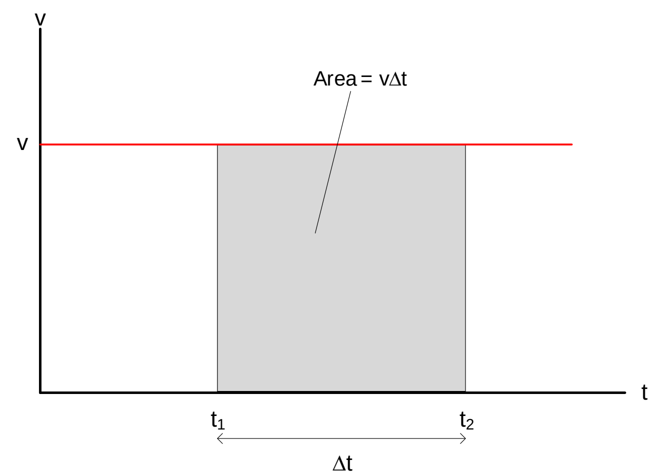
Integration – The Area Problem

Having dealt with the first major area of calculus, differential calculus, we now study the second major area called integral calculus. We mentioned earlier that integral calculus deals with the accumulation of one quantity as it varies with respect to another. This explanation is likely unclear at the moment, but hopefully will become clearer as we progress. Let's begin with a simple question: "What is the total distance traveled for a vehicle that has traveled in a straight line at a constant velocity of 60 miles per hour for 6 hours?". The answer is obtained by multiplying the distance per hour, velocity, by the number of hours, i.e. $D = 60 \cdot 6 = 360\ miles$. In general, the distance traveled over a time interval, $[t_1, t_2]$, for an object moving along a straight line with a constant velocity, v, is given as:

Straight Line Constant Velocity Distance Traveled
$$D = v\Delta t$$ Where, $\Delta t = t_2 - t_1$

We can explore this seemingly simple question further by looking at the velocity versus time graph shown below.

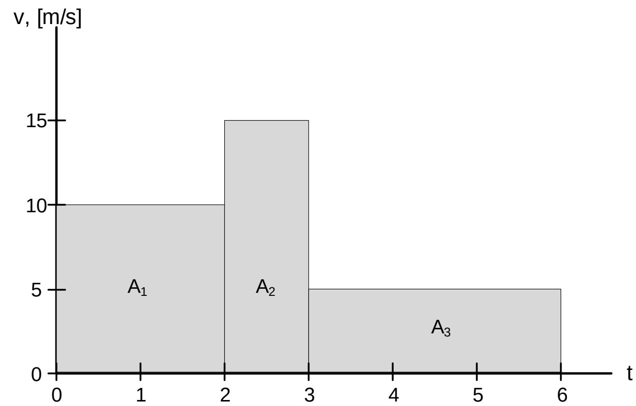


Notice that the distance traveled, D, is also equal to the area under the graph from t_1 to t_2.

Distance Traveled
$$D = Area\ under\ the\ graph\ of\ velocity\ over\ [t_1, t_2]$$

This may seem like a trivial realization, however as we will soon show this observation is key to the notion of integration. To start, we can ask another question: "How can we compute the distance traveled when we have non-constant velocity motion?". Obviously, we cannot use the first equation, $D = v\Delta t$, since there is no single value for v over the interval. What we will eventually prove is that the second method of computing the distance, using the area under the graph, does represent the actual distance traveled.

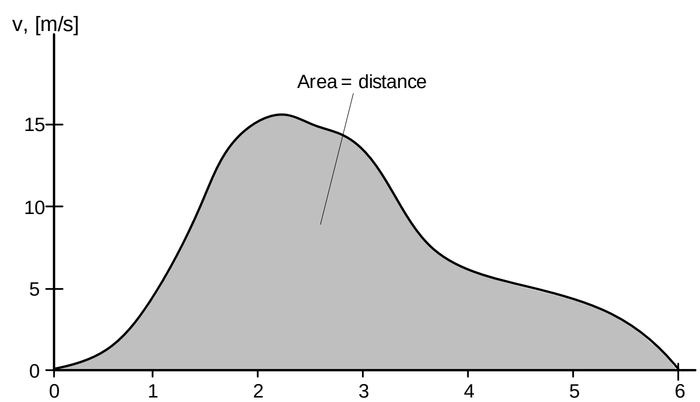
Let's begin to explore this with a perhaps unrealistic but simple example of non-constant velocity motion. The graph below represents an object with a velocity that changes over time but is constant over certain intervals. We say unrealistic because of the abrupt velocity changes between intervals.



