Differentiation – The Chain Rule

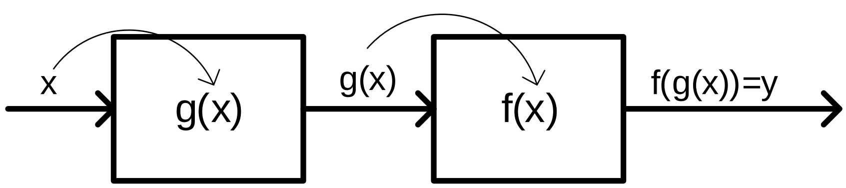
Two key rules we initially developed for our "toolbox" of differentiation rules were the power rule and the constant multiple rule. Together these rules allow us to differentiate functions of the form $f(x) = Cx^n$.

$$\frac{d}{dx}(Cx^n) = C\frac{d}{dx}(x^n) = Cnx^{n-1}$$

Next, we introduced algebraic rules that enable the differentiation of an algebraic combination of functions of the form above. The four methods we can use to algebraically combine functions are addition, subtraction, multiplication, and division. These operations and the corresponding differentiation rules developed for them are shown below.

Algebraic Operation	***Differentiation Rule***
Addition/Subtraction	$\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$
Multiplication	$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$
Division	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

There is another way in which functions can be combined, which is referred to as a composition. The composition of functions is a very common occurrence in any practical system and is graphically depicted as shown below.



As you can see the input of one function is driven by the output of another function. Using the symbols from the diagram we may write the output of the composite function as $y = f\left(g(x)\right)$, where x is the input to the first function, which produces an output, $g(x)$, which is then used as an input to the second function, producing the final output, $y = f\left(g(x)\right)$. Our goal would be to find a general rule to differentiate composite functions. We start by looking at a particular example.

Suppose oil is being poured onto the ground so that it creates an expanding circle. Suppose further that we measure the radius of the circle and find that it's increasing at a rate of $3\,m/s$. i.e., $\frac{dr}{dt} = 3$. We can now ask: "What is the rate of change of the area of the circle, $\frac{dA}{dt}$?". We can proceed as follows:

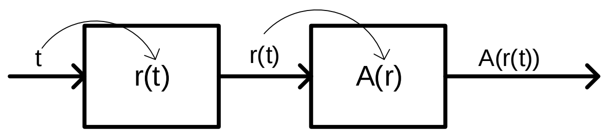
The radius of the circle at time, t, can be represented as:

$$r(t) = 3t.$$

The area of a circle, which is a function of the radius, can be written as:

$$A(r) = \pi r^2$$

Since the radius is a function of time, we can compose these functions to find the area of the circle as a function of time, as shown below.



$$\begin{aligned} A\big(r(t)\big) &= \pi\big(r(t)\big)^2 \\ A\big(r(t)\big) &= \pi(3t)^2 \\ A(t) &= \pi 9t^2 \end{aligned}$$

Now, to find the rate of change of the area with respect to time we simply differentiate $A(t)$.

$$\frac{dA}{dt} = 18\pi t$$

However, to find a general rule for differentiating composite functions we will need to investigate further. Let's see what happens if we differentiate the radius and area functions separately.

Rate of change of radius with respect to time	**Rate of change of area with respect to the radius**
$\frac{dr}{dt} = 3$	$\frac{dA}{dr} = 2\pi r$

Notice if we multiply these two rates and substitute $r = 3t$ we get the following:

$$\begin{aligned} \frac{dA}{dr} \cdot \frac{dr}{dt} &= (2\pi r) \cdot (3) \\ &= \big(2\pi(3t)\big) \cdot (3) \\ &= 18\pi t \end{aligned}$$

Which is what we found above for $\frac{dA}{dt}$! Therefore, for this example we may write the following:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

Although we will not prove it here, this relationship is true for the general case of composite functions and it is what we refer to as the chain rule. This rule will prove to be very powerful and will greatly extend the types of functions we can differentiate.

Writing the rule in Leibniz notation, as we have done above, has the advantage of being easy to memorize because it *appears* the intermediate variable differential, dr in the case above, cancels. We use the term "*appears"* because literal cancelation of differentials is not always possible. The conditions for this to be true are beyond the scope of an introductory calculus lesson, however we may still use it as an aid to memorize the rule. Of course, the rule can also be written in Lagrange notation, which as it turns out is usually preferred by students. We state the rule using both notations below.

