Derivative Applications – Related Rates

In related rate problems we are trying to find the time rate of change of one quantity based on knowing the time rate of change of another *related* quantity. For example, let's assume we have two quantities that are functions of time, $x(t)$ and $y(t)$, and are related by the function $y(t) = f(x(t))$. Taking the time derivative on both sides of this equation we write

$$\frac{d}{dt}\big(y(t)\big) = \frac{d}{dt}\Big(f\big(x(t)\big)\Big)$$

Using Leibniz notation and the chain rule for the right side of the equation we can write

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Therefore, when two quantities are related by an explicit relationship, i.e., $y(t) = f(x(t))$, the time rate of change of the dependent variable, in this case $y(t)$, is related to the time rate of change of the independent variable, $x(t)$, by the derivative of the functional relationship between the two variables, $\frac{df}{dx}$. Even if the relationship between the two variables is given implicitly, a relationship between their time derivatives can still be established. However, for these cases the relationship cannot be stated by a single formula as shown above. Instead, a process can be established for finding these relationships. We summarize this process below and illustrate with various examples.

Related Rate Problem Summary
We are generally presented with two variables, e.g., x, and y, that are both functions of time and that can be related either explicitly or implicitly, e.g., explicit: $y = f(x)$. We are then given the time rate of change of one of the variables, e.g. dx/dt, and we are asked to find the time rate of change of the other variable, e.g. dy/dt. Solving this problem generally involves the following steps: 1. Determine the relationship, if not given, between the two variables and write as either an explicit or implicit equation. 2. Take the time derivative of the entire equation from step 1. 3. Solve the equation from step 2 for the unknown rate. 4. Evaluate for the unknown rate using the values given in the problem.

Let's illustrate the procedure with some example problems.

Example 1:

Suppose there is an oil spill which is spreading in an ever-increasing circle. We measure the rate at which the radius of the circle is increasing to be $0.5\ m/min$. Find the rate at which the area of the circle is increasing when the radius is at 6 meters.

The two variables are the radius, r, and the area, A, of a circle. We are given the time rate of change of the radius as

$$\frac{dr}{dt} = 0.5\ m/min$$

And we are asked to the find time rate of change of the area, $\frac{dA}{dt}$.

According to the problem summary above, we need to first determine a relationship between the two variables. In this case an explicit relationship can be written.

$$A = \pi r^2$$

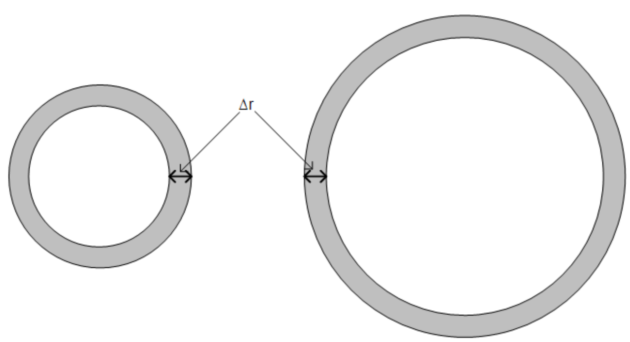
Following again the steps from the problem summary we take the time derivative of this equation using implicit differentiation.

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

This equation is already solved for the unknown quantity, dA/dt. Notice however that r is a function of time, which means that although the time rate of change of the radius is constant the time rate of change of the area is not. As a matter of fact, it is directly proportional to the radius. We can write the relationship more explicitly as follows.

$$\frac{dA}{dt}(r) = \pi 2r(0.5)$$
$$A'(r) = \pi r$$

This makes sense since, as illustrated below, the same small change in the radius results in a larger change in the area, the gray part shown, when the circle is bigger.



The question asked us for the time rate of change of the area when the radius in 6 meters.

$$A'(6) = 6\pi\ m^2/min$$

Example 2:

Helium is being pumped into a spherical balloon at a rate of $50\ cm^3/sec$. How fast is the radius of the balloon increasing when the radius is $60\ cm$?

The helium being pumped into the balloon is causing the volume to change at the rate it is being pumped.

$$\frac{dV}{dt} = 50\ cm^3/sec$$

The volume of a sphere can be written as a function of the radius as follows:

$$V = \frac{4}{3}\pi r^3$$

Differentiating with respect to time and solving for rate of change of the radius we have

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \left(\frac{1}{4\pi r^2}\right)\frac{dV}{dt}$$

$$\frac{dr}{dt} = \left(\frac{50}{4\pi r^2}\right)$$

Finally, we evaluate the rate when the radius is $60\ cm$.

