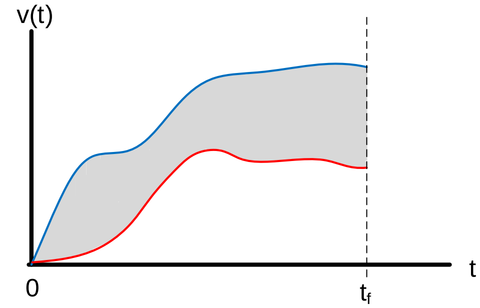
Integral Applications – Area Between Curves

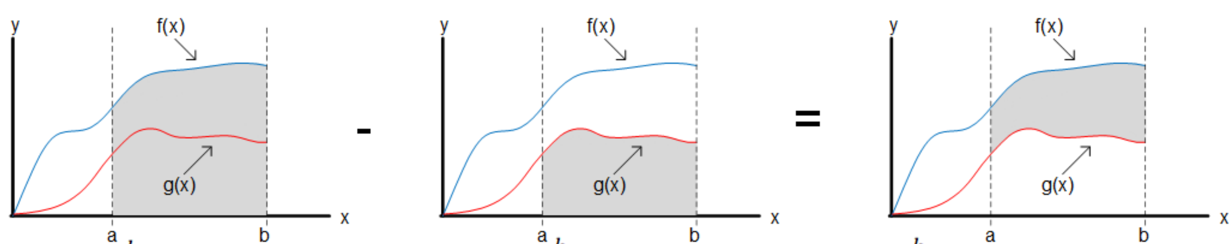
In some instances, we may be interested in finding the area between two curves. For example, suppose we wanted to determine the extra distance a vehicle could travel on a straight road in a given time if less cars were on the road. For this scenario let's assume we have a way to model the velocity of the vehicle as a function of the number of other cars on the road. The graph below shows two specific velocity functions. The red is the one with more cars on the road and the blue is the one with less cars on the road. Since the distance traveled is equal to the area under the velocity curve, the extra distance that the vehicle with less cars on the road was able to travel is given by the shaded region between the two graphs.



In this example it may see easy to visualize the area, however there are various ways two curves may create a "between" region. Two major categories are whether the region is created with curves that are vertically oriented to each other, (i.e. we integrate along the x-axis), or with curves that are horizontally oriented to each other, (i.e. we integration along the y-axis). Let's review these two cases below.

Integration along the x-axis

Given two functions, e.g. $f(x), g(x)$, we integrate along the x-axis to find the area between the two curves. In the simplest case $f(x) \geq g(x)$ for all x in an interval $[a, b]$ and the between region is called *vertically simple*. We illustrate how to compute the area with the figure below.



$$\int_a^b f(x)dx \quad - \quad \int_a^b g(x)dx \quad = \quad \int_a^b \big(f(x) - g(x)\big)dx$$

As the figure illustrates to find the area in a vertically simple region, we simply integrate the top function minus the bottom function. If we consider $f(x)$ as the top curve and $g(x)$ as the bottom curve, we obtain the following definition.

Area Between Curves that form a Vertically Simple Region
If $f(x) \geq g(x)$ for all x in the interval $[a, b]$, i.e. they form a vertically simple region, then the area between the two curves is given as $$\int_a^b \big(f(x) - g(x)\big)dx$$

Example 1: Find the area of the region between the following two functions over the given interval.

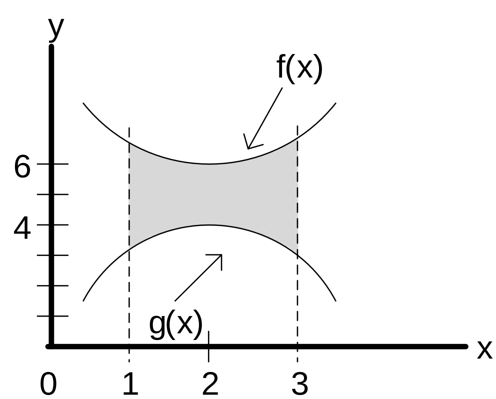
$f(x) = x^2 - 4x + 10$ $\qquad$ $g(x) = -x^2 + 4x$ $\qquad$ $[1,3]$

Solution: The first thing we need to do is determine what the region looks like by sketching or graphing the function in at least the interval under consideration. We can sketch the functions if we first complete the squares.

$$\begin{aligned} f(x) &= x^2 - 4x + 10 \\ &= (x-2)^2 - 4 + 10 \\ &= (x-2)^2 + 6 \end{aligned}$$

$$\begin{aligned} g(x) &= -x^2 + 4x \\ &= -((x-2)^2 - 4) \\ &= -(x-2)^2 + 4 \end{aligned}$$

Parabola with minimum vertex at $(2,6)$. $\qquad$ Parabola with maximum vertex at $(2,4)$.



