<u>Differentiation – Differentiation Rules</u>

In the previous section we computed derivatives by using the fundamental definition based on the limit of a difference quotient. This method of evaluation is generally very tedious, and in this section, we establish basic formulas and rules that will greatly speed up the calculation of derivatives. Many of the rules will come directly from the limit laws we have previous learned. As a reminder, before we begin, we state the formal definition of the derivative below.

The Definition of the Derivative as a Function
The derivative of a function, $f(x)$, with respect to x is another function, $f'(x)$, defined as: $$f'(x) = \lim_{h \to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}$$ The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit above exists. We say that $f(x)$ is differentiable wherever $f'(x)$ exists.

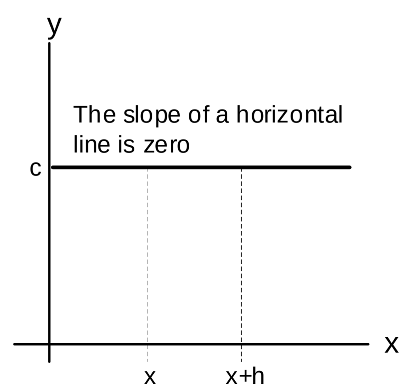
Derivative of a Constant:

The first rule comes from the interpretation of the derivative as the slope of the tangent line. We know from basic function analysis that the slope of a constant function, e.g. $f(x) = c,$ is zero. A simple mathematic proof along with a graphical interpretation is shown below.

$$f'(x) = \lim_{h \to 0}\left\{\frac{f(x+h) - f(x)}{h}\right\}$$

$$f'(x) = \lim_{h \to 0}\left\{\frac{c - c}{h}\right\}$$

$$f'(x) = 0$$



Derivative of a Constant Rule
$$\frac{d}{dx}(C) = 0$$

Linearity Rules:

The next two rules are best explained by interpreting differentiation as a process that gets applied to a function to produce a new function. In this sense we can think of the derivative as an operator, $\mathcal{L}$. To show the derivative operating on a function, $f(x)$, to produce a new function, $f'(x)$, we write the following:

$$\frac{d}{dx}\big(f(x)\big) = f'(x)$$

Where, $\frac{d}{dx}$ is considered the operator, $\mathcal{L}$.

An operator is linear if the following two conditions are met.

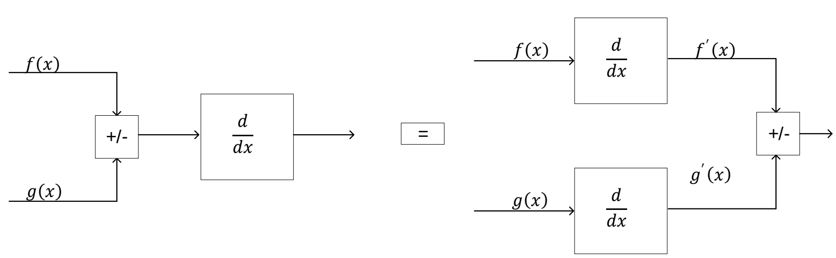
1. $\mathcal{L}\big(f(x) \pm g(x)\big) \; = \; \mathcal{L}\big(f(x)\big) \pm \mathcal{L}\big(g(x)\big)$
2. $\mathcal{L}\big(Cf(x)\big) \; = \; C\mathcal{L}\big(f(x)\big)$

The derivative is indeed a linear operator, which follows directly from the sum and difference law and the constant multiple law of limits from the previous section. The fact that the derivative is a linear operator is of considerable significance and these rules will be consistently used throughout our studies.

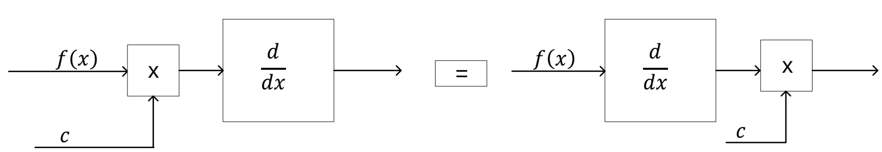
Linearity Rules of the Derivative
Sum and Difference Rule:
$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx}\big(f(x)\big) \pm \frac{d}{dx}\big(g(x)\big)$
Constant Multiple Rule:
$\frac{d}{dx}\big(Cf(x)\big) = C\frac{d}{dx}\big(f(x)\big)$

For illustrative purposes we show the two rules in the form of block diagrams below.