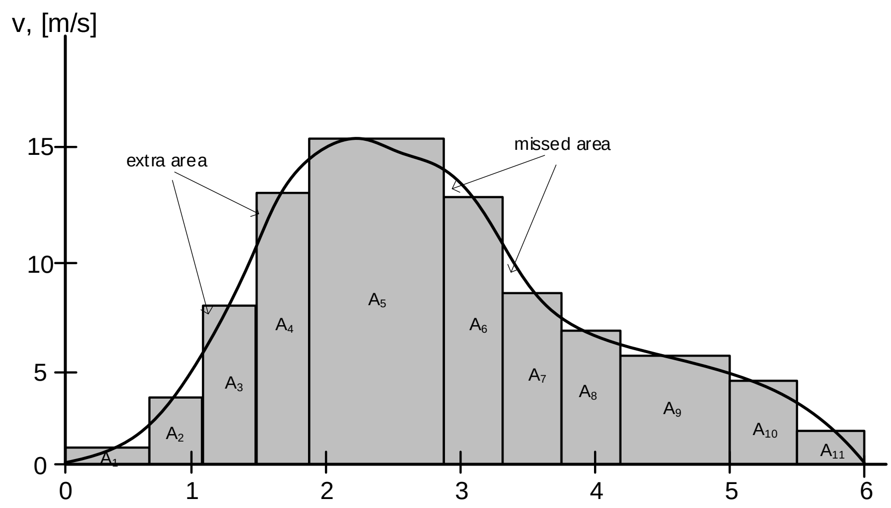
The distance traveled in each interval is the area of the rectangle in that interval. Therefore, the total distance traveled is the sum of the areas.

$$D = (A_1 + A_2 + A_3)$$
$$D = (10 \cdot 2) + (15 \cdot 1) + (5 \cdot 3)$$
$$D = 50\ m$$

To make this example a bit more realistic we remove the abrupt changes and have the velocity change continuously, such as in the graph shown below.



The distance is equal to the area under the graph, however we do not have a strategy for computing the area of an arbitrarily curved shape. Instead we propose to *approximate* the area using sufficiently small rectangles as illustrated below.



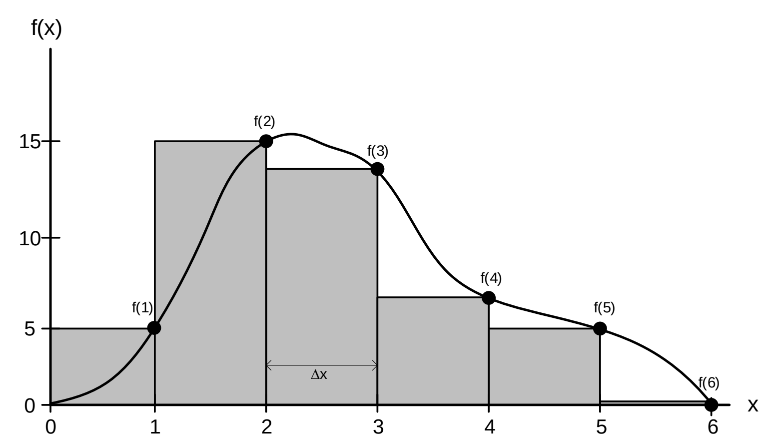
In this case we can approximate the area by summing the area from each rectangle. As you can see in some cases, we may overestimate the area while others we underestimate. In order to use this method of approximation we need to develop a systematic way for its computation, which we explore below.

Approximating the Area Under a Graph:

The distance example we used above is just one of many examples of the need to compute the area under a graph. With this in mind we switch to using a generic function, $f(x)$ for the following descriptions. As mentioned, one way to approximate the area under a graph is to construct rectangles that generally follow the curve and sum the area of each of the rectangles. To make this process more systematic we should choose a simple way to decide on both the width and height of the rectangles. For the width we shall choose a whole number, N, and divide the entire interval, $[a, b]$, into N subintervals of equal width. For the height we develop three different options as discussed below:

1. Use the ***right endpoint*** of each subinterval.
2. Use the ***left endpoint*** of each subinterval.
3. Use the ***midpoint*** of each subinterval.

Let's develop the method of computing the area using the right endpoint first.



