Integral Applications – Net Change as the Integral of Rate of Change

In previous lessons we focused on the fundamentals of computing the integral. Integration, similar to differentiation, also has far reaching practical applications. The next few lessons will focus on some of these applications. In this first lesson we focus on finding the net change of a certain parameter by computing the definite integral of the derivative of that parameter.

Net Change as the Integral of Rate of Change

We can derive the desired relationship, albeit somewhat heuristically, by starting with a simple equation relating two forms of the derivative of a function.

$$\frac{df}{dx} = f'(x)$$

Next, we multiple through by dx and integrate both sides over a given interval, e.g. , $[x_1, x_2]$ or equivalently , $[f(x_1), f(x_2)]$.

$$df = f'(x)dx$$

$$\int_{f(x_1)}^{f(x_2)} df = \int_{x_1}^{x_2} f'(x)dx$$

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)dx$$

Which is the desired expression. It shows that net change of a function over some interval is equivalent to the integral of the derivative of that function over the same interval. The theorem is formally stated below.

Net Change as the Integral of the Rate of Change
The net change of $f(x)$ over an interval, $[x_1, x_2]$, is given by the following definite integral. $$\int_{x_1}^{x_2} f'(x)dx = f(x_2) - f(x_1)$$

This relationship is most clearly seen using the example of one dimensional speed and distance. Imagine a vehicle traveling at a constant speed, v, over a given time interval, Δt. We know from basic algebra that the change in the position of the vehicle, Δx, is given as

$\Delta x = v\Delta t$ $\rightarrow$ Traveling 50 mph for 1 hour results in a change of distance of 50 miles

However, if the speed varies as a function of time, i.e., $v(t)$, we then need to use integration, just as we saw in our first lesson on integration. With this we can write the following

$$\Delta x = x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)dt$$

Getting back to the general case, let's look at a simple example to see how the general theorem is applied.

Example 1: Water is leaking from a tank into a storage closet at a rate of $0.5t^2 + 2$ liters/hour, where t is measured starting at 7 AM, when the leak started. Maintenance indicates that the tank will not be fixed until 11 AM. How much water will have leaked from the tank at this time?

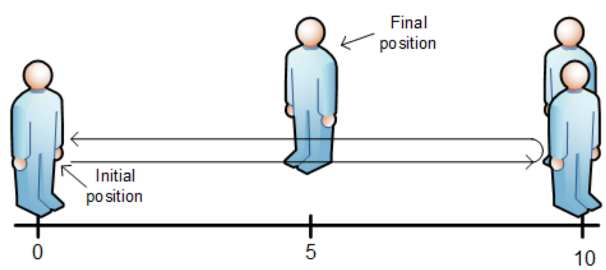
Solution: Let the amount of water that leaks out of the tank be represented as $W(t)$. Water is filling the storage closet, i.e., leaking out of the tank, at a **rate** of $W'(t) = 0.5t^2 + 2$. If we set 7 AM to $t = 0$, we want to find the net change of the water in the storage closet from $t = 0$ to $t = 4$.

$$W(4) - W(0) = \int_0^4 W'(t)dt$$
$$= \int_0^4 \left(\frac{1}{2}t^2 + 2\right) dt = \frac{1}{6}t^3 + 2t\Big|_0^4 \cong 18.7 \; liters$$

The tank will lose about 19 liters of water by 11 AM.

The integral of Velocity (Displacement versus Distance Traveled)

Let's now expand on the notion of speed and distance mentioned above. The motion of an object in one dimension is usually one of the first topics studied in physics. For this topic, it is important to understand the difference between the displacement of an object versus the distance that object has traveled. The diagram below illustrates the difference.



The person begins at the position, $x = 0$, then travels to position, $x = 10$. Next, they turn around and walk back to the position, $x = 5$, where x is measured in meters. The displacement, i.e., net change in position, is given as.

$$\Delta x = x_f - x_i = 5 - 0 = 5\; m$$

Based on the theorem above this value can also be computed as the integral of the rate of change of position, i.e., velocity: $\frac{dx}{dt} = v(t)$.

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} v(t)dt$$

On the other hand, for the distance traveled, D, we see they traveled a total of 15 meters.

$$D = |10| + |-5| = 10 + 5 = 15\ m$$

As you can see, we need to consider movement in both directions as positive to compute the distance traveled. Therefore, since the velocity is a signed quantity, we need to modify the theorem above to use the absolute value of the velocity as shown below.

$$D = \int_{t_i}^{t_f} |v(t)| dt$$

These results are captured in the theorem below.

The Integral of Velocity
If an object is in linear motion, i.e. one-dimensional motion, with a velocity of $v(t)$, then The **displacement**, Δx , i.e. , net change in position, during $[t_1, t_2]$ is given as: $\Delta x = \int_{t_1}^{t_2} v(t) dt$ The **distance traveled**, D, during $[t_1, t_2]$ is given as: $D = \int_{t_1}^{t_2} |v(t)| dt$

Let's do an example to illustrate.