Sum and Difference Rule



Constant Multiple Rule



The Power Rule:

The power rule will prove to be incredibly useful for differentiating polynomials. Instead of proving this rule generally we will show three examples that we could use to convince ourselves that the rule is indeed true.

Let's compute the derivative of the following function with $n = 1, 2$ and 3.

$$f_n(x) = x^n$$

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$\boldsymbol{f_1(x) = x^1}$

$$f_1{}'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{x} + \cancel{h} \cancel{- x}}{\cancel{h}} \right\}$$

$$f_1{}'(x) = \lim_{h \to 0} \{1\}$$

$$f_1{}'(x) = 1x^0$$

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$\boldsymbol{f_2(x) = x^2}$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{(x + h)^2 - x^2}{h} \right\}$$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h} \right\}$$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{h}(2x + h)}{\cancel{h}} \right\}$$

$$f_2{}'(x) = 2x^1$$

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$\boldsymbol{f_3(x) = x^3}$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{(x + h)^3 - x^3}{h} \right\}$$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{x^3} + 3x^2h + 3xh^2 - h^3 \cancel{- x^3}}{h} \right\}$$

$$f_2{}'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{h}(3x^2 + 3xh - h^2)}{\cancel{h}} \right\}$$

$$f_2{}'(x) = 3x^2$$

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Based on the pattern shown in the three examples above we state the power rule as shown.

The Power Rule for the Derivative
For all exponents, n: $$\frac{d}{dx}(x^n) = nx^{n-1}$$

Before moving on to additional rules let's do a few examples to illustrate the usefulness of the above rules.

Examples: Calculate the derivative of the following functions.

a. $f(x) = 4x^9 - 3x^4 + 2x$

b. $f(x) = 5x^{-2} - 7x^{-1} + 9$

c. $f(x) = x^{5/4} + 4x^{-3/2} + 11x$

d. $f(t) = 6\sqrt{t} + \frac{1}{\sqrt{t}}$

e. $f(r) = (1 - 2r)(3r + 5)$

f. $f(s) = \frac{s^2 + 4s^{1/2}}{s^2}$

Solutions:

a. A combination of the above differentiation rules can be applied to evaluate this derivative.

$$\begin{aligned} \frac{d}{dx}f(x) &= \frac{d}{dx}(4x^9 - 3x^4 + 2x) \\ &= 4\frac{d}{dx}(x^9) - 3\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^1) \\ &= (4)9x^8 - (3)4x^3 + (2)1x^0 \\ &= 36x^8 - 12x^3 + 2 \end{aligned}$$

b. Note that the power rule can just as easily be applied to negative exponents.

$$\begin{aligned} \frac{d}{dx}f(x) &= \frac{d}{dx}(5x^{-2} - 7x^{-1} + 9) \\ &= 5\frac{d}{dx}(x^{-2}) - 7\frac{d}{dx}(x^{-1}) + \frac{d}{dx}(9) \\ &= (5)(-2x^{-3}) - (7)(-1x^{-2}) + 0 \\ &= -10x^{-3} + 7x^{-2} \end{aligned}$$

c. The power rule can also be applied to fractional exponents.

$$\begin{aligned} \frac{d}{dx}f(x) &= \frac{d}{dx}\left(x^{5/4} + 4x^{-3/2} + 11x\right) \\ &= \frac{d}{dx}\left(x^{5/4}\right) + 4\frac{d}{dx}\left(x^{-3/2}\right) + 11\frac{d}{dx}(x^1) \\ &= \left(5/4 \cdot x^{1/4}\right) + (4)\left(-3/2 \cdot x^{-5/2}\right) + (11)(1x^0) \\ &= 5/4 \cdot x^{1/4} - 6x^{-5/2} + 11 \end{aligned}$$

d. Convert radicals to fractional exponents so that the power rule can more easily be applied.

