Differentiation – Implicit Differentiation

The equations we have worked with until now were defined by *explicit* relationships, in the sense that y is given *explicitly* in terms of x, (e.g. $y = x^4 + 2x + 5$). However, not all relationships can be written explicitly. Let's look at the equation of a unit circle centered at the origin.

$$y^2 + x^2 = 1$$

If we tried to solve this equation for y we get the following:

$$y = \pm\sqrt{1 - x^2}$$

Which is ambiguous due to the addition and subtraction sign. Instead, we generally leave the equation as it was and refer to this type of equation as an *implicit* relationship, in the sense that y is known only *implicitly* through the relationship but cannot be algebraically isolated to create a function. You'll notice that implicit equations do not generally pass the vertical line test and therefore are not strictly functions. Nevertheless, we may still want to find the slope of a tangent line to these curves. For this we use what is referred to as *implicit differentiation*. Let's illustrate this technique using the circle equation from above.

$$y^2 + x^2 = 1$$

We start by taking the derivative with respect to x on both sides of the equation.

$$\frac{d}{dx}(y^2 + x^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) = 0$$
$$\frac{d}{dx}(y^2) + 2x = 0$$

The derivative of x^2 with respect to x was straightforward, but what is the derivative of y^2 with respect to x? To determine this, we note that y is a function of x and deliberately show this by writing $y = g(x)$. Rewriting the derivate with this substitution we have:

$$\frac{d}{dx}\left(\left(g(x)\right)^2\right)$$

For which we need to apply the chain rule to evaluate since it is a composite function with the inside function being $g(x) = y$, and the outside function being $f(x) = x^2$.

$$\begin{aligned}\left(f\left(g(x)\right)\right)' &= f'\left(g(x)\right)g'(x)\\ &= \left(2\left(g(x)\right)^1\right)g'(x)\\ &= (2y)y'\\ &= 2y\frac{dy}{dx}\end{aligned}$$

We can also use Leibniz notation for the chain rule by representing the inside function as $y(x) = y$, and the outside function by $f(y) = y^2$. Therefore, we may write the following

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}$$
$$= 2y\frac{dy}{dx}$$

Using these results to complete differentiating the original equation we have

$$\frac{d}{dx}(y^2) + 2x = 0$$
$$2y\frac{dy}{dx} + 2x = 0$$

Lastly, we solve for $\frac{dy}{dx}$, which represents the slope of the tangent line for a unit circle.

$$\frac{dy}{dx} = \frac{-x}{y}$$

To illustrate let's find the equation of the tangent line for the following points:

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \qquad P_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \qquad P_3 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \qquad P_4 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

With the corresponding slopes given as:

$$m_n = \left.\frac{dy}{dx}\right|_{x=x_n, y=y_n} = \frac{-x_n}{y_n}$$

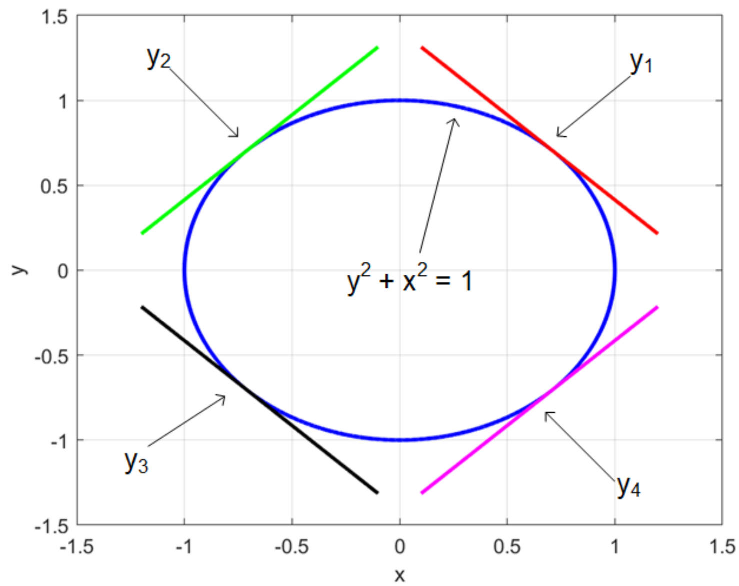
$$m_1 = -1 \qquad m_2 = 1 \qquad m_3 = -1 \qquad m_4 = 1$$

The tangent lines, indexed by i, are then given by:

$$y_i = m_n(x - x_n) + y_n$$
$$y_i = m_n x - m_n x_n + y_n$$

$$y_1 = -x + \sqrt{2} \qquad y_2 = x + \sqrt{2} \qquad y_3 = -x - \sqrt{2} \qquad y_4 = x - \sqrt{2}$$

The figure below shows the circle along with each of the four tangent lines found above.