Using the figure above as reference, we choose the interval, $[a, b]$, to be $[0,6]$, and the number of intervals, N, to be 6. Therefore, the size of each interval, i.e., the width of each rectangle, is

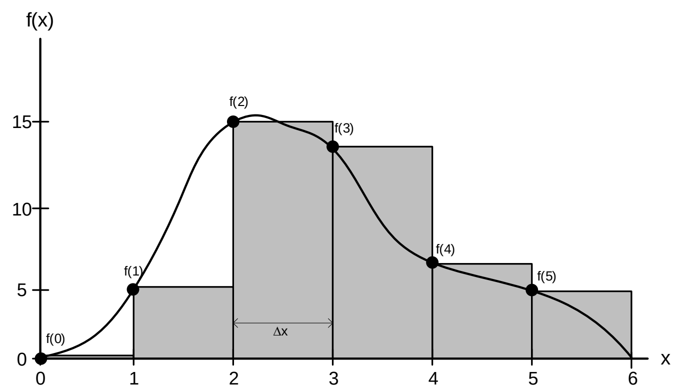
$$\Delta x = \frac{b-a}{N} = \frac{6-0}{6} = 1$$

Next, since we are using the right endpoints, the height of each rectangle is shown in the figure to be the value of the function evaluated at $x = 1, 2, 3, 4, 5, 6$.

The approximate area, $\tilde{A}$, is then given as

$$\tilde{A} = f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x + f(6)\Delta x$$
$$\tilde{A} = \Delta x\big(f(1) + f(2) + f(3) + f(4) + f(5) + f(6)\big)$$

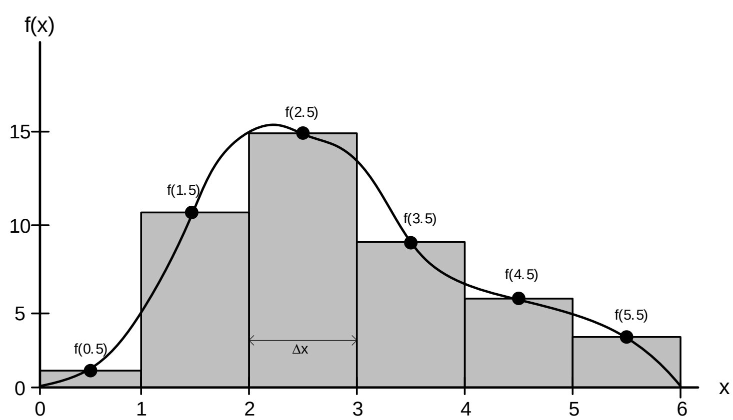
If we instead choose the left endpoints the rectangles created are as shown below.



And the approximate area, $\tilde{A}$, is then given as

$$\tilde{A} = f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x$$
$$\tilde{A} = \Delta x\big(f(0) + f(1) + f(2) + f(3) + f(4) + f(5)\big)$$

Finally, if we choose the midpoint, the graph is as follows:



And the approximate area, $\tilde{A}$, is then given as

$$\tilde{A} = f(0.5)\Delta x + f(1.5)\Delta x + f(2.5)\Delta x + f(3.5)\Delta x + f(4.5)\Delta x + f(5.5)\Delta x$$
$$\tilde{A} = \Delta x\big(f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)\big)$$

The main component in each of the above area approximations is the summing of N function values. We can generalize these expressions, and write them more compactly, using summation notation as shown, (see appendix for summation review if needed).

Area Approximation Using Right Endpoints
$$R_N = \Delta x \sum_{j=0}^{N-1} f(x_{j+1})$$
Area Approximation Using Left Endpoints
$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$
Area Approximation Using Midpoints
$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$
Where the following definitions apply to all three approximations. If the interval over which the area is computed is defined as $[a, b]$, the subinterval width is $$\Delta x = \frac{b-a}{N}$$ and the evaluation points are $$x_j = a + j\Delta x$$

Example 1

Let's use these formulas to approximate the area of $f(x) = x^2 + 2$ over the interval $[0,4]$ using $N = 8$ subintervals.