Example 2: A particle has a velocity, $v(t) = t^3 - 10t^2 + 24t$ m/s. Compute the displacement and the total distance traveled in the interval, $[0,6]$.

Solution: We can start by sketching the velocity function. After factoring as shown below, we find the roots at $t = 0, 4$, and 6.

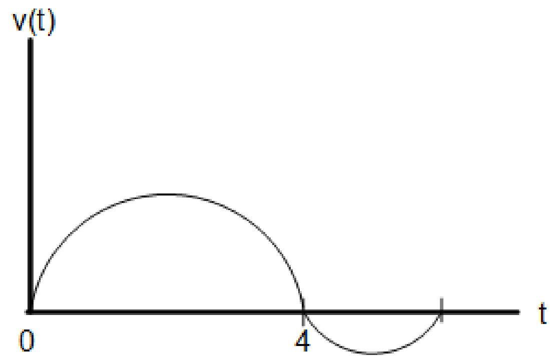
$$t^3 - 10t^2 + 24t = 0$$
$$t(t^2 - 10t + 24) = 0$$
$$t(t-4)(t-6) = 0$$

Next, we can check this sign of $v(t)$ between 0 and 4 and then between 4 and 6.

$$v(1) = 1^3 - 10 \cdot 1^2 + 24 \cdot 1 = 15$$

$$v(5) = 5^3 - 10 \cdot 5^2 + 24 \cdot 5 = -5$$

Based on the above we can sketch the graph of $v(t)$ as shown below.



The displacement, Δx, is found using $v(t)$ directly as follows

$$\Delta x = \int_0^6 v(t)dt$$

$$\Delta x = \int_0^6 (t^3 - 10t^2 + 24t)dt$$

$$= \frac{1}{4}t^4 - \frac{10}{3}t^3 + \frac{24}{2}t^2\Big|_0^6 = 36\ m$$

For the distance we need to consider, $|v(t)|$. For this we use $-v(t)$ for the interval $[4,6]$.

$$D = \int_{t_i}^{t_f} |v(t)|dt$$

$$= \int_0^4 v(t)dt + \int_4^6 -v(t)dt$$

$$= \int_0^4 (t^3 - 10t^2 + 24t)dt - \int_4^6 (t^3 - 10t^2 + 24t)dt$$

$$= \left(\frac{1}{4}t^4 - \frac{10}{3}t^3 + \frac{24}{2}t^2\Big|_0^4\right) - \left(\frac{1}{4}t^4 - \frac{10}{3}t^3 + \frac{24}{2}t^2\Big|_4^6\right)$$

$$= \left(\frac{128}{3}\right) - \left(\frac{20}{3}\right)$$

$$= \frac{148}{3} \cong 49.3\ m$$

As you can see the distance traveled results in a larger value than the displacement of the particle since it begins to retrace its steps after $t = 4$.

Total versus Marginal Cost

The net change concept discussed above can also be applied in business applications. To illustrate we consider the dollar cost of producing x units of a product as $C(x)$. The derivative, $C'(x)$, is called the *marginal cost*, and it measures the rate of change of the cost as a function of the number of units produced. For example, the change in cost for producing 1 versus 2 units may be less than the cost of producing 20 versus 21 units. The net change in cost, ΔC, to go from producing x_a units to producing $x_b$units is then

$$\Delta C = \int_{x_a}^{x_b} C'(x)dx$$

Let's again do an example to illustrate.

Example 3: The marginal cost of producing x widgets, (in units of 1000), is given as:

$$C'(x) = 300x^2 - 4{,}000x + 40{,}000 \text{ (dollars per 1000 widgets)}$$

a) Find the cost of increasing production from 10,000 to 15,000 widgets.
b) Determine the total cost of producing 15,000 widgets assuming this it costs \$30,000 to set up the manufacturing run, i.e. $C(0) = \$30{,}000$.

Solution:

a) The cost of increasing production from 10,000 to 15,000 widgets is given as

$$\Delta C = \int_{10}^{15} (300x^2 - 4{,}000x + 40{,}000)dx$$

$$= \frac{300}{3}x^3 - \frac{4000}{2}x^2 + 40000x\Big|_{10}^{15}$$

$$= (100 \cdot 15^3 - 2000 \cdot 15^2 + 40000 \cdot 15) - (100 \cdot 10^3 - 2000 \cdot 10^2 + 40000 \cdot 10)$$

$$= (487500) - (300000) \ = \ \$187{,}500$$

b) In this case we want the net change from 0 to 15,000 widgets.

$$\Delta C = \int_{0}^{15} (300x^2 - 4{,}000x + 40{,}000)dx$$

$$= \frac{300}{3}15^3 - \frac{4000}{2}15^2 + 40000 \cdot 15\Big|_{0}^{15} \ = \ \$487{,}500$$

The total cost, however, would include the initial cost of \$30,000.

$$T \ = \ \$487{,}500 + \$30{,}000 = \$517{,}500$$

We will finish this lesson with a few more examples using the applications discussed above.

