we first find the radius function using similar triangles as shown below.



$$\frac{r}{H-h} = \frac{R}{H}$$

$$r(h) = \frac{R}{H}(H-h)$$

Next, we write the volume of the infinitesimal disk and integrate to find the total volume.

$$dV = A(h)dh$$

$$dV = \pi r^2(h)dh$$

$$dV = \frac{\pi R^2}{H^2}(H-h)^2 dh$$

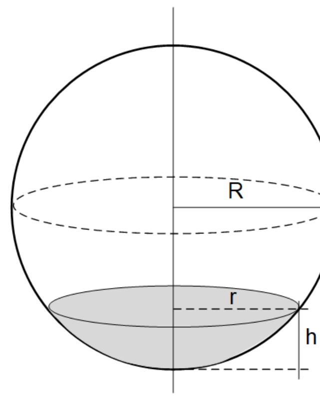
$$\rightarrow$$

$$V = \frac{\pi R^2}{H^2}\int_0^H (H-h)^2 dh$$

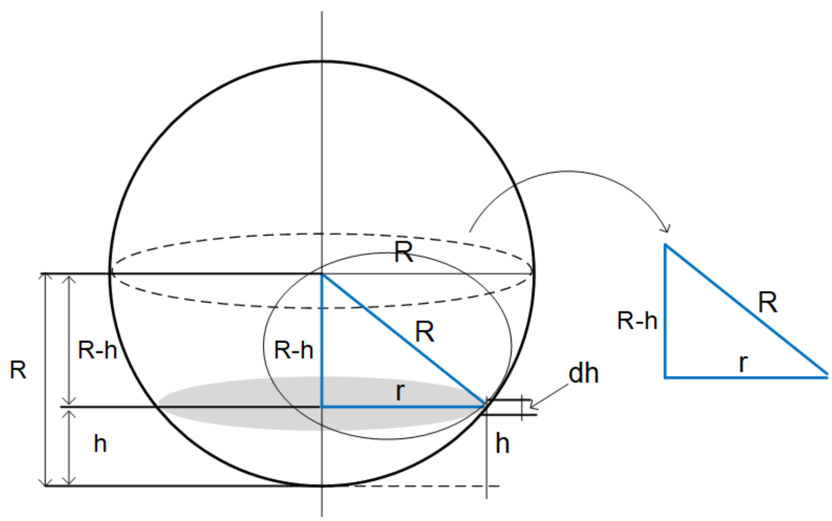
$$V = -\frac{\pi R^2}{H^2}\int_H^0 (u)^2 du = \frac{\pi R^2}{H^2}\left(\frac{u^3}{3}\bigg|_0^H\right) = \frac{1}{3}\pi R^2 H$$

Where, we used the substitution with $u = H - h$ and $du = -dh$.

Example 5: Find the volume of liquid needed to fill a metal sphere of radius R to a height of H.



Solution: We again need to find the radius, r, as a function of the height of the liquid in the sphere. We redraw the sphere below with a triangle from which we can use the Pythagorean theorem to write an expression for r as a function of the height, h.



$$r^2 = R^2 - (R-h)^2$$

$$r^2 = \cancel{R^2 - R^2} + 2Rh - h^2$$

$$r(h) = \sqrt{2Rh - h^2}$$

With this relationship we can now write an expression for the volume of an infinitesimal disk and integrate to find the total volume.

