Differentiation – Trigonometric Derivatives

Algebraic functions, the ones we have worked with until now, are functions that can be expressed as a finite sequence of algebraic operations, (e.g. polynomials). We have developed rules that allow us to differentiate a great number of these types of functions. Transcendental functions, on the other hand, are functions that "transcend" algebra, in the sense that they cannot be expressed as a finite sequence of algebraic operations. These include exponential functions, logarithmic functions, and trigonometric functions, which we focus on in this section. Unfortunately, none of the differentiation rules we have developed so far can help us differentiate the basic trigonometric functions. Consequently, we will need to return to the fundamental definition of the derivative. Luckily, as we will see shortly, the derivative of trigonometric functions are trigonometric functions themselves. Even further, since all trigonometric functions can be written in terms of $sin(x)$ and $cos(x)$, once we determine the derivatives of these two trigonometric functions, we can then use the rules we have learned to evaluate the derivatives of the other four basic trigonometric functions.

Let's start with the derivative of $f(x) = sin(x)$.

$$\frac{d}{dx}(sin(x)) = \lim_{h\to 0}\left\{\frac{sin(x+h) - sin(x)}{h}\right\}$$

We apply the addition formula for $sin(x+h)$ to help us evaluate this limit.

$$\begin{aligned}
\frac{d}{dx}(sin(x)) &= \lim_{h\to 0}\left\{\frac{sin(x)\,cos(h) + cos(x)\,sin(h) - sin(x)}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{sin(x)\,[cos(h) - 1] + [cos(x)\,sin(h)]}{h}\right\} \\
&= \left(\lim_{h\to 0}\left\{\frac{sin(x)\,[cos(h) - 1]}{h}\right\}\right) + \left(\lim_{h\to 0}\left\{\frac{cos(x)\,sin(h)}{h}\right\}\right) \\
&= sin(x)\left(\lim_{h\to 0}\left\{\frac{[cos(h) - 1]}{h}\right\}\right) + cos(x)\left(\lim_{h\to 0}\left\{\frac{sin(h)}{h}\right\}\right) \\
&= sin(x)\,(0) + cos(x)\,(1)
\end{aligned}$$

$$\frac{d}{dx}(sin(x)) = cos(x)$$

Where, we used the following two trigonometric limits we learned in a previous lesson.

$$\lim_{h\to 0}\left\{\frac{[cos(h) - 1]}{h}\right\} = 0 \qquad\qquad \lim_{h\to 0}\left\{\frac{sin(h)}{h}\right\} = 1$$

A similar procedure can be used to find the derivative of $cos(x)$.

$$\frac{d}{dx}(cos(x)) = \lim_{h\to 0}\left\{\frac{cos(x+h) - cos(x)}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{cos(x)\,cos(h) - sin(x)\,sin(h) - cos(x)}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{cos(x)\,[cos(h) - 1] - sin(x)\,sin(h)}{h}\right\}$$

$$= cos(x)\left(\lim_{h\to 0}\left\{\frac{[cos(h) - 1]}{h}\right\}\right) - sin(x)\left(\lim_{h\to 0}\left\{\frac{[sin(h)]}{h}\right\}\right)$$

$$= cos(x)\,(0) - sin(x)\,(1)$$

$$\frac{d}{dx}(cos(x)) = -\,sin(x)$$

The derivatives for $sin(x)$ and $cos(x)$, summarized below, are fundamental derivatives that should be memorized.

Derivatives of Sine and Cosine	
$\frac{d}{dx}(sin(x)) = cos(x)$	$\frac{d}{dx}(cos(x)) = -\,sin(x)$
Where, x is measured in radians.	

As mentioned, we can now evaluate the remaining trigonometric functions without the need to evaluate the limit definition. Instead we will make use of the quotient rule. We begin with the tangent function.

$$\frac{d}{dx}(tan(x)) = \frac{d}{dx}\left(\frac{sin(x)}{cos(x)}\right)$$

$$= \frac{(sin(x))'(cos(x)) - (sin(x))(cos(x))'}{(cos(x))^2}$$

$$= \frac{(cos(x))(cos(x)) - (sin(x))(-\,sin(x))}{(cos(x))^2}$$

$$= \frac{cos^2(x) + sin^2(x)}{(cos(x))^2}$$

$$= \frac{1}{cos^2(x)}$$

$$\frac{d}{dx}(tan(x)) = sec^2(x)$$

The cotangent is similar.