$$\begin{aligned}
\frac{d}{dt}f(t) &= \frac{d}{dt}\left(6\sqrt{t}+\frac{1}{\sqrt{t}}\right) \\
&= 6\frac{d}{dt}\left(t^{1/2}\right) + \frac{d}{dt}\left(t^{-1/2}\right) \\
&= (6)\left(1/2\cdot t^{-1/2}\right) + \left(-1/2\cdot t^{-3/2}\right) \\
&= \frac{3}{\sqrt{t}} - \frac{1}{2\sqrt{t^3}}
\end{aligned}$$

e. Expand first before attempting to differentiate.

$$\begin{aligned}
\frac{d}{dr}f(r) &= \frac{d}{dr}\big((1-2r)(3r+5)\big) \\
&= \frac{d}{dr}(3r+5-6r^2-10r) \\
&= \frac{d}{dr}(-6r^2-7r+5) \\
&= -6\frac{d}{dr}(r^2) - 7\frac{d}{dr}(r^1) + \frac{d}{dr}(5) \\
&= (-6)(2r^1) - 7(1r^0) + 0 \\
&= -(12r+7)
\end{aligned}$$

f. Simplify first before attempting to differentiate.

$$\begin{aligned}
\frac{d}{ds}f(s) &= \frac{d}{ds}\left(\frac{s^2+4s^{1/2}}{s^2}\right) \\
&= \frac{d}{ds}\left(1+4s^{-3/2}\right) \\
&= \frac{d}{ds}(1) + 4\frac{d}{ds}\left(s^{-3/2}\right) \\
&= 0 + (4)\left(-3/2\cdot s^{-5/2}\right) \\
&= -\frac{6}{\sqrt{s^5}}
\end{aligned}$$

The Product Rule:

The product rule and the quotient rule are explained below. These two rules, combined with the previous ones, rules allow for the differentiation of many complex functions.

The Product Rule for the Derivative
If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)g(x)$ is also differentiable and: $$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$ Dropping the independent variable in the notation we can write this more compactly as: $$(fg)' = f'g + fg'$$

We can prove the product rule by starting with the definition of the derivative as follows.

$$\big(f(x)g(x)\big)' = \lim_{h\to 0}\left\{\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\right\}$$

We can start with a 'trick', shown below, where we add the term in brackets, $[.]$ to the numerator inside the limit.

$$f(x+h)g(x+h) + [-f(x+h)g(x) + f(x+h)g(x)] - f(x)g(x)$$

Next, we take a common factor from the first two and last two terms as follows:

$$\big[f(x+h)\big(g(x+h) - g(x)\big)\big] + \big[g(x)\big(f(x+h) - f(x)\big)\big]$$

The proof is complete by placing these results back into the original limit expression and using the fact that the limit operator is linear.

$$= \lim_{h\to 0}\left\{\frac{\big[f(x+h)\big(g(x+h) - g(x)\big)\big] + \big[g(x)\big(f(x+h) - f(x)\big)\big]}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)\big(g(x+h) - g(x)\big)}{h}\right\} + \lim_{h\to 0}\left\{\frac{g(x)\big(f(x+h) - f(x)\big)}{h}\right\}$$

$$= \left(\lim_{h\to 0}\{f(x+h)\}\right)\left(\lim_{h\to 0}\left\{\frac{\big(g(x+h) - g(x)\big)}{h}\right\}\right) + \left(\lim_{h\to 0}\{g(x)\}\right)\left(\lim_{h\to 0}\left\{\frac{\big(f(x+h) - f(x)\big)}{h}\right\}\right)$$

$$= f(x)\left(\lim_{h\to 0}\left\{\frac{\big(g(x+h) - g(x)\big)}{h}\right\}\right) + g(x)\left(\lim_{h\to 0}\left\{\frac{\big(f(x+h) - f(x)\big)}{h}\right\}\right)$$

$$= f(x)g'(x) + g(x)f'(x)$$

Which is equivalent to the product rule stated above with the terms slightly rearranged.

