Differentiation – Derivatives of Exponentials and Logarithms

In an earlier lesson we learned that trigonometric functions are a type of transcendental function, which required a return to the fundamental definition of the derivative for differentiation. In that lesson we mentioned that exponential and logarithm functions are also transcendental functions. Consequently, we will need to return to the fundamental definition again to differentiate these types of transcendental functions. The goal will be to derive general rules that we can apply to exponential and logarithmic functions of any base. Let's begin with the generic exponential function, $f(x) = b^x$, and attempt to evaluate the derivative using the fundamental definition.

$$\begin{aligned} \frac{d}{dx}(b^x) &= \lim_{h\to 0}\left\{\frac{b^{x+h} - b^x}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{b^x b^h - b^x}{h}\right\} \\ &= b^x \lim_{h\to 0}\left\{\frac{b^h - 1}{h}\right\} \end{aligned}$$

We state, without proof, that $\lim_{h\to 0}\left\{\frac{b^h-1}{h}\right\}$ exists and that its value depends on the base, b.

Rewriting the derivative with $m(b) = \lim_{h\to 0}\left\{\frac{b^h-1}{h}\right\}$, we have the very interesting result.

$$\frac{d}{dx}(b^x) = m(b)b^x$$

Which says that the derivative of an exponential function is equal to the *same exponential function* scaled by a constant value that depends on the base, b. Note that if we can find a value of b such that $m(b) = 1$, we will have found an exponential function whose derivative is itself! Calling this base b^*, we would have the following derivative rule.

$$\frac{d}{dx}((b^*)^x) = (b^*)^x$$

As it turns out there is indeed such a special base, and it is none other than Euler's number, e! So, without proof, we claim that $b^* = e$, which as we know is an irrational number similar to π that has a value of approximately 2.718. As an exercise you can compute the limit below using small values of h to convince yourself that this is indeed true. As a matter of fact, one of the fundamental definitions of Euler's number is precisely given by this limit, which is stated below.

Euler's Number
Euler's number, e, is the precise value that makes the following limit equation true. $$\lim_{h\to 0}\left\{\frac{e^h - 1}{h}\right\} = 1$$

We have now established a very important derivative rule in calculus.

Derivative of the Natural Exponential Function
The derivative of an exponential function with base, e, is equal to the function itself. $$\frac{d}{dx}(e^x) = e^x$$

We can now extend this rule to other bases by using the following change of base formula for exponentials.

$$b^x = \left(e^{ln(b)}\right)^x = e^{ln(b)x}$$

For which we can employ the chain rule to differentiate.

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{ln(b)x}\right)$$
$$\frac{d}{dx}(b^x) = \left(\frac{d}{dx}(e^x)|_{x=ln(b)x}\right)\left(\frac{d}{dx}(ln(b)\,x)\right)$$
$$\frac{d}{dx}(b^x) = \left(e^{ln(b)x}\right)(ln(b))$$
$$\frac{d}{dx}(b^x) = ln(b)\,b^x$$

Note if $b = e$, then $ln(e) = 1$, which agrees with our earlier result for $\frac{d}{dx}(e^x)$.

With this we state the derivative rule for a general exponential function below.

Derivative of a General Exponential Function
If $f(x) = b^x$ for any $b > 0$ $$\frac{d}{dx}(b^x) = ln(b)\,b^x$$

Derivative of Logarithm Functions:

To derive a derivative rule for logarithmic functions we proceed as follows. Starting with the natural logarithm, $f(x) = ln(x)$ we define the equation, $y = ln(x)$, and exponentiating both sides as follows:

$$y = ln(x)$$
$$e^y = x$$

Next, we use implicit differentiation to find $\frac{dy}{dx}$.

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$
$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$
$$\frac{d}{dx}(ln(x)) = \frac{1}{x}$$

Therefore, we have found a simple rule for the derivative of the natural logarithm.

Derivative of the Natural Logarithm
$$\frac{d}{dx}(ln(x)) = \frac{1}{x}$$ For $x > 0$

Next, we again extend this rule to other bases using the change of base formula for logarithms.

$$log_b(x) = \frac{ln(x)}{ln(b)}$$

Which can be easily differentiated since $ln(b)$ is a constant.

$$\frac{d}{dx}(log_b(x)) = \frac{d}{dx}\left(\frac{ln(x)}{ln(b)}\right)$$
$$\frac{d}{dx}(log_b(x)) = \frac{1}{ln(b)}\frac{d}{dx}(ln(x))$$
$$\frac{d}{dx}(log_b(x)) = \frac{1}{xln(b)}$$

Derivative of a General Logarithmic Function
If $f(x) = log_b(x)$ for any $b > 0$ $$\frac{d}{dx}(log_b(x)) = \frac{1}{xln(b)}$$

With the above rules established let's do some examples.