$$\begin{aligned}
\frac{d}{dx}(cot(x)) &= \frac{d}{dx}\left(\frac{cos(x)}{sin(x)}\right) \\
&= \frac{(cos(x))'(sin(x)) - (cos(x))(sin(x))'}{(sin(x))^2} \\
&= \frac{(-sin(x))(sin(x)) - (cos(x))(cos(x))}{(sin(x))^2} \\
&= -\frac{(sin^2(x) + cos^2(x))}{(sin(x))^2} \\
&= -\frac{1}{sin^2(x)} \\
\frac{d}{dx}(cot(x)) &= -csc^2(x)
\end{aligned}$$

Finally, the cosecant and secant are similarly evaluated.

$$\begin{aligned}
\frac{d}{dx}(csc(x)) &= \frac{d}{dx}\left(\frac{1}{sin(x)}\right) \\
&= \frac{(1)'(sin(x)) - (1)(sin(x))'}{(sin(x))^2} \\
&= \frac{(0)(cos(x)) - (1)(cos(x))}{(sin(x))^2} \\
&= \frac{-(cos(x))}{(sin(x))^2} \\
&= -\frac{1}{sin(x)} \cdot \frac{cos(x)}{sin(x)} \\
\frac{d}{dx}(csc(x)) &= -csc(x)\,cot(x)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(sec(x)) &= \frac{d}{dx}\left(\frac{1}{cos(x)}\right) \\
&= \frac{(1)'(cos(x)) - (1)(cos(x))'}{(cos(x))^2} \\
&= \frac{(0)(cos(x)) - (1)(-sin(x))}{(cos(x))^2} \\
&= \frac{sin(x)}{(cos(x))^2} \\
&= \frac{1}{cos(x)} \cdot \frac{sin(x)}{cos(x)} \\
\frac{d}{dx}(sec(x)) &= sec(x)\,tan(x)
\end{aligned}$$

The derivatives for all six trigonometric functions are listed below.

Derivatives of Trigonometric Functions	
$\frac{d}{dx}(sin(x)) = cos(x)$	$\frac{d}{dx}(cos(x)) = -sin(x)$
$\frac{d}{dx}(tan(x)) = sec^2(x)$	$\frac{d}{dx}(cot(x)) = -csc^2(x)$
$\frac{d}{dx}(csc(x)) = -csc(x)\,cot(x)$	$\frac{d}{dx}(sec(x)) = sec(x)\,tan(x)$
Where, x is measured in radians.	

We can now add these trigonometric "rules" to the algebraic rules we have already learned to solve a much wider range of functions. We demonstrate with some examples below.

Example 1: Calculate the derivative of the following functions.

a. $f(x) = sin(x)\,cos(x)$	*b.* $f(x) = \frac{3\,cos(x)-4}{sin(x)}$
c. $f(x) = (2x^4 - 4x^{-1})\,sec(x)$	*d.* $f(x) = tan(x) + cot(x)$

Solutions:

a. We use the product rule.

$$\begin{aligned}\frac{d}{dx}(sin(x)\,cos(x)) &= (sin(x))'(cos(x)) + (sin(x))(cos(x))' \\ &= cos(x)\,cos(x) + sin(x)\,(-sin(x)) \\ &= cos^2(x) - sin^2(x) \\ \frac{d}{dx}(sin(x)\,cos(x)) &= cos(2x)\end{aligned}$$

b. We use the quotient rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{3\,cos(x)-4}{sin(x)}\right) &= \frac{(3\,cos(x)-4)'(sin(x)) - (3\,cos(x)-4)(sin(x))'}{(sin(x))^2} \\ &= \frac{-3\,sin(x)\,(sin(x)) - (3\,cos(x)-4)\,cos(x)}{(sin(x))^2} \\ &= \frac{-3\,sin^2(x) - 3\,cos^2(x) + 4\,cos(x)}{sin^2(x)} \\ &= \frac{-3(sin^2(x) + cos^2(x)) + 4\,cos(x)}{sin^2(x)} \\ &= \frac{4\,cos(x) - 3}{sin^2(x)}\end{aligned}$$

c. We again use the product rule.

$$\frac{d}{dx}\left((2x^4 - 4x^{-1})\,sec(x)\right) = (2x^4 - 4x^{-1})'(sec(x)) + (2x^4 - 4x^{-1})(sec(x))'$$
$$= (8x^3 + 4x^{-2})(sec(x)) + (2x^4 - 4x^{-1})(sec(x)\,tan(x))$$

d. In this case we can simply use the sum rule.

$$\frac{d}{dx}(tan(x) + cot(x)) = \frac{d}{dx}(tan(x)) + \frac{d}{dx}(cot(x))$$
$$= sec^2(x) - csc^2(x)$$

Example 2: Find the equation of the tangent line at the point indicated.