The area is then found as follows:

$$\begin{aligned} A &= \int_1^3 \{(x^2 - 4x + 10) - (-x^2 + 4x)\}dx \\ &= \int_1^3 (2x^2 - 8x + 10)\,dx \\ &= \frac{2}{3}x^3 - 4x^2 + 10x\Big|_1^3 \\ &= \left(\frac{2}{3}3^3 - 4 \cdot 3^2 + 10 \cdot 3\right) - \left(\frac{2}{3}1^3 - 4 \cdot 1^2 + 10 \cdot 1\right) \\ &= (12) - \left(\frac{20}{3}\right) = \frac{16}{3} \end{aligned}$$

Example 2: Find the area of the region enclosed by the two functions.

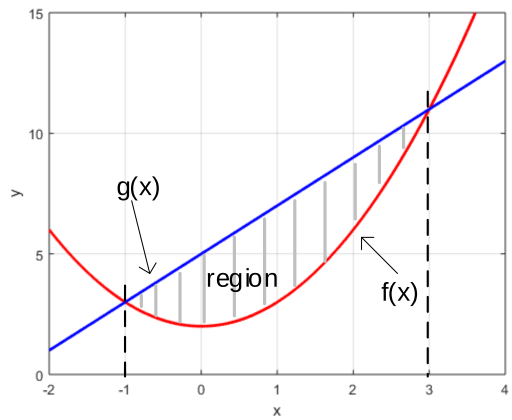
$$f(x) = x^2 + 2 \qquad\qquad g(x) = 2x + 5$$

Solution:

Since we are not given a specific interval, we assume the curves intersect such that a region is formed. We can find the points of intersection by setting the functions equal to each other.

$$x^2 + 2 = 2x + 5$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

Therefore, curves intersect at $x = -1$ and $x = 3$. We sketch the region below.



The area is then found as follows:

$$A = \int_{-1}^{3} \big(g(x) - f(x)\big) dx$$

$$= \int_{-1}^{3} \{(2x + 5) - (x^2 + 2)\} dx$$

$$= \int_{-1}^{3} (-x^2 + 2x + 3) dx$$

$$= -\frac{1}{3}x^3 + x^2 + 3x\Big|_{-1}^{3}$$

$$= (9) - \left(-\frac{5}{3}\right) = \frac{32}{3}$$

Example 3: Find the area of the region enclosed by the two functions.

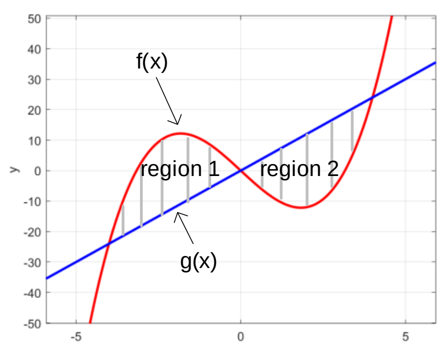
$$f(x) = x^3 - 10x \qquad\qquad g(x) = 6x$$

Solution:

Again, we are not given an interval, so we first need to find any intersection points.

$$\begin{aligned} x^3 - 10x &= 6x \\ x^3 - 16x &= 0 \\ x(x^2 - 16) &= 0 \\ x(x-4)(x+4) &= 0 \end{aligned}$$

These curves intersect at three locations, $x = -4$, $x = 0$, and $x = 4$, therefore there is likely two separate regions to consider. To determine which function is the top we could either test values within the two intervals, sketch the curves by hand, or graph the curves directly with a graphing calculator. Below we show a graph of the curves.