Example 1: Differentiate the following exponential functions.

a. $y = 2^{x^3}$ b. $f(x) = 5^{(x^2-2x)}$ c. $f(x) = e^{(ln(x))^2}$

a. We combine the chain rule with our new exponential differentiation rule.

$$\begin{aligned}\frac{d}{dx}\left(2^{x^3}\right) &= \left(\frac{d}{dx}(2^x)|_{x=x^3}\right)\left(\frac{d}{dx}(x^3)\right)\\ &= \left(ln(2)\,2^{x^3}\right)(3x^2)\\ &= 3\,ln(2)\,x^2 2^{x^3}\end{aligned}$$

Note, we can also differentiate this function by first changing to base e as shown below.

$$\begin{aligned}\frac{d}{dx}\left(2^{x^3}\right) &= \frac{d}{dx}\left(e^{ln(2)x^3}\right)\\ &= \left(\frac{d}{dx}(e^x)|_{x=ln(2)x^3}\right)\left(\frac{d}{dx}(ln(2)\,x^3)\right)\\ &= \left(e^{ln(2)x^3}\right)(3\,ln(2)\,x^2)\\ &= \left(2^{x^3}\right)(3\,ln(2)\,x^2)\\ &= 3\,ln(2)\,x^2 2^{x^3}\end{aligned}$$

b. We apply the same technique from above.

$$\begin{aligned}\frac{d}{dx}\left(5^{(x^2-2x)}\right) &= \left(\frac{d}{dx}(5^x)|_{x=(x^2-2x)}\right)\left(\frac{d}{dx}(x^2-2x)\right)\\ &= \left(ln(5)\,5^{(x^2-2x)}\right)(2x-2)\\ &= ln(5)\,(2x-2)5^{(x^2-2x)}\end{aligned}$$

c. This problem requires multiple uses of the chain rule and knowledge of the derivative of the natural logarithm.

$$\begin{aligned}\frac{d}{dx}\left(e^{(ln(x))^2}\right) &= \left(\frac{d}{dx}(e^x)|_{x=((ln(x))^2)}\right)\left(\frac{d}{dx}((ln(x))^2)\right)\\ &= \left(\frac{d}{dx}(e^x)|_{x=((ln(x))^2)}\right)\left(\frac{d}{dx}(x^2)|_{x=(ln(x))}\right)\left(\frac{d}{dx}(ln(x))\right)\\ &= \left(e^{(ln(x))^2}\right)(2\,ln(x))\left(\frac{1}{x}\right)\\ &= \left(\frac{2\,ln(x)\,e^{(ln(x))^2}}{x}\right)\end{aligned}$$

Example 2: Differentiate the following logarithmic type functions.

a. $y = log_5(2x^2 + 7)$ b. $f(x) = ln(9x^2 - 8)$ c. $f(x) = (ln(ln(x)))^3$

a. We combine the chain rule with the logarithm differentiation rule.

$$\begin{aligned}\frac{d}{dx}(log_5(2x^2+7)) &= \left(\frac{d}{dx}(log_5(x))|_{x=2x^2+7}\right)\left(\frac{d}{dx}(2x^2+7)\right)\\ &= \left(\frac{1}{(2x^2+7)\,ln(5)}\right)(4x)\\ &= \frac{4x}{(2x^2+7)\,ln(5)}\end{aligned}$$

Similar to what we did in the exponential example above we can differentiate this function by first changing to base e as shown below.

$$\begin{aligned}\frac{d}{dx}(log_5(2x^2+7)) &= \left(\frac{d}{dx}\frac{ln(2x^2+7)}{ln(5)}\right)\\ &= \frac{1}{ln(5)}\left(\frac{d}{dx}(ln(x))|_{x=2x^2+7}\right)\left(\frac{d}{dx}(2x^2+7)\right)\\ &= \frac{1}{ln(5)}\left(\frac{1}{(2x^2+7)}\right)(4x)\\ &= \frac{4x}{(2x^2+7)\,ln(5)}\end{aligned}$$

b. We apply the same technique from the first method above.

$$\begin{aligned}\frac{d}{dx}(ln(9x^2-8)) &= \left(\frac{d}{dx}(ln(x))|_{x=9x^2-8}\right)\left(\frac{d}{dx}(9x^2-8)\right)\\ &= \left(\frac{1}{(9x^2-8)}\right)(18x)\\ &= \frac{18x}{(9x^2-8)}\end{aligned}$$

c. This problem requires multiple uses of the chain rule.