The Quotient Rule:

The quotient rule is very similar to the product rule, which will be helpful for memorization. You'll notice that the terms of the numerator remain unchanged, but a minus sign is used between them. Additionally, a single denominator term is used.

The Quotient Rule for the Derivative
If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)/g(x)$ is also differentiable and: $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$ Dropping the independent variable in the notation we can write this more compactly as: $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The quotient rule is proven via the product rule.

We start by defining a new function, $q = f/g$. Therefore $f = qg$. Using the product rule on f we have.

$$(f)' = q'g + qg'$$

Solving for q' we have.

$$q' = \frac{f' - qg'}{g}$$

And multiplying the numerator and denominator by g

$$q' = \frac{f' - \frac{f}{g}g'}{g}\left(\frac{g}{g}\right)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Which is equivalent to the quotient rule stated above. Let's now do some examples to get more familiar with the product and quotient rule.

Examples: Calculate the derivative of the following functions.

a. $h(r) = (1 - 2r)(3r + 5)$

b. $h(x) = (1 + x^2 - x^3)(1 - x^2 + x^3)$

c. $h(x) = \frac{1+x^2}{1-x^2}$

d. $h(t) = \frac{t^3}{1+2t+t^2}$

e. $h(s) = \frac{s^2+4s^4}{s}$

f. $h(x) = x^2(1 - x^2)(2 + x^2)$

Solutions:

a. This problem is the same one we did previously before learning the product rule. In that case we started by expanding. Let's solve it now using the product rule and verify we get the same results, which was: $-(12r + 7)$.

$$\begin{aligned}((1-2r)(3r+5))' &= f'g + fg' \\ &= \left(\frac{d}{dr}(1-2r)\right)(3r+5) + (1-2r)\left(\frac{d}{dr}(3r+5)\right) \\ &= (0-2)(3r+5) + (1-2r)(3+0) \\ &= (-6r-10) + (3-6r) \\ &= -(12r+7)\end{aligned}$$

b. We directly apply the product rule with $f(x) = (1 + x^2 - x^3)$ and $g(x) = (1 - x^2 + x^3)$

$$\begin{aligned}\big(f(x)g(x)\big)' &= (1+x^2-x^3)'(1-x^2+x^3) + (1+x^2-x^3)(1-x^2+x^3)' \\ &= (0+2x-3x^2)(1-x^2+x^3) + (1+x^2-x^3)(0-2x+3x^2) \\ &= \cancel{2x} - 2x^3 + 2x^4 \cancel{- 3x^2} + 3x^4 - 3x^5 \cancel{- 2x} - 2x^3 + 2x^4 \cancel{+ 3x^2} + 3x^4 - 3x^5 \\ &= -6x^5 + 10x^4 - 4x^3 \\ &= -2x^3(3x^2 - 5x + 2)\end{aligned}$$

c. We directly apply the quotient rule.

$$\begin{aligned}
\left(\frac{1+x^2}{1-x^2}\right)' &= \frac{f'g - fg'}{g^2} \\
&= \frac{(1+x^2)'(1-x^2) - (1+x^2)(1-x^2)'}{(1-x^2)^2} \\
&= \frac{(0+2x)(1-x^2) - (1+x^2)(0-2x)}{(1-x^2)^2} \\
&= \frac{2x \cancel{- 2x^3} + 2x \cancel{+ 2x^3}}{(1-x^2)^2} \\
&= \frac{4x}{(1-x^2)^2}
\end{aligned}$$

d. Again, we directly apply the quotient rule.