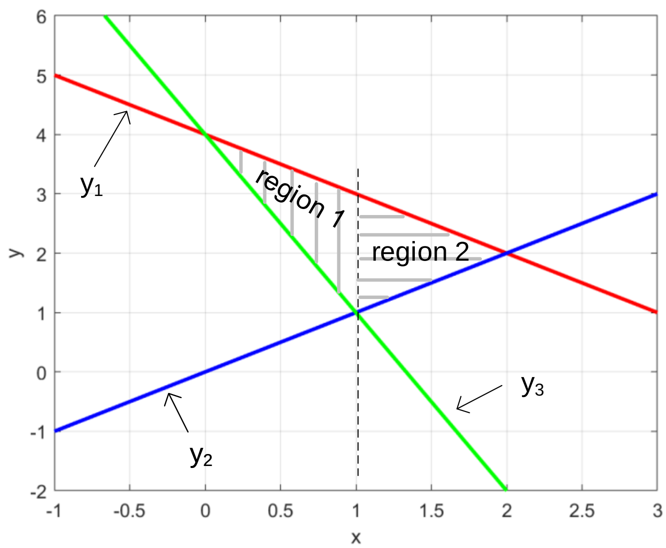
As you can see there are indeed two separated regions. Additionally, in the first region $f(x)$ is the top function and $g(x)$ is the bottom function, however this is reversed in the second region. In this case we do not have a *vertically simple region*, and to find the total area we need subdivide the area into two simple regions and integrate as shown below.

$$\begin{aligned} A &= \int_{-4}^{0} \big((x^3 - 10x) - 6x\big)dx + \int_{0}^{4} \big(6x - (x^3 - 10x)\big)dx \\ &= \int_{0}^{-4} (-x^3 + 16x)dx + \int_{0}^{4} (-x^3 + 16x)dx \\ &= \left(-\frac{1}{4}x^4 + 8x^2\Big|_0^{-4}\right) + \left(-\frac{1}{4}x^4 + 8x^2\Big|_0^{4}\right) \\ &= (64) + (64) = 128 \end{aligned}$$

Example 4: Find the area of the region enclosed by the following functions.

$y_1 = 4 - x$ $\qquad\qquad$ $y_2 = x$ $\qquad\qquad$ $y_3 = 4 - 3x$

Solution: We sketch the graphs first to find the region enclosed.



In this case we also need to subdivide the area into two regions since the top and bottom functions change at $x = 1$. In the first region the top function is y_1 and the bottom function is y_3, but in the second region the top function is y_1 and the bottom function is y_2. The area in both regions is found as follows.

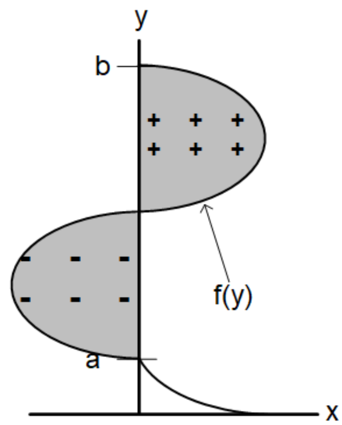
$$
\begin{aligned}
A &= \int_0^1 (y_1 - y_3)dx + \int_1^2 (y_1 - y_2)dx \\
&= \int_0^1 \big((4 - x) - (4 - 3x)\big)dx + \int_1^2 \big((4 - x) - (x)\big)dx \\
&= \int_0^1 2x dx + \int_1^2 (4 - 2x)dx \\
&= \left(\frac{2}{2}x^2\Big|_0^1\right) + \left(4x - \frac{2}{2}x^2\Big|_1^2\right) \\
&= (1) + \big((8 - 4) \; - \; (4 - 1)\big) = \; 2
\end{aligned}
$$

Integration along the y-axis

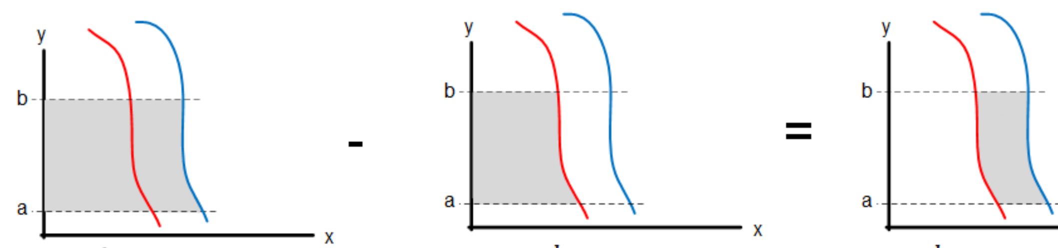
In some cases, we may be given a curve as a function of y, i.e. $f(y)$, instead as functions of x, i.e. $f(x)$. If so, we may ask what is the meaning of the following integral?

$$\int_a^b f(y)\,dy$$

Extending what we know about x-axis integration this integral can also be interpreted as the signed area, where the region to the left of the y-axis has negative area and the region to the right of the y-axis gives positive area as illustrated below.