The Chain Rule of Differentiation
If $f(x)$ and $g(x)$ are differentiable functions, then the composite function, $f\big(g(x)\big)$, is differentiable and Using Leibniz notation: $$\frac{df}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$$ Using Lagrange notation: $$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$$ Using this notation, we usually refer to g as the inside function and f as the outside function. In the above form we can state the operations in words as follows: "The derivative of $f\big(g(x)\big)$ is equal to the derivative of the outside function evaluated at the inside function, multiplied by the derivative of the inside function."

We now move to examples starting with some basic examples to help us begin to feel more comfortable applying the chain rule.

Example 1: Calculate the derivative of $y = sin(x^3)$.

We start by identifying the composite function as: $f\big(g(x)\big) = sin(x^3)$

Inside Function	Outside Function
$g(x) = x^3$	$f(x) = sin(x)$

We differentiate first using Lagrange notation according to: $\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$.

Derivative of outside function evaluated at inside function	Derivative of inside function
$f'\big(g(x)\big) = cos(x)\vert_{x=g(x)} = cos(x^3)$	$g'(x) = 3x^2$

Putting the results from above together we have:

$$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$$
$$\Big(f\big(g(x)\big)\Big)' = cos(x^3)\,3x^2$$
$$\Big(f\big(g(x)\big)\Big)' = 3x^2\,cos(x^3)$$

Now let's differentiate using Leibniz notation according to: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

In this case we write the outside function as a function of g only.

$$f(g) = sin(g)$$

And the inside function as a function of x only.

$$g(x) = x^3$$

$\boldsymbol{{}^{df}/_{dg}}$	$\boldsymbol{{}^{dg}/_{dx}}$
$\frac{df}{dg} = \frac{d}{dg}(sin(g)) = cos(g)$	$\frac{dg}{dx} = \frac{d}{dx}(x^3) = 3x^2$

Putting the results from above together and substituting $g = g(x) = x^3$ we have:

$$\frac{df}{dx} = cos(x^3) \cdot 3x^2$$
$$\frac{df}{dx} = 3x^2\,cos(x^3)$$

Example 2: Calculate the derivative of $y = (x + sin(x))^4$.

As before, we identify the composite function as: $f\big(g(x)\big) = (x + sin(x))^4$

Inside Function	Outside Function
$g(x) = x + sin(x)$	$f(x) = x^4$

We differentiate first using Lagrange notation according to: $\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$.

Derivative of outside function evaluated at inside function	Derivative of inside function
$f'\big(g(x)\big) = 4(x)^3\vert_{x=g(x)} = 4(x + sin(x))^3$	$g'(x) = 1 + cos(x)$

Putting the results from above together we have:

$$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$$
$$\Big(f\big(g(x)\big)\Big)' = (4(x + sin(x))^3)(1 + cos(x))$$

Now using Leibniz notation according to: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

We again write the outside function as a function of g only.

$$f(g) = g^4$$

And the inside function as a function of x only.

$$g(x) = x + sin(x)$$

$\boldsymbol{df}/\boldsymbol{dg}$	$\boldsymbol{dg}/\boldsymbol{dx}$
$\frac{df}{dg} = \frac{d}{dg}(g^4) = 4g^3$	$\frac{dg}{dx} = \frac{d}{dx}(x + sin(x)) = 1 + cos(x)$

Putting the results from above together and subsisting $g = g(x) = x + sin(x)$ we have:

$$\frac{df}{dx} = (4(x + sin(x))^3)(1 + cos(x))$$

The next few examples involve a combination of the chain rule and the product and/or quotient rule. Additionally, we will begin using the Lagrange notation only.

Example 3: Calculate the derivative of $y = \sqrt{sin(x)\, cos(x)}$.

Inside Function	Outside Function
$g(x) = sin(x)\, cos(x)$	$f(x) = x^{1/2}$

$$\begin{aligned}
\big(f(g(x))\big)' &= f'(g(x))g'(x) \\
&= 1/2\,(sin(x)\,cos(x))^{-1/2} \cdot ((sin(x))'(cos(x)) + (sin(x))(cos(x))') \\
&= \frac{1}{2\sqrt{sin(x)\,cos(x)}} \cdot \big(cos(x)\,(cos(x)) + (sin(x))(-sin(x))\big) \\
&= \frac{1}{2\sqrt{(1/2)\,sin(2x)}} \cdot (cos^2(x) - sin^2(x)) \\
&= \frac{cos(2x)}{\sqrt{2\,sin(2x)}}
\end{aligned}$$

Where, we used the product rule to differentiate $g(x)$, and the following two trigonometric identities to simplify the expression.

$$cos^2(x) - sin^2(x) = cos(2x)$$
$$2\,sin(x)\,cos(x) = sin(2x)$$

Example 4: Calculate the derivative of $y = \left(\frac{x+1}{x-1}\right)^4$.