Let's do a few more examples to get practice with the mechanics of implicit differentiation.

Example 1:

Find $\frac{dy}{dx}$ when y is given implicitly as: $x\,sin(y) + y = 0$.

Differentiating the equation with respect to x we have:

$$\frac{d}{dx}(x\,sin(y)) + \frac{d}{dx}(y) = \frac{d}{dx}(0)$$

$$\left(\left(\frac{d}{dx}(x)\right)(sin(y)) \;+\; (x)\left(\frac{d}{dx}(sin(y))\right)\right) + \frac{dy}{dx} = 0$$

$$\left(1(sin(y)) \;+\; (x)\,cos(y)\frac{dy}{dx}\right) + \frac{dy}{dx} = 0$$

$$sin(y) + \; x\,cos(y)\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x\,cos(y) + 1) = -\,sin(y)$$

$$\frac{dy}{dx} = -\frac{sin(y)}{x\,cos(y) + 1}$$

Where, we used the product rule, in conjunction with the chain rule, to differentiate $(x)(sin(y))$.

Example 2:

Find $\frac{dy}{dx}$ when y is given implicitly as: $xy + x^2y^2 = 6$.

$$\frac{d}{dx}(xy) + \frac{d}{dx}(x^2y^2) = \frac{d}{dx}(6)$$

$$\left(\left(\frac{d}{dx}(x)\right)(y) + (x)\frac{d}{dx}(y)\right) + \left(\left(\frac{d}{dx}(x^2)\right)(y^2) + (x^2)\frac{d}{dx}(y^2)\right) = 0$$

$$\left((1)(y) + (x)\left(\frac{dy}{dx}\right)\right) + \left((2x)(y^2) + (x^2)\left(2y\frac{dy}{dx}\right)\right) = 0$$

$$y + x\frac{dy}{dx} + 2xy^2 + x^2 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x + x^2 2y) = -2xy^2 - y$$

$$\frac{dy}{dx} = -\frac{y\cancel{(2xy + 1)}}{x\cancel{(1 + 2xy)}}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Example 3:

Find $\frac{dy}{dx}$ when y is given implicitly as: $\frac{y}{x} + \frac{x}{y} = 2y$.

$$\frac{d}{dx}\left(\frac{y}{x}\right) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(2y)$$

$$\left(\frac{\left(\frac{d}{dx}(y)\right)(x) - (y)\frac{d}{dx}(x)}{x^2}\right) + \left(\frac{\left(\frac{d}{dx}(x)\right)(y) - (x)\frac{d}{dx}(y)}{y^2}\right) = 2\frac{dy}{dx}$$

$$\left(\frac{\frac{dy}{dx}x - y}{x^2}\right) + \left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = 2\frac{dy}{dx}$$

$$\left(\frac{xy^2\frac{dy}{dx} - y^3 + x^2y - x^3\frac{dy}{dx}}{x^2y^2}\right) = 2\frac{dy}{dx}$$

$$\frac{dy}{dx}(xy^2 - x^3 - 2x^2y^2) = y^3 - x^2y$$

$$\frac{dy}{dx} = \frac{y(y^2 - x^2)}{x(y^2 - x^2 - 2xy^2)}$$

Derivative of Inverse Trigonometric Functions

Implicit differentiation can be used to find the derivative of inverse trigonometric functions. Let's start with the inverse sine function.

$$\frac{d}{dx}(sin^{-1}(x))$$

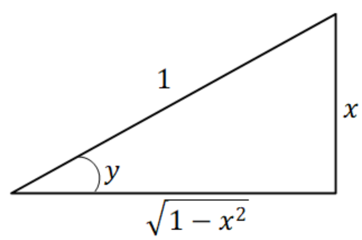
Since we do not have any rules to differentiate this function, we instead rewrite the relationship as below.