In this case, when we are given two functions, e.g. $f(y), g(y)$, we integrate along the y-axis to find the area between the two curves. In the simplest case $f(y) \geq g(y)$ for all y in an interval $[a, b]$ and the between region is called *horizontally simple*. We illustrate how to compute the area with the figures below.



$$\int_a^b f(y)dy \quad - \quad \int_a^b g(y)dy \quad = \quad \int_a^b \big(f(y) - g(y)\big)dx$$

As the figure illustrates to find the area in a horizontally simple region, we simply integrate the *right* function minus the *left* function. If we consider $f(y)$ as the right curve and $g(y)$ as the left curve, we obtain the following definition.

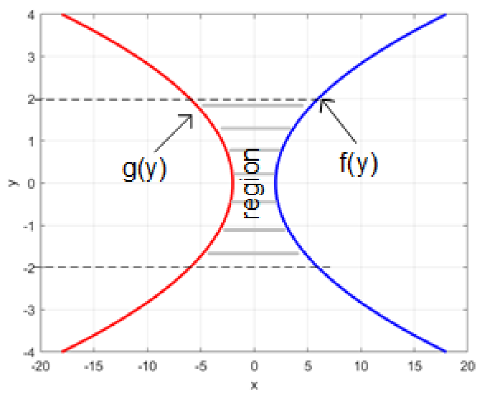
Area Between Curves that form a Horizontally Simple Region
If $f(y) \geq g(y)$ for all y in the interval $[a, b]$, i.e. they form a horizontally simple region, then the area between the two curves is given as $$\int_a^b (f(y) - g(y))dy$$

Example 5: Find the area of the region between the following two functions over the given interval.

$$f(y) = y^2 + 2 \qquad g(y) = -(y^2 + 2) \qquad -2 \leq y \leq 2$$

Solution:

From the equations alone, we see the region is horizontally simple, i.e. $f(y) \geq g(y)$, since $f(y)$ is always positive and $g(y)$ is always negative. Therefore, we can directly apply the theorem from above to find the area of the region. Nonetheless, for illustrative purposes we plot the region below.



$$A = \int_{-2}^{2} (y^2 + 2) + (y^2 + 2)dy$$

$$= \int_{-2}^{2} (2y^2 + 4)dy$$

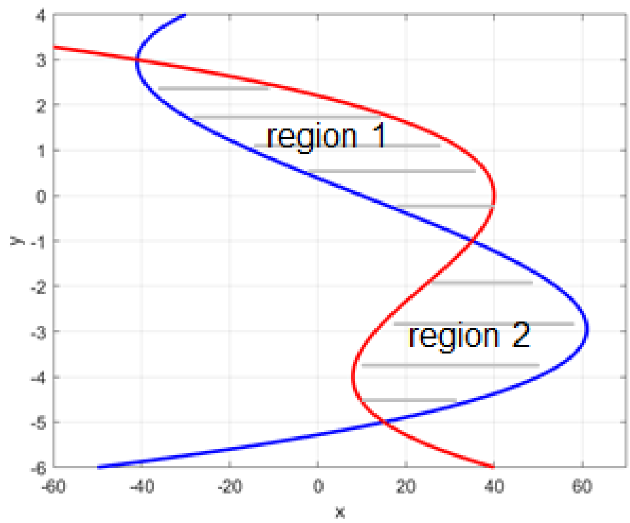
$$= \frac{2}{3}y^3 + 4y\Big|_{-2}^{2}$$

$$= \left(\frac{2^4}{3} + 8\right) - \left(-\frac{2^4}{3} - 8\right)$$

$$= \frac{2^5}{3} + \frac{48}{3} = \frac{80}{3}$$

Example 6: The figure below shows the region enclosed by the functions given. Match the equations with the correct curves and find the area of the region.

$$f(y) = y^3 - 26y + 10 \qquad\qquad g(y) = -y^3 - 6y^2 + 40$$



Solution:

The figure shows two different regions, where each one is horizontally simple. To identify the curves, we can choose $y = 0$. In this case we can see that $f(0) = 10$, which refers to the blue curve, while $g(0) = 40$ refers to the red curve. The total area is then

$$A = \int_{-5}^{-1} \big(f(y) - g(y)\big)dy + \int_{-1}^{3} \big(g(y) - f(y)\big)dy$$

$$= \int_{-5}^{-1} (2y^3 + 6y^2 - 26y - 30)dy + \int_{-1}^{3} (-2y^3 - 6y^2 + 26y + 30)dy$$