$$\begin{aligned}\frac{d}{dx}((ln(ln(x)))^3) &= \left(\frac{d}{dx}(x^3)|_{x=ln(ln(x))}\right)\left(\frac{d}{dx}(ln(ln(x)))\right)\\ &= \left(\frac{d}{dx}(x^3)|_{x=ln(ln(x))}\right)\left(\frac{d}{dx}(ln(x))|_{x=ln(x)}\right)\left(\frac{d}{dx}(ln(x))\right)\\ &= (3(ln(ln(x)))^2)\left(\frac{1}{ln(x)}\right)\left(\frac{1}{x}\right)\\ &= \frac{3(ln(ln(x)))^2}{xln(x)}\end{aligned}$$

Example 3: Differentiate the following functions.

a. $y = x^x$

b. $f(x) = x^{e^x}$

c. $f(x) = x^{cos(x)}$

a. We start by rewriting the function using e so that we can differentiate using the exponential differentiation rule.

$$x^x = \left(e^{ln(x)}\right)^x = e^{x\,ln(x)}$$

$$\begin{aligned}\frac{d}{dx}\left(e^{x\,ln(x)}\right) &= \left(\frac{d}{dx}(e^x)|_{x=x\,ln(x)}\right)\left(\frac{d}{dx}(x\,ln(x))\right)\\ &= \left(\frac{d}{dx}(e^x)|_{x=x\,ln(x)}\right)\left(\left(\frac{d}{dx}(x)\right)(ln(x)) + x\left(\frac{d}{dx}(ln(x))\right)\right)\\ &= \left(e^{x\,ln(x)}\right)\left(ln(x) + \frac{x}{x}\right)\\ &= (x^x)(ln(x)+1)\end{aligned}$$

b. We use a similar approach for this problem.

$$x^{e^x} = \left(e^{ln(x)}\right)^{e^x} = e^{e^x\,ln(x)}$$

$$\begin{aligned}\frac{d}{dx}\left(e^{e^x\,ln(x)}\right) &= \left(\frac{d}{dx}(e^x)|_{x=e^x\,ln(x)}\right)\left(\left(\frac{d}{dx}(e^x)\right)(ln(x)) + e^x\left(\frac{d}{dx}(ln(x))\right)\right)\\ &= \left(e^{e^x\,ln(x)}\right)\left(e^x(ln(x)) + e^x\left(\frac{1}{x}\right)\right)\\ &= x^{e^x}e^x\left(ln(x) + \frac{1}{x}\right)\end{aligned}$$

c. We again use the same approach as previous problem.

$$\frac{d}{dx}\left(x^{cos(x)}\right) = \frac{d}{dx}\left(e^{cos(x)\,ln(x)}\right)$$
$$= \left(\frac{d}{dx}(e^x)|_{x=cos(x)\,ln(x)}\right)\left(\left(\frac{d}{dx}(cos(x))\right)(ln(x)) + cos(x)\left(\frac{d}{dx}(ln(x))\right)\right)$$
$$= \left(e^{cos(x)\,ln(x)}\right)\left(-\,sin(x)\,(ln(x)) + cos(x)\left(\frac{1}{x}\right)\right)$$
$$= x^{cos(x)}\left(\frac{cos(x)}{x} - sin(x)\,ln(x)\right)$$

Logarithmic Differentiation:

Logarithmic differentiation is a technique that can be used to save time when differentiating certain types of functions. Generally, when presented with the task of computing $f'(x)$, we can instead compute $\left(ln\left(f(x)\right)\right)'$, the derivative of the logarithm of the function. Let's do this symbolically using the chain rule.

$$\left(ln\left(f(x)\right)\right)' = \frac{1}{f(x)} \cdot f'(x)$$

Solving for $f'(x)$ we have

$$f'(x) \;=\; f(x)\left(ln\left(f(x)\right)\right)'$$

Which means to compute the derivative of a function, we can instead differentiate the logarithm of the function and multiply it by the function itself. Although the right-hand side of this equation seems more complex, the advantage to using this method relies mainly on the following two logarithm properties.

$$ln(A \cdot B) \;=\; ln(A) + ln(B)$$
$$ln(A/B) \;=\; ln(A) - ln(B)$$

Let's first illustrate the procedure by doing a small example.

Example 4:

Compute the derivative of $f(x) = x^2(x^3 - 4)$.