$$y = sin^{-1}(x)$$
$$sin(y) = x$$

This is an implicit relationship; therefore, we can apply implicit differentiation.

$$\frac{d}{dx}(sin(y)) = \frac{d}{dx}(x)$$
$$cos(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{cos(y)}$$

Next, we replace $cos(y)$ with an algebraic expression by using a right triangle that represents the original expression, $sin(y) = \frac{x}{1}$. The adjacent side is found with the Pythagorean theorem so that we can then represent $cos(y)$ as shown to the right.



$$cos(y) = \frac{\sqrt{1-x^2}}{1}$$

Substituting for $cos(y)$ we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

And since $y = sin^{-1}(x)$ we can write

$$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

Note: Since the range of $sin^{-1}(x)$ is from $-\pi/2$ to $\pi/2$, for which the cosine function is positive, we take the positive square root rather than the negative.

The other inverse trigonometric functions can be done in a similar fashion. We demonstrate the inverse tangent function below and leave the rest as an exercise.

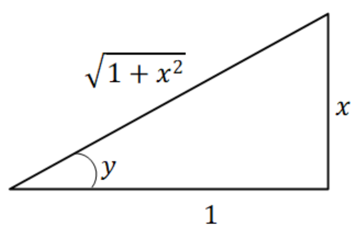
We start by solving for x in the below inverse tangent expression.

$$y = tan^{-1}(x)$$
$$tan(y) = x$$

Then, applying implicit differentiation we can find $\frac{dy}{dx}$.

$$\frac{d}{dx}(tan(y)) = \frac{d}{dx}(x)$$
$$sec^2(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{sec^2(y)}$$

Next, we draw a right triangle representing $tan(y) = \frac{x}{1}$, solve for the hypotenuse, and then write an expression for $sec^2(y)$.



$$sec^2(y) = \left(\frac{\sqrt{1+x^2}}{1}\right)^2$$
$$sec^2(y) = 1 + x^2$$

Substituting for $sec^2(y)$ we write the derivative of $tan^{-1}(x)$ as shown.

$$\frac{d}{dx}(tan^{-1}(x)) = \frac{1}{1+x^2}$$

The remaining inverse trigonometric derivatives are shown in the table below.

Derivative of Inverse Trigonometric Functions	
$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(cot^{-1}(x)) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(csc^{-1}(x)) = -\frac{1}{\lvert x\rvert\sqrt{1-x^2}}$	$\frac{d}{dx}(sec^{-1}(x)) = \frac{1}{\lvert x\rvert\sqrt{1-x^2}}$

Derivative of General Inverse Functions

We can generalize the results above for any inverse function. To do this we start with a relationship we know is true based on the fundamental definition of inverse functions.

$$f\left(f^{-1}(x)\right) = x$$

Next, we use implicit differentiation, along with the chain rule, to find a general formula for the derivative of inverse functions.

$$\frac{d}{dx}\left(f\left(f^{-1}(x)\right)\right) = \frac{d}{dx}(x)$$

$$f'\left(f^{-1}(x)\right)\frac{d}{dx}\left(f^{-1}(x)\right) = 1$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Recall when finding the derivative of $y = sin^{-1}(x)$, we derived the following relationship.

$$\frac{dy}{dx} = \frac{1}{cos(y)}$$

Using the f notation we can write

$$f^{-1}(x) = sin^{-1}(x) \qquad\qquad f(x) = sin(x)$$

Therefore, using the formula above we have

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$\frac{d}{dx}(sin^{-1}(x)) = f'(x)|_{x=sin^{-1}(x)}$$

$$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{cos(sin^{-1}(x))}$$

Which is identical to what we derived earlier since $y = sin^{-1}(x)$! We formally state the formula below.

Derivative of Inverse Functions
Suppose f has an inverse function, f^{-1}. If f is differentiable at $f^{-1}(x)$ and $f'\left(f^{-1}(x)\right)$ is not zero, then f^{-1} is differentiable at x and we may apply the following formula to evaluate. $$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Before ending this section, let's do a few more examples.