For all three cases we have

$$\Delta x = \frac{4-0}{8} = \frac{1}{2} \qquad\qquad x_j = 0 + j\frac{1}{2} = \frac{j}{2}$$

<u>Right Endpoint Approximation:</u>

$$\begin{aligned} R_8 &= \frac{1}{2}\sum_{j=0}^{7} f(x_{j+1}) \\ &= \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f\left(\frac{2}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{8}{2}\right)\right) \\ &= \frac{1}{2}(2.25 + 3 + 4.25 + 6 + 8.25 + 11 + 14.25 + 18) = 33.5 \end{aligned}$$

<u>Left Endpoint Approximation:</u>

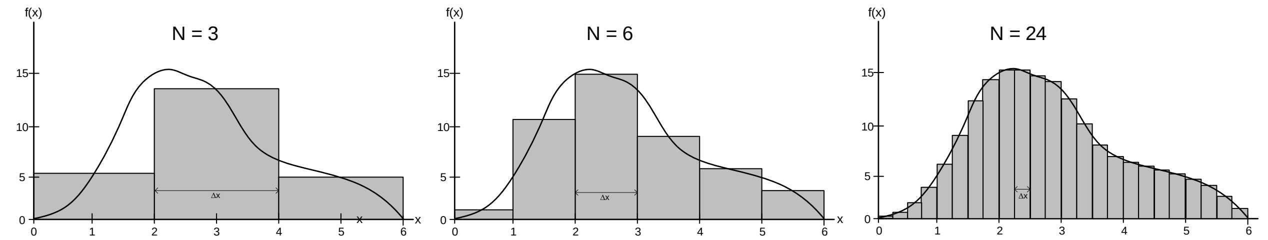
$$\begin{aligned} L_8 &= \frac{1}{2}\sum_{j=0}^{7} f(x_j) \\ &= \frac{1}{2}\left(f(0) + f\left(\frac{1}{2}\right) + f\left(\frac{2}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right) + f\left(\frac{7}{2}\right)\right) \\ &= \frac{1}{2}(2 + 2.25 + 3 + 4.25 + 6 + 8.25 + 11 + 14.25) = 25.5 \end{aligned}$$

<u>Midpoint Approximation:</u>

$$\begin{aligned} M_8 &= \frac{1}{2}\sum_{j=0}^{7} f\left(\frac{x_j + x_{j+1}}{2}\right) \\ &= \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right) \\ &= \frac{1}{2}(2.0625 + 2.5625 + 3.5625 + 5.0625 + 7.0625 + 9.5625 + 12.5625 + 16.0625) \\ &= 29.25 \end{aligned}$$

The exact answer turns out to be is $29.\overline{3}$, therefore the midpoint approximation turns out to be a good approximation.

Next, we ask the question: "How can we make the approximations more accurate?". To answer this, we can take a look at the three figures below.



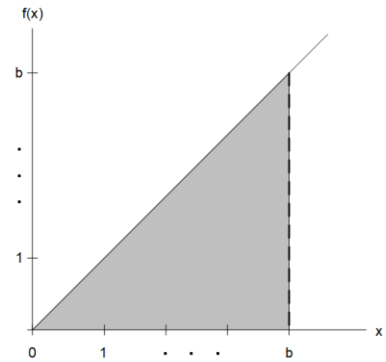
The figures make it clear that as N gets larger the error in our approximation becomes smaller. It seems that if we consider the limit as $N \to \infty$, the approximation error should approach zero. This is indeed true as the theorem below formally states.

Area Under the Graph
If f is continuous on $[a,b]$, then the endpoint and midpoint approximations approach the same value, A, in the limit as $N \to \infty$. In other words: $$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = A$$ If $f(x) > 0$ on $[a,b]$, then A represents the area under the graph of $f(x)$ on $[a,b]$ Note: If $f(x)$ takes on negative values the above theorem still holds, but we must interpret A as the *signed area,* which is discussed in the next section.

Let's see if we can compute the exact area in the next few examples using the above theorem. First, we provide, without proof, the following two summation formulas, which we will find useful in the upcoming examples.

Summation Formulas
$$\sum_{j=0}^{N-1} j = 0 + 1 + 2 \cdots (N-1) = \frac{N(N-1)}{2} = \frac{N^2}{2} - \frac{N}{2}$$ $$\sum_{j=0}^{N-1} j^2 = 0 + 1^2 + 2^2 \cdots (N-1)^2 = \frac{N(N-1)(2N-1)}{6} = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}$$

Example 2: Compute the area under the graph of $f(x) = x$ on $[0, b]$.