$$dV = A(h)dh$$

$$dV = \pi r^2(h)dh$$

$$dV = \pi(2Rh - h^2)dh$$

$$V = \pi \int_0^H (2Rh - h^2)dh$$

$$V = \pi\left(Rh^2 - \frac{h^3}{3}\bigg|_0^H\right)$$

$$V = \pi\left(RH^2 - \frac{H^3}{3}\right)$$

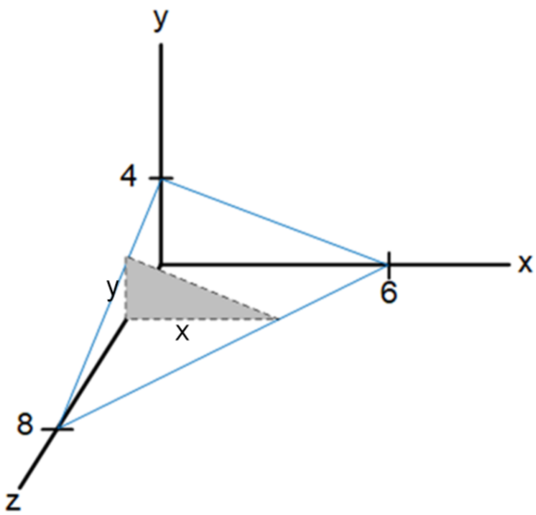
Since the volume of a sphere is known to be $\frac{4}{3}\pi R^3$, let's check our expression by letting $H = 2R$.

$$V|_{H=2R} = \pi\left(R(2R)^2 - \frac{(2R)^3}{3}\right)$$

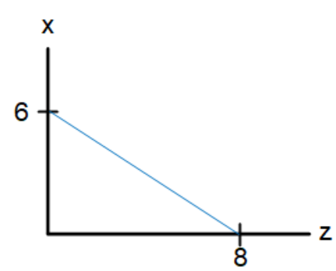
$$= \pi\left(\frac{12R^3}{3} - \frac{8R^3}{3}\right)$$

$$= \pi\left(\frac{4R^3}{3}\right)$$

Example 6: Find the volume of the wedge in the figure below using the vertical cross section shown.

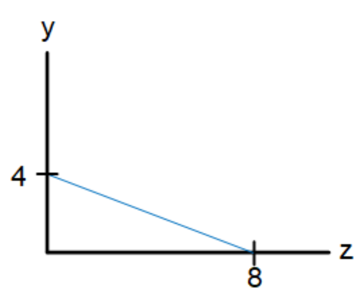


Solution: Using the gray triangle in the figure as our cross section we need to find expressions for the base, x, and the height, y, as functions of z. We can do that by finding the linear equations in the x-z plane for the base and the y-z plane for the height. We do this below for each.



$$m = \frac{6}{-8} = -\frac{3}{4}$$

$$x(z) - x(8) = -\frac{3}{4}(z-8)$$
$$x(z) = \frac{3}{4}(8-z)$$



$$m = \frac{4}{-8} = -\frac{1}{2}$$

$$y(z) - y(8) = -\frac{1}{2}(z-8)$$
$$y(z) = \frac{1}{2}(8-z)$$

The area of the triangle is then given as

$$A(z) = \frac{1}{2}x(z)y(z)$$
$$A(z) = \frac{3}{16}(8-z)^2$$

Finally, we express the infinitesimal volume of the triangle with a thickness of dz and integrate to find the total volume of the wedge.