Inside Function	Outside Function
$g(x) = \dfrac{x+1}{x-1}$	$f(x) = x^4$

$$\begin{aligned}
\big(f(g(x))\big)' &= f'(g(x))g'(x) \\
&= 4\left(\frac{x+1}{x-1}\right)^3 \cdot \frac{(x+1)'(x-1) - (x+1)(x-1)'}{(x-1)^2} \\
&= 4\frac{(x+1)^3}{(x-1)^3} \cdot \frac{(x-1) - (x+1)}{(x-1)^2} \\
&= 4\frac{(x+1)^3}{(x-1)^5} \cdot \left(\frac{-2}{1}\right) \\
&= -8\frac{(x+1)^3}{(x-1)^5}
\end{aligned}$$

Where, we used the quotient rule to differentiate $g(x)$.

Example 5: Calculate the derivative of $y = sin(x^2)\, cos(x^2)$.

Inside Functions	Outside Functions
$g_1(x) = x^2$ $g_2(x) = x^2$	$f_1(x) = sin(x)$ $f_2(x) = cos(x)$

In this case we have a product of two composite functions.

$$\begin{aligned}\frac{d}{dx}(sin(x^2)\,cos(x^2)) &= (sin(x^2))'(cos(x^2)) + (sin(x^2))(cos(x^2))' \\ &= (2x\,cos(x^2))(cos(x^2)) + (sin(x^2))(-2x\,sin(x^2)) \\ &= 2x(cos^2(x^2) - sin^2(x^2)) \\ &= 2x(cos(2x^2)) \\ &= 2x\,cos(x^2)\end{aligned}$$

Where, we used the chain rule for both derivates required for the product rule, and the following trigonometric identity to simplify the expression.

$$cos^2(x) - sin^2(x) = cos(2x)$$

The next few examples involve repeated use of the chain rule.

Example 6: Calculate the derivative of $y = sin(cos(sin(x)))$.

In this case we have three functions that are composed: $y = \left(f\left(h\big(g(x)\big)\right)\right)$

For this first case of repeated use of the chain rule we will evaluate using both Lagrange and Leibniz notations.

Lagrange Notation:		
Inside Function	**Middle Function**	**Outside Function**
$g(x) = sin(x)$	$h(x) = cos(x)$	$f(x) = sin(x)$

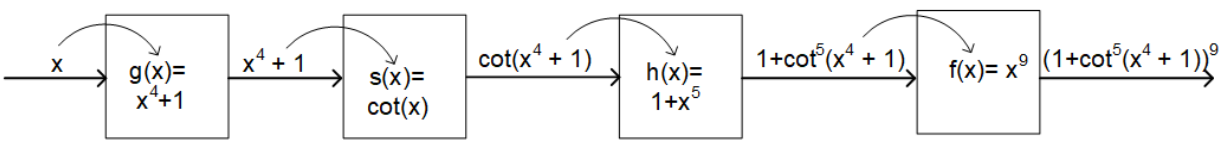
$$\begin{aligned}\left(f\left(h\big(g(x)\big)\right)\right)' &= f'\left(h\big(g(x)\big)\right)\left(h\big(g(x)\big)\right)' \\ &= f'\left(h\big(g(x)\big)\right)h'\big(g(x)\big)g'(x) \\ &= cos(cos(sin(x)))\cdot(-\,sin(sin(x)))\cdot cos(x) \\ &= -\,cos(cos(sin(x)))\cdot(sin(sin(x)))\cdot cos(x)\end{aligned}$$

Leibniz Notation:		
Inside Function	**Middle Function**	**Outside Function**
$g(x) = sin(x)$	$h(g) = cos(g)$	$f(h) = sin(h)$

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{dh}\cdot\frac{dh}{dg}\cdot\frac{dg}{dx} \\ &= cos(h)\cdot(-sin(g))\cdot cos(x) \\ &= -cos(cos(g))\cdot(sin(sin(x)))\cdot cos(x) \\ &= -cos(cos(sin(x)))\cdot(sin(sin(x)))\cdot cos(x)\end{aligned}$$

Example 7: Calculate the derivative of $y = (1 + cot^5(x^4 + 1))^9$.