Example 4:
A projectile is released with an initial (vertical) velocity of 100 m/s. Use the formula $v(t) = 100 - 9.8t$ to find the displacement and the distance traveled in the first 15 seconds.

Solution:

The vertical displacement, Δy, of the projectile is given by our standard formula.

$$\Delta y = \int_0^{15} v(t)dt$$
$$\Delta y = \int_0^{15} (100 - 9.8t\,)dt$$
$$= 100t - \frac{9.8}{2}t^2\Big|_0^{15} \quad = \; 397.5\,m$$

However, to find the total distance, we need to determine if the projectile reached its peak and began falling within the 15 seconds. The projectile will reach its peak when the velocity is zero.

$$v(t) = 0 = 100 - 9.8t$$
$$t = \frac{100}{9.8} \cong 10.2\ s$$

Therefore, the projectile begins to fall back to the ground after about 10.2 seconds. To find the total distance we need to solve the following.

$$D = \int_0^{15} |v(t)\,|dt$$
$$= \int_0^{\frac{100}{9.8}} (100 - 9.8t\,)dt - \int_{\frac{100}{9.8}}^{15} (100 - 9.8t\,)dt$$
$$= \left(100t - \frac{9.8}{2}t^2\Big|_0^{\frac{100}{9.8}}\right) - \left(100t - \frac{9.8}{2}t^2\Big|_{\frac{100}{9.8}}^{15}\right)$$
$$= (510.2) - (-112.7)$$
$$\cong 623\ m$$

Example 5:
The marginal cost of producing x tablet computers is $C'(x) = 120 - 0.06x + 0.00001x^2$. What is the cost of producing 3,000 units if the set-up cost is $90,000? If production is set at 3,000 units what is the cost of producing 200 extra units? Compare this to the cost of the first 200 units.

Solution: The net change in cost from 0 to 3,000 units is given as.

$$\Delta C = \int_0^{3000} (120 - 0.06x + 0.00001x^2)dx$$
$$= 120x - 0.03x^2 + \frac{0.00001}{3}x^3 \Big|_0^{3000}$$
$$= \$180{,}000$$

To find the total cost we need to add the initial set-up cost.

$$T = \$180{,}000 + \$90{,}000$$
$$T = \$270{,}000$$

To find the additional cost in producing 200 units more than the 3,000, we solve the following integral

$$\Delta C = \int_{3000}^{3200} (120 - 0.06x + 0.00001x^2)dx$$
$$= 120x - 0.03x^2 + \frac{0.00001}{3}x^3 \Big|_{3000}^{3200} \cong \$6{,}026.70$$

Comparing this cost to the cost of the first 200 units we have

$$\Delta C = \int_0^{200} (120 - 0.06x + 0.00001x^2)dx$$
$$= 120x - 0.03x^2 + \frac{0.00001}{3}x^3 \Big|_0^{200} \cong \$22{,}826.70$$

The first 200 units are **much** more costly!

Example 6: Water flows into an empty reservoir at a rate of $3000 + 20t$ liters/hour. What is the quantity of water in the reservoir after 5 hours?

Solution: The rate of change of the water in the reservoir is

$$w'(t) = 3000 + 20t$$

Therefore, the amount of water after 5 hours is given as

$$W = \int_0^5 (3000 + 20t)dt$$
$$= 3000t + \frac{20}{2}t^2 \Big|_0^5 = 15250\ liters$$

Final Summary for Integral Applications – Net Change as the Integral of a Rate of Change

Net Change as the Integral of the Rate of Change
The net change of $f(x)$ over an interval, $[x_1, x_2]$, is given by the following definite integral. $$\int_{x_1}^{x_2} f'(x)dx = f(x_2) - f(x_1)$$
The Integral of Velocity
If an object is in linear motion, i.e. one-dimensional motion, with a velocity, $v(t)$, then   The **displacement**, Δx , i.e. , net change in position, during $[t_1, t_2]$ is given as: $$\Delta x = \int_{t_i}^{t_f} v(t)dt$$ The **distance traveled**, D, during $[t_1, t_2]$ is given as: $$D = \int_{t_i}^{t_f} \lvert v(t) \rvert dt$$
Production Costs
The cost, C, to produce x units of a certain product is represented as $C(x)$.   The rate of change of cost with respect to the number of units produced is referred to as the *marginal cost* and is given $C'(x)$.   The net change in cost, ΔC, to go from producing x_a units to x_b units is given as: $$\Delta C = \int_{x_a}^{x_b} C'(x)dx$$

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