Let's start by differentiating the function directly using the product rule.

$$\begin{aligned}\frac{d}{dx}\left(x^2(x^3-4)\right) &= \frac{d}{dx}(x^2)(x^3-4) + x^2\frac{d}{dx}\left((x^3-4)\right)\\ &= (2x)(x^3-4) + x^2(3x^2)\\ &= (2x^4 - 8x) + (3x^4)\\ &= 5x^4 - 8x\end{aligned}$$

Now let's differentiate using the logarithmic differentiation technique.

$$\begin{aligned}\frac{d}{dx}f(x) &= f(x)\frac{d}{dx}\left(ln\left(f(x)\right)\right)\\ &= \left(x^2(x^3-4)\right)\left[\frac{d}{dx}\left(ln\left(x^2(x^3-4)\right)\right)\right]\\ &= \left(x^2(x^3-4)\right)\left[\frac{d}{dx}(ln(x^2)) + \frac{d}{dx}(ln(x^3-4))\right]\\ &= \left(x^2(x^3-4)\right)\left[\frac{2x}{x^2} + \frac{3x^2}{x^3-4}\right]\\ &= \cancel{\left(x^2(x^3-4)\right)}\left[\frac{2x(x^3-4)+3x^2(x^2)}{\cancel{x^2(x^3-4)}}\right]\\ &= (2x^2 - 8x) + 3x^4\\ &= 5x^4 - 8x\end{aligned}$$

For this example, it may have seemed like the second method was more time consuming. However, for functions containing a product or quotient with *several* factors this method will prove to be advantageous since the product and quotient derivatives are converted to sum and difference derivatives using the logarithm property mentioned above. Let's look at a more complex example to better illustrate the advantage of using this method.

Example 5:

Compute the derivative of $f(x) = \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}$

Let's start by taking the logarithm of the function and expanding.

$$\begin{aligned}ln\left(f(x)\right) &= ln\left(\frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}\right)\\ &= ln((x+1)^2) + ln\left((2x^2-3)\right) - ln\left(\sqrt{x^2+1}\right)\\ &= 2\,ln\left((x+1)\right) + ln\left((2x^2-3)\right) - \frac{1}{2}ln(x^2+1)\end{aligned}$$

Now we differentiate the above function.

$$\frac{d}{dx}\left(ln\left(f(x)\right)\right) = 2\frac{d}{dx}\left(ln\left((x+1)\right)\right) + \frac{d}{dx}\left(ln\left((2x^2-3)\right)\right) - \frac{1}{2}\frac{d}{dx}(ln(x^2+1))$$
$$= \frac{2}{(x+1)} + \frac{4x}{(2x^2-3)} - \frac{x}{(x^2+1)}$$

The final step is to multiply by $f(x)$.

$$\frac{d}{dx}\left(f(x)\right) = f(x)\frac{d}{dx}\left(ln\left(f(x)\right)\right)$$
$$= \left(\frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}\right)\left(\frac{2}{(x+1)} + \frac{4x}{(2x^2-3)} - \frac{x}{(x^2+1)}\right)$$

In this case the differentiation after taking the logarithm became mush more straightforward. We can also use this method for functions of the type $f(x)^{g(x)}$. Note that we differentiated these types of function in example 3 using a different technique. Let's differentiate part a. from example 3 using logarithmic differentiation to illustrate.

$$\frac{d}{dx}f(x) = f(x)\frac{d}{dx}\left(ln\left(f(x)\right)\right)$$
$$\frac{d}{dx}(x^x) = x^x\frac{d}{dx}(ln(x^x))$$
$$= x^x\frac{d}{dx}(x\,ln(x))$$
$$= x^x\left(\frac{d}{dx}(x)\,ln(x) + (x)\frac{d}{dx}(ln(x))\right)$$
$$= x^x\left(ln(x) + (x)\frac{1}{x}\right)$$
$$= x^x(ln(x)+1)$$

Final Summary for Differentiation – Derivatives of Exponentials and Logarithms

Derivative of Exponential Functions
If $f(x) = b^x$ for any $b > 0$, $$\frac{d}{dx}(b^x) = ln(b)\, b^x$$ For $b = e$, $ln(e) = 1$, which results in a function whose derivative is the function itself. $$\frac{d}{dx}(e^x) = e^x$$
Derivative of Logarithmic Functions
If $f(x) = log_b(x)$ for any $b > 0$, and for $x > 0$ $$\frac{d}{dx}(log_b(x)) = \frac{1}{xln(b)}$$ For $b = e$, $ln(e) = 1$, therefore the derivative of the natural logarithm is given as $$\frac{d}{dx}(ln(x)) = \frac{1}{x}$$
Logarithmic Differentiation
Logarithmic differentiation can be used to save time when differentiating functions containing a product and/or quotient with *several* factors or functions of the type, $f(x)^{g(x)}$. The procedure calls for first taking the logarithm of the function and then using various logarithm properties to simplify the differentiation process. The derivative of the original function is then computed as $$\frac{d}{dx}\big(f(x)\big) = \big(f(x)\big)\frac{d}{dx}\Big(ln\big(f(x)\big)\Big)$$

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