$$dV = A(z)dz$$

$$dV = \frac{3}{16}(8-z)^2 dz$$

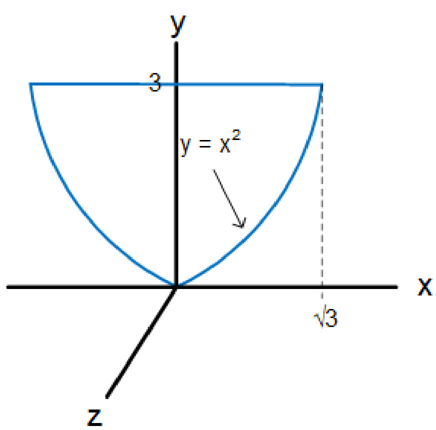
$$V = \frac{3}{16}\int_0^8 (8-z)^2 dz$$

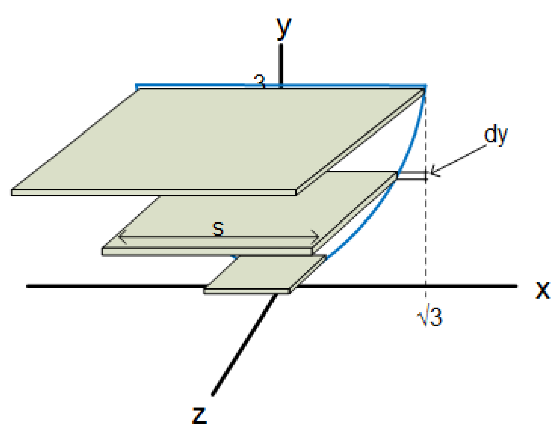
$$V = -\frac{3}{16}\int_8^0 (u)^2 du$$

$$V = \frac{\cancel{3}}{16}\left(\frac{u^3}{\cancel{3}}\Bigg|_0^8\right) = \frac{512}{16} = 32$$

Example 7: Find the volume of a solid when the base is a region enclosed by $y = x^2$ and $y = 3$ and the cross sections perpendicular to the y-axis are squares.

Solution: We start by graphing the base in the x-y plane. We then add square cross sections perpendicular to the y-axis extending into the z dimension.





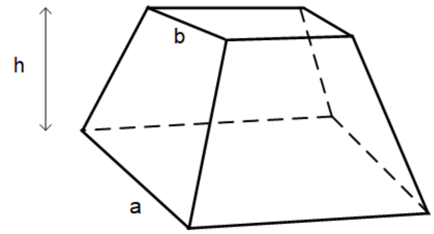
The side length of each square is equal to twice the x value in the first equation, $y = x^2$, therefore $s(y) = 2\sqrt{y}$. The infinitesimal volume for each cross section with thickness dy is

$$dV = s^2(y)dy$$
$$dV = 4ydy$$

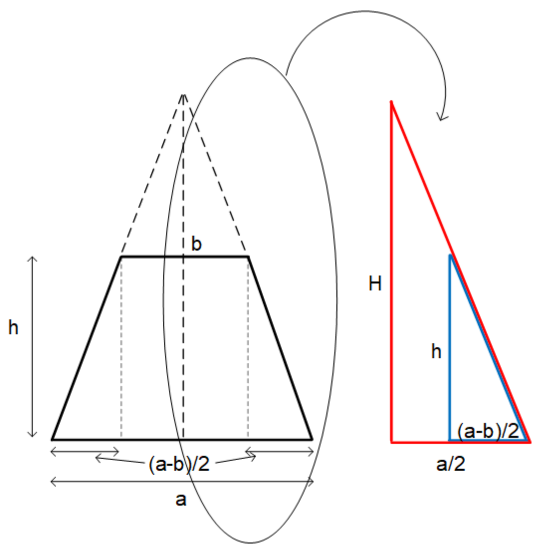
Integrating we find the total volume as follows:

$$V = \int_0^3 4ydy = 4\frac{1}{2}y^2\Big|_0^3 = 18$$

Example 8: A frustum is a pyramid with its top cut off. Find the volume of the frustum shown below.



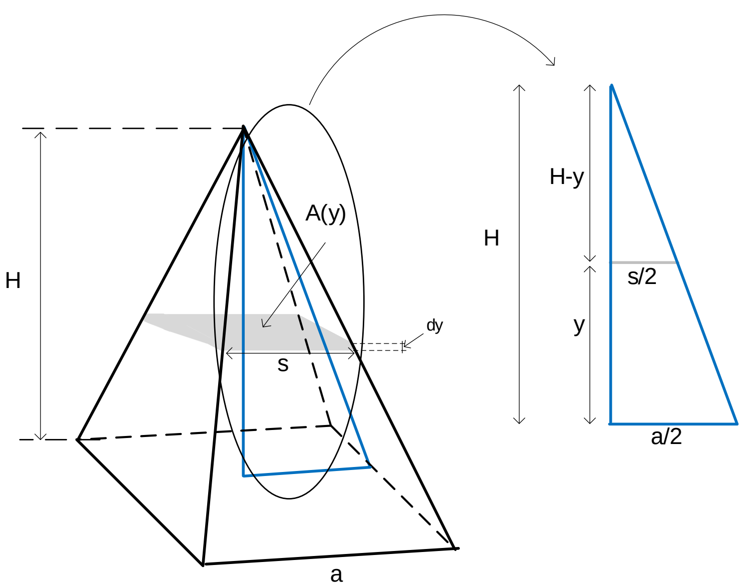
Solution: We start by imagining the frustrum as a full pyramid with total height, H. Next, we express H as a function of the given parameters using similar triangles shown below.