a. $f(x) = x^3 + cos(x)\,,\; x = 0$	*b.* $f(x) = \frac{sin(x)}{1+cos(x)}\,,\; x = \frac{\pi}{3}$
c. $f(x) = tan(x)\,,\; x = \frac{\pi}{4}$	*d.* $f(x) = sin(x) + 3cos(x)\,,\; x = 0$

Solutions:

a. We start by finding the derivative, i.e. the slope of the tangent line, at $x = 0$.

$$\frac{d}{dx}(x^3 + cos(x))\Big|_{x=0} = 3x^2 - sin(x)|_{x=0}$$
$$m = 0 - sin(0)$$
$$m = 0$$

The tangent line is at the point, $P = (x, x^3 + cos(x))$. With $x = 0$ we obtain $P = (0, 1)$

Since the slope is 0, the tangent line is a horizontal line at $y = 1$, however we can also use the point-slope formula to derive this result explicitly.

$$y - y_P = m(x - x_P)$$
$$y - 1 = 0(x - 0)$$
$$y = 1$$

b. We follow the same procedure as the previous problem for the remaining problems.

$$\frac{d}{dx}\left(\frac{sin(x)}{1+cos(x)}\right)\bigg|_{x=\frac{\pi}{3}} = \frac{(sin(x))'(1+cos(x)) - (sin(x))(1+cos(x))'}{(1+cos(x))^2}\bigg|_{x=\frac{\pi}{3}}$$

$$= \frac{cos(x)\,(1+cos(x)) - (sin(x))(-sin(x))}{(1+cos(x))^2}\bigg|_{x=\frac{\pi}{3}}$$

$$= \frac{(cos(x) + cos^2(x)) + (sin^2(x))}{(1+cos(x))^2}\bigg|_{x=\frac{\pi}{3}}$$

$$= \frac{cos(x)+1}{(1+cos(x))^2}\bigg|_{x=\frac{\pi}{3}}$$

$$= \frac{1}{1+cos(x)}\bigg|_{x=\frac{\pi}{3}}$$

$$= \frac{1}{1+cos\left(\frac{\pi}{3}\right)}$$

$$m = \frac{2}{3}$$

$$P = \left(\frac{\pi}{3}, \frac{sin\left(\frac{\pi}{3}\right)}{1+cos\left(\frac{\pi}{3}\right)}\right) = \left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$$

$$y - y_P = m(x - x_P)$$

$$y - \frac{\sqrt{3}}{3} = \frac{2}{3}\left(x - \frac{\pi}{3}\right)$$

$$y = \frac{2}{3}x + \left(\frac{3\sqrt{3} - 2\pi}{9}\right)$$

c.

$$\frac{d}{dx}(tan(x))\bigg|_{x=\frac{\pi}{4}} = sec^2(x)|_{x=\frac{\pi}{4}}$$

$$m = 2$$

$$P = \left(\frac{\pi}{4}, tan\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, 1\right)$$

$$y - y_P = m(x - x_P)$$

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 2x + \left(\frac{2-\pi}{2}\right)$$

d.

$$\frac{d}{dx}(sin(x) + 3cos(x))\bigg|_{x=0} = cos(x) - 3\,sin(x)|_{x=0}$$

$$m = cos(0) - 3\,sin(0)$$

$$m = 1$$

$$P = (0, sin(0) + 3cos(0)) = (0, 3)$$

$$y - y_P = m(x - x_P)$$

$$y - 3 = 1(x - 0)$$

$$y = x + 3$$

Final Summary for Differentiation – Trigonometric **Derivatives**

Trigonometric Derivatives
Trigonometric functions are transcendental functions, which prevents us from using the algebraic differentiation rules we have developed previously. The two main trigonometrics function are $sin(x)$ and $cos(x)$, in the sense that the other four can be written in terms of these. After determining the derivatives of the sine and cosine function, via the limit definition of the derivative, we can find the other four using our algebraic rules. The derivative of trigonometric functions are themselves trigonometric functions. Considering the derivatives of the six trigonometric functions as "trigonometric differentiation rules", we can now use them, together with the algebraic ones, to differentiate a much wider range of functions.
Derivative of Trigonometric Functions
$\frac{d}{dx}(sin(x)) = cos(x)$ $\frac{d}{dx}(cos(x)) = -sin(x)$ $\frac{d}{dx}(tan(x)) = sec^2(x)$ $\frac{d}{dx}(cot(x)) = -csc^2(x)$ $\frac{d}{dx}(csc(x)) = -csc(x)\,cot(x)$ $\frac{d}{dx}(sec(x)) = sec(x)\,tan(x)$ Where, x is measured in radians.

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