$$\begin{aligned}
\left(\frac{t^3}{1+2t+t^2}\right)' &= \frac{f'g - fg'}{g^2} \\
&= \frac{(t^3)'(1+2t+t^2) - (t^3)(1+2t+t^2)'}{(1+2t+t^2)^2} \\
&= \frac{(3t^2)(1+2t+t^2) - (t^3)(0+2+2t)}{(1+2t+t^2)^2} \\
&= \frac{3t^2 + 6t^3 + 3t^4 - 2t^3 - 2t^4}{(1+2t+t^2)^2} \\
&= \frac{t^4 + 4t^3 + 3t^2}{(1+2t+t^2)^2} \\
&= \frac{t^2(3+4t+t^2)}{(1+2t+t^2)^2}
\end{aligned}$$

e. Sometimes it may be easier to algebraically manipulate an equation first to before trying to evaluate the derivative. Assuming the algebra is done properly the derivatives will be the same. For this example the easiest method is to distribute the denominator before we differentiate. However, for illustrative purposes we will compute the derivative three ways: 1.) Distributing the denominator, 2.) Using the quotient rule, 3.) Using the product rule.

1.) Distributing the denominator

$$\left(\frac{s^2+4s^4}{s}\right)' = (s+4s^3)'$$
$$= 1+12s^2$$

2.) Quotient Rule

$$\left(\frac{s^2+4s^4}{s}\right)' = \frac{f'g - fg'}{g^2}$$
$$= \frac{(2s+16s^3)(s) - (s^2+4s^4)(1)}{s^2}$$
$$= \frac{2s^2+16s^4-s^2-4s^4}{s^2}$$
$$= \frac{12s^4+s^2}{s^2}$$
$$= 12s^2+1$$

3.) Product Rule

$$\left((s^2+4s^4)(s^{-1})\right)' = f'g + fg'$$
$$= (2s+16s^3)(s^{-1}) + (s^2+4s^4)(-1s^{-2})$$
$$= 2+16s^2-1-4s^2$$
$$= 12s^2+1$$

f. In this last example we could expand the expression and differentiate, but for illustrative purposes we will first solve by using the product rule twice. We will also evaluate the derivative a second time after expanding to verify the solution.

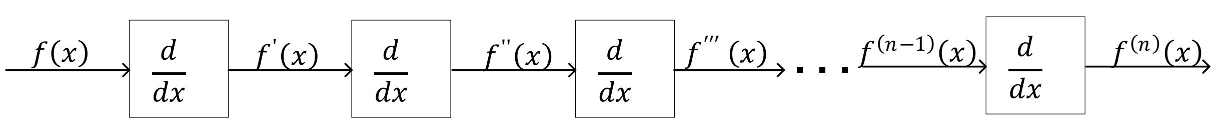
$$
\begin{aligned}
&([(x^2)(1-x^2)][2+x^2])' \\
&= \big((x^2)(1-x^2)\big)'(2+x^2) + \big((x^2)(1-x^2)\big)(2+x^2)' \\
&= ((x^2)'(1-x^2) + (x^2)(1-x^2)')(2+x^2) + (x^2)(1-x^2)(2+x^2)' \\
&= \big((2x)(1-x^2) + (x^2)(-2x)\big)(2+x^2) + (x^2)(1-x^2)(2x) \\
&= (2x - 2x^3 - 2x^3)(2+x^2) + (2x^3 - 2x^5) \\
&= (4x - 4x^3 \cancel{- 4x^3 + 2x^3} - 2x^5 - 2x^5) + (\cancel{2x^3} - 2x^5) \\
&= (-6x^5 - 4x^3 + 4x)
\end{aligned}
$$

We expand now and then differentiate.

$$
\begin{aligned}
([(x^2)(1-x^2)][2+x^2])' &= ([(x^2-x^4)][2+x^2])' \\
&= (2x^2 + x^4 - 2x^4 - x^6)' \\
&= (-x^6 - x^4 + 2x^2)' \\
&= (-6x^5 - 4x^3 + 4x)
\end{aligned}
$$

Higher Derivatives:

Our final topic in this will section will be on higher derivatives, which are obtained by repeatedly differentiating a function. The block diagram below illustrates this process.