$$\frac{H}{a/2} = \frac{h}{(a-b)/2}$$

$$H = \frac{ha}{a-b}$$

Now that we have the height of a what the full pyramid would be, H, we can use the same technique from example 1, but this time only integrate from 0 to h. We redraw the pyramid from example 1 using H as the height and a as the base side length. The side length, s, of the square cross section is found below.



$$\frac{a/2}{H} = \frac{s/2}{H-y}$$

$$s(y) = \frac{a(H-y)}{H}$$

The area of the cross section as a function of y is then given as

$$A(y) = s^2(y) = \left(\frac{a(H-y)}{H}\right)^2 = \frac{a^2}{H^2}(H-y)^2$$

The volume integral is then given as

$$V = \frac{a^2}{H^2}\int_0^h (H-y)^2\,dy$$

Which we can solve with the substitution method by letting $u = H - y$ and $du = -dy$.

$$V = -\frac{a^2}{H^2}\int_{u=H}^{u=H-h} u^2\,du = \frac{a^2}{H^2}\int_{H-h}^{H} u^2\,du$$

$$V = \left(\frac{a^2}{H^2}\right)\cdot\frac{u^3}{3}\Bigg|_{H-h}^{H} = \left(\frac{a^2}{3H^2}\right)(H^3 - (H-h)^3)$$

$$V = \left(\frac{a^2}{3H^2}\right)(\cancel{H^3 - H^3} + 3H^2h - 3Hh^2 + h^3)$$

$$V = \left(\frac{h}{3}\right)\left(3a^2 - \frac{3a^2h}{H} + \frac{a^2h^2}{H^2}\right)$$

The final step is to substitute for $H = \frac{ha}{a-b}$.

$$V = \left(\frac{h}{3}\right)\left(3a^2 - \frac{3a^{\cancel{2}}\cancel{h}(a-b)}{\cancel{ha}} + \frac{\cancel{h^2a^2}(a-b)^2}{\cancel{h^2a^2}}\right)$$

$$V = \left(\frac{h}{3}\right)(\cancel{3a^2 - 3a^2} + 3ab + a^2 - 2ab + b^2)$$

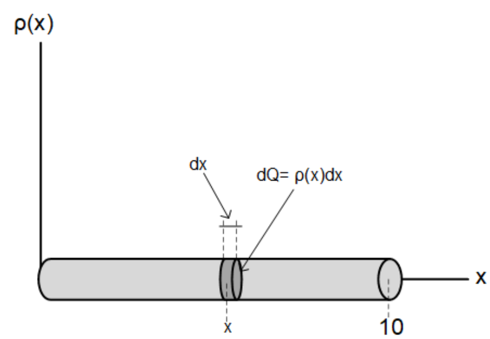
$$V = \left(\frac{h}{3}\right)(a^2 + b^2 + ab)$$

Note the formula for the volume of a pyramid with a square base is $\frac{s^2h}{3}$, which matches our expression of we let $b = 0$ and $a = s$.

Example 9: Charge is distributed along a glass tube of length 10 cm with a linear charge density of $\rho(x) = x(x^2+1)^{-2} \cdot 10^{-4}$ coulombs per centimeter. Find the total charge of the rod.