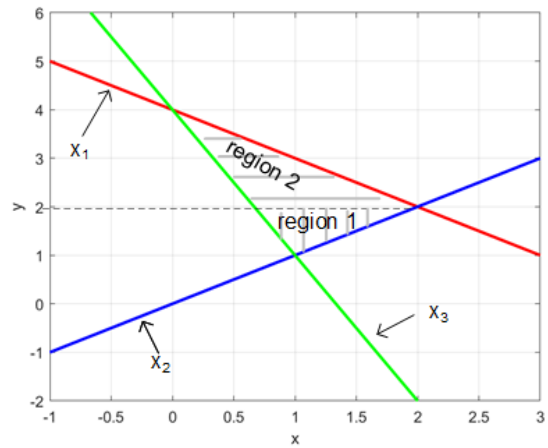
$$= \left(\frac{1}{2}y^4 + 2y^3 - 13y^2 - 30y\Big|_{-5}^{-1}\right) + \left(-\frac{1}{2}y^4 - 2y^3 + 13y^2 + 30y\Big|_{-1}^{3}\right)$$

$$= (128) + (128) = 256$$

Example 7: Use the equations from example 4 to find the area of the region by employing integration along the y-axis.

Solution: We can start by redrawing the figure with redefined regions according the "new" functions, which are the original functions solved for x.

$$x_1 = 4 - y \qquad x_2 = y \qquad x_3 = \frac{4-y}{3}$$



The area can now be computed as follows:

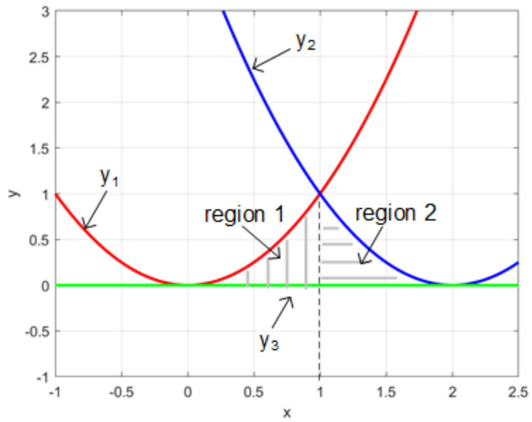
$$\begin{aligned}
A &= \int_1^2 (x_2 - x_3)dy + \int_2^4 (x_1 - x_3)dy \\
&= \int_1^2 \left(y - \left(\frac{4-y}{3}\right)\right) dy + \int_2^4 \left((4-y) - \left(\frac{4-y}{3}\right)\right) dy \\
&= \frac{4}{3}\int_1^2 (y-1)dy + \frac{2}{3}\int_2^4 (4-y)dy \\
&= \frac{4}{3}\left(\frac{1}{2}y^2 - y\Big|_1^2\right) + \frac{2}{3}\left(4y - \frac{1}{2}y^2\Big|_2^4\right) \\
&= \frac{4}{3}\left(\frac{1}{2}\right) + \frac{2}{3}(2) \\
&= \frac{4}{6} + \frac{8}{6} = 2
\end{aligned}$$

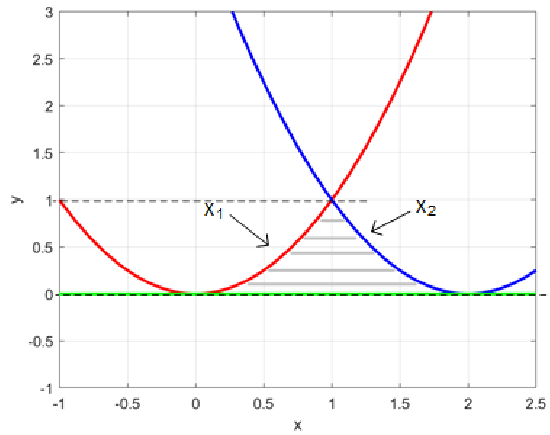
As expected, (and required), the area of the region is the same as we found in example 4! This example shows that the area between curves can be computed using either x-axis or y-axis integration, whichever is most convenient for the particular problem.