Where we refer to $f'(x)$ as the first derivative, $f''(x)$ as the second derivative, and so on. After three prime marks we usually switch to using integer values, but in parenthesis to distinguish from raising the function to a power. For example, the fourth derivative is notated as $f^{(4)}(x)$. Finally, using Leibniz notation we write the following:

$$\frac{df}{dx}, \qquad \frac{d^2f}{dx^2}, \qquad \frac{d^3f}{dx^3}, \qquad \frac{d^4f}{dx^4}, \ldots$$

As an example, lets compute the first, second, and third derivative of $f(x) = 3x^5 + 6x^3 + 2x^2$.

$$f'(x) = \frac{d}{dx}(3x^5 + 6x^3 + 2x^2) = 15x^4 + 18x^2 + 4x$$

$$f''(x) = \frac{d^2}{dx}(3x^5 + 6x^3 + 2x^2) = \frac{d}{dx}(15x^4 + 18x^2 + 4x) = 60x^3 + 36x + 4$$

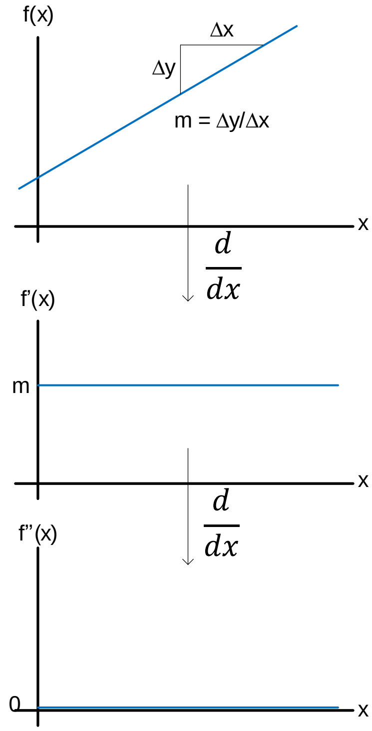
$$f'''(x) = \frac{d^3}{dx}(3x^5 + 6x^3 + 2x^2) = \frac{d}{dx}(60x^3 + 36x + 4) = 180x^2 + 36$$

To get a better understanding of higher derivatives, we recall that the derivative of a function, $f(x)$, is another function, $f'(x)$, which describes how $f(x)$ changes with respect to x. Extending this, we can say that the derivative of $f'(x)$, (i.e. the second derivative of $f(x)$), describes how $f'(x)$ changes with respect to x. Let's take the example of a linear equation, $f(x) = mx + b$. The first derivative is equal to m since the function changes at a constant rate.

$$\frac{d}{dx}(mx + b) = m$$

And since the derivative of a linear function does not change with x, (i.e. it's a constant m), the second derivative, (i.e. the rate of change of the first derivative), is zero.

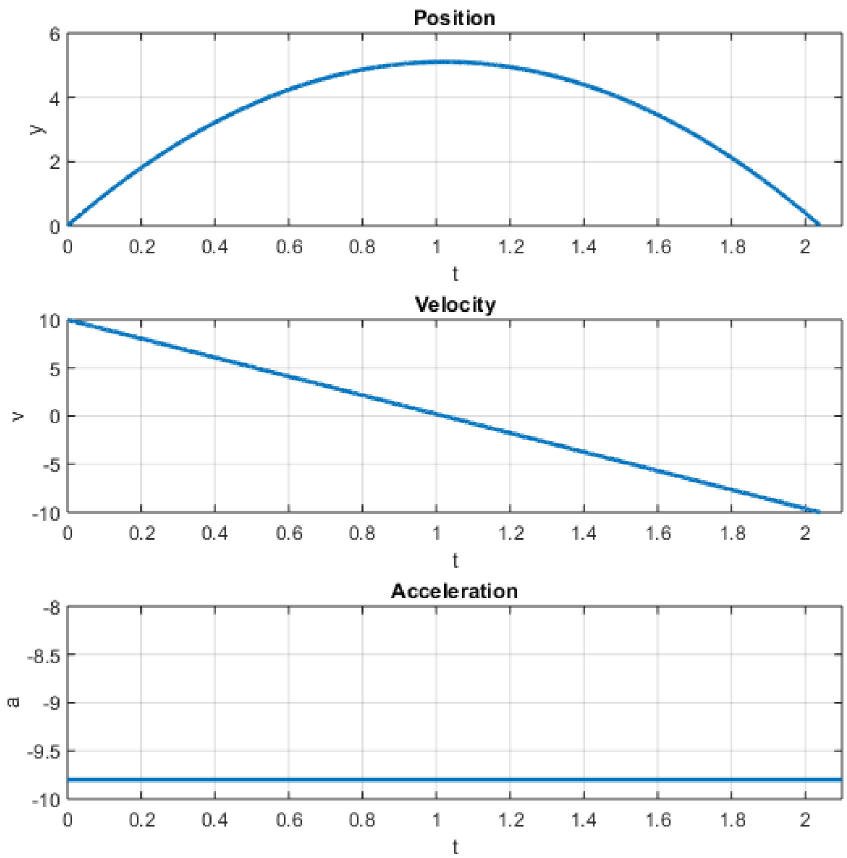
$$\frac{d^2}{dx^2}(mx + b) = \frac{d}{dx}(m) = 0$$