Example 4:

Find the equation of the tangent line to the curve, $x^3 + y^3 = 3xy$, at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3\frac{d}{dx}(xy)$$

$$\cancel{3}x^2 + \cancel{3}y^2\frac{dy}{dx} = \cancel{3}\left(y + x\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}(y^2 - x) = (y - x^2)$$

$$\frac{dy}{dx} = \frac{(y - x^2)}{(y^2 - x)}$$

The slope of the tangent line at the given point is

$$m = \left.\frac{dy}{dx}\right|_{\left(\frac{2}{3},\frac{4}{3}\right)} = \frac{\left(\frac{4}{3} - \frac{4}{9}\right)}{\left(\frac{16}{9} - \frac{2}{3}\right)}$$

$$m = \frac{4}{5}$$

And the equation of the tangent line is as follows:

$$\left(y - \frac{4}{3}\right) = \frac{4}{5}\left(x - \frac{2}{3}\right)$$

$$y = \frac{4}{5}x + \frac{4}{5}$$

Example 5:

Find $\frac{dy}{dx}$ for $y = cos^{-1}(x^2)$

We combine the chain rule with the formula for the derivative of the inverse cosine function.

$$\frac{d}{dx}(cos^{-1}(x^2)) = \frac{d}{dx}(cos^{-1}(x))|_{x=x^2} \cdot \frac{d}{dx}(x^2)$$

$$= -\frac{2x}{\sqrt{1 - x^2}}$$

Example 6:

Find $\frac{dy}{dx}$ for $y = (tan^{-1}(x))^3$

$$\frac{d}{dx}((tan^{-1}(x))^3) = \frac{d}{dx}(x^3)|_{x=tan^{-1}(x)} \cdot \frac{d}{dx}(tan^{-1}(x))$$
$$= 3(tan^{-1}(x))^2 \cdot \frac{1}{1+x^2}$$
$$= \frac{3(tan^{-1}(x))^2}{1+x^2}$$

Example 7:

Given $f(x) = 3x^5 + 6x^3 + 4$, and the fact that the inverse function $f^{-1}(x)$ exists, find the value of $\frac{d}{dx}\left(f^{-1}(13)\right)$.

Explicitly finding the inverse function in this case is not straightforward, however according the formula described above we can find the derivative of inverse functions without explicitly knowing the function!

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$
$$= \frac{1}{\frac{d}{dx}(3x^5 + 6x^3 + 4)|_{x=f^{-1}(x)}}$$
$$= \frac{1}{(15x^4 + 18x^2)|_{x=f^{-1}(x)}}$$

You may be tempted to substitute $x = 13$, however we need to substitute $x = f^{-1}(13)$.

Question: How do we find this value without knowing $f^{-1}(x)$ explicitly?

Answer: We use the fundamental relationship of an inverse function.

We need to find the value, a, shown below.

$$f^{-1}(13) = a$$

Applying f to both sides we find

$$f\left(f^{-1}(13)\right) = f(a)$$
$$13 = f(a)$$
$$13 = 3a^5 + 6a^3 + 4$$

This polynomial is not trivial to solve, however using a graphing calculator we find, $a = 1$.

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{(15x^4 + 18x^2)|_{x=1}}$$
$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{33}$$

Final Summary for Differentiation – Implicit Differentiation

Implicit Differentiation

Implicit differentiation is used to compute dy/dx when x and y are related through an implicit relation only. The steps to compute dy/dx can be summarized as follows:

1. Compute the derivative of both sides of the equation with respect to x.
2. Solve the resulting equation for dy/dx.

Remember, when differentiating any y terms, we need to use the chain rule since we are differentiating with respect to x and y is a function of x. Example shown below:

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

Derivative of Inverse Trigonometric Functions

$$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(cot^{-1}(x)) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(csc^{-1}(x)) = -\frac{1}{|x|\sqrt{1-x^2}}$$

$$\frac{d}{dx}(sec^{-1}(x)) = \frac{1}{|x|\sqrt{1-x^2}}$$

Derivative of General Inverse Functions

Suppose f has an inverse function, f^{-1}. If f is differentiable at $f^{-1}(x)$ and $f'\left(f^{-1}(x)\right)$ is not zero, then $f^{-1}$is differentiable at x and we may apply the following formula to evaluate.

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

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