You'll notice this function can be considered a composition of four functions as shown below.



Using the Lagrange notation, we evaluate as follows:

$$\begin{aligned}\Big(f\Big(h\Big(s\big(g(x)\big)\Big)\Big)\Big)' &= f'\Big(h\Big(s\big(g(x)\big)\Big)\Big)\Big(h\Big(s\big(g(x)\big)\Big)\Big)' \\ &= f'\Big(h\Big(s\big(g(x)\big)\Big)\Big)h'\Big(s\big(g(x)\big)\Big)\Big(s\big(g(x)\big)\Big)' \\ &= f'\Big(h\Big(s\big(g(x)\big)\Big)\Big)h'\Big(s\big(g(x)\big)\Big)s'\big(g(x)\big)g'(x) \\ &= (9(1 + cot^5(x^4 + 1))^8)\cdot(5\,cot^4(x^4 + 1))\cdot(-csc^2(x^4 + 1))(4x^3) \\ &= -180x^3((1 + cot^5(x^4 + 1))^8)(cot^4(x^4 + 1))(csc^2(x^4 + 1))\end{aligned}$$

Example 8: Calculate the derivative of $y = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$.

This is a composition of three functions that may be easier to evaluate if we rewrite it using powers instead of radicals.

$$y = \Big(1 + \big(1 + (x)^{1/2}\big)^{1/2}\Big)^{1/2}$$

Let’s now evaluate as we did above from the outside function to inside function.

$$y' = 1/2\left(\left(1+\left(1+(x)^{1/2}\right)^{1/2}\right)^{-1/2}\right)\cdot 1/2\left(\left(1+(x)^{1/2}\right)^{-1/2}\right)\cdot 1/2\,(x)^{-1/2}$$

$$= \frac{1}{8\sqrt{(1+(1+(x)^{1/2})^{1/2})}(1+(x)^{1/2})^{1/2}(x)^{1/2}}$$

$$= \frac{1}{8\sqrt{\left(1+\sqrt{(1+\sqrt{x})}\right)}\left(\sqrt{1+\sqrt{x}}\right)\left(\sqrt{x}\right)}$$

Let’s now try doing this example using Leibniz notation. We first identify the three composed functions.

Leibniz Notation:		
Inside Function	**Middle Function**	**Outside Function**
$g(x) = 1+\sqrt{x}$	$h(g) = 1+\sqrt{g}$	$f(h) = \sqrt{h}$

Differentiating we have:

$$\frac{df}{dx} = \frac{df}{dh}\cdot\frac{dh}{dg}\cdot\frac{dg}{dx}$$

$$= \left(\frac{1}{2\sqrt{h}}\right)\cdot\left(\left(\frac{1}{2\sqrt{g}}\right)\right)\cdot\left(\left(\frac{1}{2\sqrt{x}}\right)\right)$$

$$= \left(\frac{1}{2\sqrt{1+\sqrt{g}}}\right)\cdot\left(\left(\frac{1}{2\sqrt{1+\sqrt{x}}}\right)\right)\cdot\left(\left(\frac{1}{2\sqrt{x}}\right)\right)$$

$$= \left(\frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\right)\cdot\left(\left(\frac{1}{2\sqrt{1+\sqrt{x}}}\right)\right)\cdot\left(\left(\frac{1}{2\sqrt{x}}\right)\right)$$

$$= \left(\frac{1}{8\left(\sqrt{1+\sqrt{1+\sqrt{x}}}\right)\left(\sqrt{1+\sqrt{x}}\right)\left(\sqrt{x}\right)}\right)$$

Final Summary for Differentiation – The Chain Rule

The Chain Rule of Differentiation
If $f(x)$ and $g(x)$ are differentiable functions, then the composite function, $f\left(g(x)\right)$, is differentiable and Using Leibniz notation: $$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$ Using Lagrange notation: $$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right)g'(x)$$ Using this notation, we usually refer to g as the inside function and f as the outside function. With this the above form can be stated in words as follows: “The derivative of $f\left(g(x)\right)$ is equal to the derivative of the outside function evaluated at the inside function, multiplied by the derivative of the inside function.”

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