Let's do one last practical example regarding a ball that is thrown up in the air. The height of the ball is given as $y(t) = y(0) + v_y(0)t - 4.9t^2$. Where $y(0) = 0$ is the initial height of the ball in meters, and $v_y(0) = 10$ is the initial vertical velocity of the ball in meters per second. Let's now find the velocity and acceleration of the ball and plot all three functions.

The vertical velocity, $v_y(t)$, is the rate of change of the position, $y(t)$, i.e. $y'(t)$. The acceleration, $a(t)$, is the rate of change of the velocity, $v_y(t)$, i.e. $v_y'(t) = y''(t)$.

Velocity - First Derivative of Position	**Acceleration - Second Derivative of Position**
$v_y(t) = \frac{d}{dt}(10t - 4.9t^2)$ $v_y(t) = 10 - 9.8t \ m/s$	$a(t) = \frac{d^2}{dt}(10t - 4.9t^2)$ $a(t) = \frac{d}{dt}(10 - 9.8t)$ $a(t) = -9.8 \ m/s^2$



<u>Final Summary for Differentiation – Differentiation Rules</u>

Derivative of a Constant Rule
$$\frac{d}{dx}(C) = 0$$
Linearity Rules of the Derivative
<u>Sum and Difference Rule:</u> $$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx}\big(f(x)\big) \pm \frac{d}{dx}\big(g(x)\big)$$ *<u>Constant Multiple Rule:</u>* $$\frac{d}{dx}\big(Cf(x)\big) = C\frac{d}{dx}\big(f(x)\big)$$
The Power Rule for the Derivative
For all exponents, n: $$\frac{d}{dx}(x^n) = nx^{n-1}$$
The Product Rule for the Derivative

If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)g(x)$ is also differentiable and:

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Dropping the independent variable in the notation we can write this more compactly as:

$$(fg)' = f'g + fg'$$

The Quotient Rule for the Derivative

If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)/g(x)$ is also differentiable and:

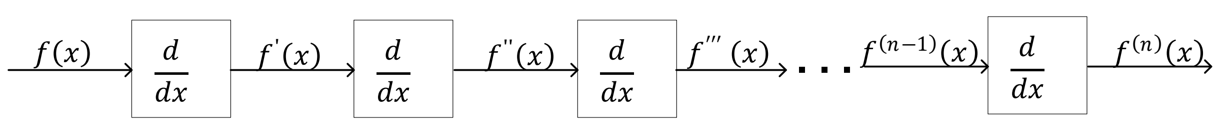
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Dropping the independent variable in the notation we can write this more compactly as:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Higher Order Derivatives

Higher derivatives are obtained by repeatedly differentiating a function according to the diagram below.



Where we refer to $f'(x)$ as the first derivative, $f''(x)$ as the second derivative, and so on. After three prime marks we usually switch to using integer values, but in parenthesis to distinguish from raising the function to a power. For example, the fourth derivative is notated as $f^{(4)}(x)$.

Using Leibniz notation, higher order differentiation is notated as follows:

$$\frac{df}{dx}, \qquad \frac{d^2f}{dx^2}, \qquad \frac{d^3f}{dx^3}, \qquad \frac{d^4f}{dx^4}, \ldots$$

By: ferrantetutoring