Since the graph is a straight line, we can compute the exact area using geometry. The area of a triangle is given by half of the base times the height, however in this case $b = h$.

$$A = \frac{1}{2}bh = \frac{1}{2}b^2$$

Now let's attempt to verify this result starting with the left endpoint approximation from above. Since our interval $[0, b]$, we have

$$\Delta x = \frac{b - 0}{N} = \frac{b}{N} \qquad\qquad x_j = a + j\Delta x = j\frac{b}{N}$$

The left endpoint approximation is then

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$L_N = \frac{b}{N}\sum_{j=0}^{N-1} x_j$$

$$L_N = \frac{b}{N}\sum_{j=0}^{N-1} j\frac{b}{N}$$

$$L_N = \frac{b^2}{N^2}\left(\sum_{j=0}^{N-1} j\right)$$

$$L_N = \frac{b^2}{N^2}\left(\frac{N^2}{2} - \frac{N}{2}\right)$$

$$L_N = \frac{b^2}{2} - \frac{b^2}{2N}$$

Where, we used the fact that $f(x_j) = x_j = j\frac{b}{N}$, together with the first summation formula from above.

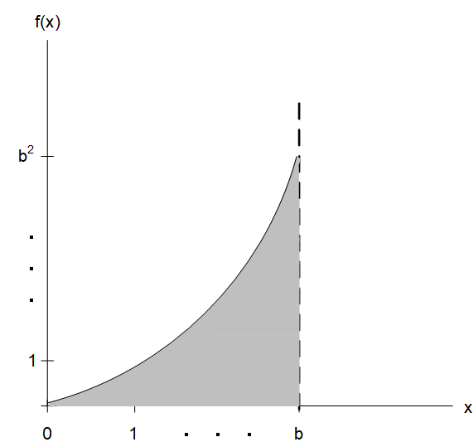
Now, according to the theorem above if we let $N \to \infty$ we get the exact area.

$$A = \lim_{N\to\infty} L_N = \lim_{N\to\infty}\left(\frac{b^2}{2}\right) - \lim_{N\to\infty}\left(\frac{b^2}{2N}\right)$$
$$A = \frac{b^2}{2} - 0 = \frac{1}{2}b^2$$

Which is the exact answer we obtained from simple geometry! Furthermore, we can generalize this result with the following formula.

The area under the graph of $f(x) = Cx$ on $[0, b]$, is given as: $$A = C\frac{1}{2}b^2$$

Example 3: Compute the area under the graph of $f(x) = x^2$ on $[0, b]$.



In this case the graph is curved and hence we cannot use geometry to compute the exact area. Therefore, we instead go directly to using the approximation technique from the previous example. The variables for the sum remain as they were in the last example.

$$\Delta x = \frac{b-0}{N} = \frac{b}{N} \qquad\qquad x_j = a + j\Delta x = j\frac{b}{N}$$

The left endpoint approximation is then

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$L_N = \frac{b}{N} \sum_{j=0}^{N-1} {x_j}^2$$

$$L_N = \frac{b}{N} \sum_{j=0}^{N-1} j^2 \left(\frac{b}{N}\right)^2$$

$$L_N = \frac{b^3}{N^3} \left(\sum_{j=0}^{N-1} j^2 \right)$$

$$L_N = \frac{b^3}{N^3} \left(\frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6} \right)$$

$$L_N = \frac{b^3}{3} - \frac{b^3}{2N} + \frac{b^3}{6N^2}$$

Where, we again used the fact that $f(x_j) = x_j = j\frac{b}{N}$, together with the second summation formula from above.

Now, we again let $N \to \infty$ to get the exact area.

$$A = \lim_{N\to\infty} L_N = \lim_{N\to\infty} \left(\frac{b^3}{3}\right) - \lim_{N\to\infty} \left(\frac{b^3}{2N}\right) + \lim_{N\to\infty} \left(\frac{b^3}{6N^2}\right)$$

$$A = \frac{b^3}{3} - 0 + 0 = \frac{1}{3} b^3$$

Once more, we can generalize these results as follows.

The area under the graph of $f(x) = Cx^2$ on $[0, b]$, is given as: $$A = C\frac{1}{3}b^3$$

Let's finish up this section with one more slightly more complex example.

Example 4: Compute the area under the graph of $f(x) = 2x^2 - 3x + 2$ on $[1,3]$.