Example 8: Find the area of the region between the following functions.

$$y_1 = x^2 \qquad y_2 = (x-2)^2 \qquad y_3 = 0$$

Solution: For illustration we solve this example using both x and y axis integration. The equations are straightforward to graph and are shown below for both cases.





x-axis integration: The figure on the left is for x-axis integration and shows two regions that need to be computed separately. Note we use the substitution method for the second integral.

$$A = \int_0^1 (y_1 - y_3)dx + \int_1^2 (y_2 - y_3)dx$$

$$= \int_0^1 (x^2 - 0)dx + \int_1^2 ((x-2)^2 - 0)dx$$

$$= \int_0^1 x^2 dx + \int_{-1}^0 u^2 du$$

$$= \left(\frac{1}{3}x^3\bigg|_0^1\right) - \left(\frac{1}{3}u^3\bigg|_0^{-1}\right) = \left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

y-axis integration: The figure on the right is for y-axis integration and shows a single region. However, we need to be careful when creating the inverse functions. For the first equation we have $x_1 = \pm\sqrt{y}$, however since the portion we want to integrate is for positive values of x we use $x_1 = \sqrt{y}$. For the second function we have $x_2 = \pm\sqrt{y} + 2$. In this case if we substitute $y = 1$ we get $x_2 = 3$ or $x_2 = 1$. From the figure we see that $x_2 = 1$ is needed and therefore the equation we use is $x_2 = -\sqrt{y} + 2$

$$A = \int_0^1 (x_2 - x_1)dy$$

$$= \int_0^1 \left((2 - \sqrt{y}) - (\sqrt{y})\right) dy$$

$$= 2\int_0^1 (1 - y^{1/2})dy$$

$$= 2\left(y - \frac{2}{3}y^{3/2}\bigg|_0^1\right) = 2\left(1 - \frac{2}{3}\right) = \frac{2}{3}$$

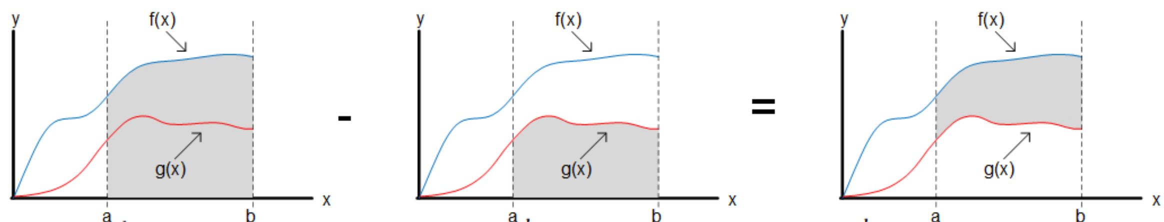
Final Summary for Integral Applications – Area Between Curves

Area Between Curves
Given two curves in the x-y plane, the area between these curves, either over a given interval or where they naturally intersect, may be of interest. The region formed may be "simple", as defined below, in which case the area can be computed with a single integral. Non-simple regions can be subdivided into multiple "simple" regions and integrated separately to find the total area. In either case, it's always best to sketch or graph the functions to visualize the region in order to properly set up the integral(s).

Area Between Curves that form a Vertically Simple Region

If $f(x) \geq g(x)$ for all x in the interval $[a, b]$, i.e. they form a vertically simple region, then the area between the two curves is given as

$$\boxed{\int_a^b \big(f(x) - g(x)\big)dx}$$

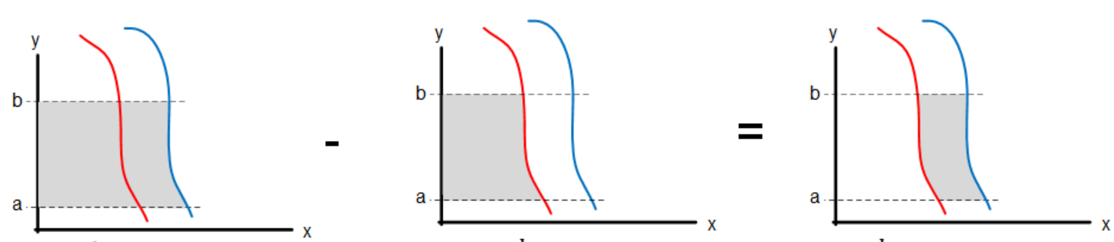


$$\int_a^b f(x)dx \quad - \quad \int_a^b g(x)dx \quad = \quad \int_a^b \big(f(x) - g(x)\big)dx$$

Area Between Curves that form a Horizontally Simple Region

If $f(y) \geq g(y)$ for all y in the interval $[a, b]$, i.e. they form a horizontally simple region, then the area between the two curves is given as

$$\boxed{\int_a^b \big(f(y) - g(y)\big)dy}$$



$$\int_a^b f(y)dy \quad - \quad \int_a^b g(y)dy \quad = \quad \int_a^b \big(f(y) - g(y)\big)dx$$

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