Solution: We can compute the total charge the same way we computed the total mass. The figure below shows the glass rod where we take an infinitesimal disk and find its change to be

$$dQ = \rho(x)dx$$
$$dQ = (x(x^2+1)^{-2} \cdot 10^{-4})dx$$



The total charge is then found by integrating along the length of the rod.

$$Q = 10^{-4} \int_0^{10} x(x^2+1)^{-2}\, dx$$

We can use substitution to solve this integral.

$$u = x^2 + 1 \quad \text{and} \quad du = 2xdx \;\rightarrow xdx = \frac{1}{2}du$$

$$Q = \frac{10^{-4}}{2} \int_1^{101} u^{-2}\, dx$$
$$= \frac{10^{-4}}{2}\left(-\frac{1}{u}\Big|_1^{101}\right)$$
$$= \frac{10^{-4}}{2}\left(1 - \frac{1}{101}\right) = \frac{50}{101} \cdot 10^{-4}\, C$$

Example 10: The temperature (in degrees Celsius) at time t (in hours) in a room varies according to $T(t) = 20 + 5\cos\left(\frac{\pi}{12}t\right)$. Find the average temperature over the time periods $[0,24]$ and $[2,6]$.

Solution: In this case we can directly apply the average value integral we derived above.

Case 1: $0 \le t \le 24$

$$\begin{aligned} T_{avg} &= \frac{1}{24}\int_0^{24} 20 + 5\,cos\left(\frac{\pi}{12}t\right)dt \\ &= \frac{1}{24}\left(20t + \frac{60}{\pi}sin\left(\frac{\pi}{12}t\right)\Big|_0^{24}\right) \\ &= \frac{1}{24}\left(\left(480 + \cancel{\frac{60}{\pi}sin(2\pi)}\right) - \left(0 + \cancel{\frac{60}{\pi}sin(0)}\right)\right) \\ &= 20\ C \end{aligned}$$

Case 1: $2 \le t \le 6$

$$\begin{aligned} T_{avg} &= \frac{1}{4}\int_2^{6} 20 + 5\,cos\left(\frac{\pi}{12}t\right)dt \\ &= \frac{1}{4}\left(20t + \frac{60}{\pi}sin\left(\frac{\pi}{12}t\right)\Big|_2^{6}\right) \\ &= \frac{1}{4}\left(\left(120 + \frac{60}{\pi}sin\left(\frac{\pi}{2}\right)\right) - \left(40 + \frac{60}{\pi}sin\left(\frac{1}{6}\right)\right)\right) \\ &= \frac{1}{4}\left(\left(120 + \frac{60}{\pi}\right) - \left(40 + \frac{60}{\pi}\cdot\frac{1}{2}\right)\right) \\ &= \frac{1}{4}\left(\left(80 + \frac{30}{\pi}\right)\right) \\ &= \left(20 + \frac{7.5}{\pi}\right) \cong 22.4\ C \end{aligned}$$

<u>Final Summary for Integral Applications – Setting up Integrals</u>

Setting up Integrals

- In general terms the integral symbol, $\int$, is used to represent a summation symbol, $\sum$, when we are summing infinitely many infinitesimal objects, e.g., $dx's, dA's$.
- The limits of integration represent the interval over which these infinitesimal objects exist, e.g., $0 \leq x \leq 3$.

In some application where integration is required the following high-level procedure can provide guidance for setting up the integral.

1. Identify the infinitesimal "segment" from which the "total amount" can be computed. For example, if the "object" is area, the infinitesimal segment is a rectangle with area dA.
2. Identify how this segment changes as we compute the total amount. For example, in the case of the area under a curve, the rectangle height, $f(x)$, changes as a function of x.
3. Write an expression for the infinitesimal segment in terms of the variable you will change to compute the total amount. As an example, for the area under a curve the infinitesimal rectangle has an area of $dA = f(x)dx$.
4. Integrate the relationship from step 3 over an interval, e.g. $[a, b]$, to find the total amount, e.g. $A = \int_a^b f(x)dx$

Volume as the Integral of the Cross-Sectional Area

Let $A(y)$ be the area of a x-z plane cross section at height y of a solid body that extends from $y = a$ to $y = b$. The volume of the object, V, can then be computed as

$$V = \int_a^b A(y)dy$$

Total Mass

The total mass of an object with a linear mass density, $\rho(x)$, in $mass/length$, and a length of L is given by

$$M = \int_0^L \rho(x)dx$$

Average Value

The average value of a continuous function, $f(x)$, over $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

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