In this case:

$$\Delta x = \frac{3-1}{N} = \frac{2}{N} \qquad\qquad x_j = 1 + j\Delta x = 1 + j\frac{2}{N}$$

The left endpoint approximation is then

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

$$L_N = \frac{2}{N} \sum_{j=0}^{N-1} 2{x_j}^2 - 3x_j + 2$$

$$L_N = \frac{2}{N} \sum_{j=0}^{N-1} 2\left(1 + j\frac{2}{N}\right)^2 - 3\left(1 + j\frac{2}{N}\right) + 2$$

$$L_N = \frac{2}{N} \sum_{j=0}^{N-1} 2 + \frac{8}{N}j + j^2\frac{8}{N^2} - 3 - \frac{6}{N}j + 2$$

$$L_N = \frac{2}{N} \sum_{j=0}^{N-1} j^2\frac{8}{N^2} + \frac{2}{N}j + 1$$

$$L_N = \frac{16}{N^3} \sum_{j=0}^{N-1} j^2 + \frac{4}{N^2} \sum_{j=0}^{N-1} j + \frac{2}{N} \sum_{j=0}^{N-1} 1$$

$$L_N = \frac{16}{N^3}\left(\frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}\right) + \frac{4}{N^2}\left(\frac{N^2}{2} - \frac{N}{2}\right) + \frac{2}{N}(N)$$

$$L_N = \left(\frac{16}{3} - \frac{8}{N} + \frac{8}{3N^2} + 2 - \frac{2}{N} + 2\right)$$

$$L_N = \left(\frac{28}{3} - \frac{8}{N} + \frac{8}{3N^2} - \frac{2}{N}\right)$$

Finally, we let $N \to \infty$.

$$A = \lim_{N\to\infty} L_N = \lim_{N\to\infty}\left(\frac{28}{3}\right) - \lim_{N\to\infty}\left(\frac{8}{N}\right) + \lim_{N\to\infty}\left(\frac{8}{3N^2}\right) - \lim_{N\to\infty}\left(\frac{2}{N}\right)$$

$$A = \frac{28}{3} - 0 + 0 - 0$$

$$A = \frac{28}{3}$$

Final Summary for Integration – The Area Problem

Area Approximation Using Right Endpoints

$$R_N = \Delta x \sum_{j=0}^{N-1} f(x_{j+1})$$

Area Approximation Using Left Endpoints

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

Area Approximation Using Midpoints

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$

Where the following definitions apply to all three approximations.
If the interval over which the area is computed is defined as $[a, b]$, the subinterval width is

$$\Delta x = \frac{b-a}{N}$$

and the evaluation points are

$$x_i = a + i\Delta x$$

Summation Formulas

$$\sum_{j=0}^{N-1} j = 0 + 1 + 2 \cdots (N-1) = \frac{N(N-1)}{2} = \frac{N^2}{2} - \frac{N}{2}$$

$$\sum_{j=0}^{N-1} j^2 = 0 + 1^2 + 2^2 \cdots (N-1)^2 = \frac{N(N-1)(2N-1)}{6} = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}$$

Area Under the Graph

If f is continuous on $[a, b]$, then the endpoint and midpoint approximations approach the same value, A, in the limit as $N \to \infty$. In other words:

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = A$$

If $f(x) > 0$ on $[a, b]$, then A represents the area under the graph of $f(x)$ on $[a, b]$

The area under the graph of $f(x) = Cx$ on $[0, b]$, is given as:

$$A = C\frac{1}{2}b^2$$

The area under the graph of $f(x) = Cx^2$ on $[0, b]$, is given as:

$$A = C\frac{1}{3}b^3$$

Appendix: Summation Notation

It is often the case in mathematics that a problem requires us to add the values of some variable. If the number of values to be added is large the *Summation,* or *Sigma,* notation is a convenient shorthand way to represent the operation. Assume we have a set of 8 values as shown below.

$$\{x_0, x_1, x_2, x_3, x_4, x_6, x_7\}$$

Using the conventional addition operator, we would represent the summation, S, of these values as follows.

$$S = (x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7)$$

As the set of values to be added gets larger this notation becomes inconvenient. Using *Sigma Notation* we can instead represent the above summation as follows:

$$S = \sum_{j=0}^{7} x_j$$

Where, j is used to index the variable to be added, the lower value, 0, is the starting index, the upper value, 7, is the stopping index, $\sum$, is the notation used to represent the summation, and x_j is the element to be added.

In general, we represent the summation of a set of N values, $\{x_0, x_1, \cdots, x_{N-1}\}$ as follows:

$$S = \sum_{j=0}^{N-1} x_j$$

Two important properties of the summation notation, shown below, follow from the fact that we are simply adding a set of values.

A Constant can be Factored out of a Summation
$\sum_{j=0}^{N-1} Cx_j = C\sum_{j=0}^{N-1} x_j$
Internal Sums and Differences can be Split Across Multiple Summations
$\sum_{j=0}^{N-1} (x_j \pm y_j) = \sum_{j=0}^{N-1} x_j \pm \sum_{j=0}^{N-1} y_j$

By: ferrantetutoring