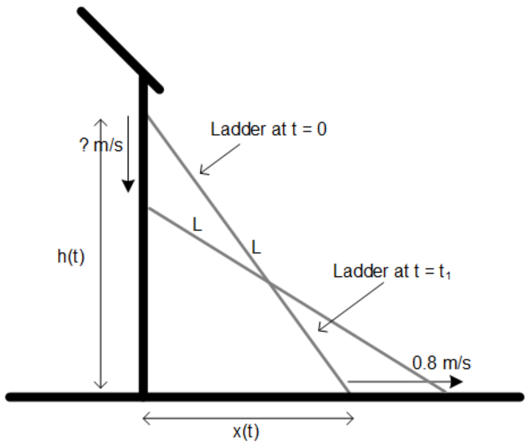
$$\left.\frac{dr}{dt}\right|_{r=60} = \left(\frac{50}{4\pi(60)^2}\right)$$

$$\left.\frac{dr}{dt}\right|_{r=60} = \frac{1}{288\pi}\ cm/sec$$

Example 3:

A $5\ m$ ladder leans against a wall such that the bottom of the ladder is $1.5\ m$ from the wall. The ladder begins to slide such that the bottom of the ladder is moving away from the wall at a rate of $0.8\ m/s$. Find the velocity of the ladder 1 second after it starts to slide.

Let's start by drawing a picture.



The two variables involved are the distance of the bottom of the ladder from the wall, x, and the distance of the top of the ladder from the ground, h. The length of the ladder, L, is constant and the ladder forms a right triangle with the wall at all times. Therefore, we can relate the two variables *implicitly* using the Pythagorean Theorem.

$$x^2 + h^2 = L^2$$

Differentiating with respect to time and solving for the rate of change of the height we have:

$$\begin{aligned}
\frac{d}{dt}(x^2) + \frac{d}{dt}(h^2) &= \frac{d}{dt}(L^2) \\
\cancel{2}x\frac{dx}{dt} + \cancel{2}h\frac{dh}{dt} &= 0 \\
\cancel{2}x\frac{dx}{dt} + \cancel{2}h\frac{dh}{dt} &= 0 \\
\frac{dh}{dt} &= \frac{dx}{dt}\left(-\frac{x}{h}\right) \\
\frac{dh}{dt} &= -0.8\frac{x}{h}
\end{aligned}$$

And since $h = \sqrt{L^2 - x^2}$, we have:

$$\frac{dh}{dt} = -0.8\frac{x}{\sqrt{L^2 - x^2}}$$

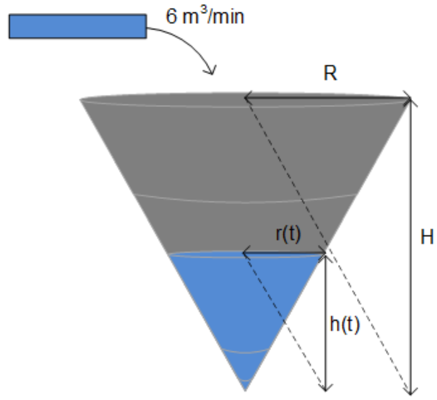
The bottom of the ladder started at 1.5 meters from the wall and moves away at a rate of $0.8\ m/s$. Therefore at $t = 1\ sec$ we have $x = 1.5 + 0.8 = 2.3\ m$.

$$\left.\frac{dh}{dt}\right|_{x=2.3} = -0.8\frac{2.3}{\sqrt{(5)^2 - (2.3)^2}}$$

$$\left.\frac{dh}{dt}\right|_{x=2.3} = -0.414\ m/s$$

Example 4:

Water pours into a canonical tank of height, $H = 10\ m$, and radius, $R = 4\ m$, at a rate of $6\,m^3/min$. At what rate is the level rising when the level is $5\ m$ high?



We can relate our two variables, the volume and height, using the volume of a cone as follows.

$$V = \frac{1}{3}\pi h r^2$$

Unfortunately, this relationship is between three variables. However, we can use similar triangles shown in the figure to write the radius, r, as a function of the height, h.

$$\frac{r}{h} = \frac{R}{H}$$

$$r = \left(\frac{R}{H}\right)h$$

Substituting for r in the volume equation, differentiating, and solving for dh/dt, we have

$$V = \left(\frac{\pi R^2}{3H^2}\right)h^3$$

$$\frac{dV}{dt} = \left(\frac{\pi R^2}{\cancel{3}H^2}\right)\cancel{3}h^2\frac{dh}{dx}$$

$$\frac{dh}{dt} = \left(\frac{dV}{dx}\cdot\frac{H^2}{\pi R^2}\right)\frac{1}{h^2}$$

$$\frac{dh}{dt} = \left(6\cdot\frac{100}{\pi 16}\right)\frac{1}{h^2}$$

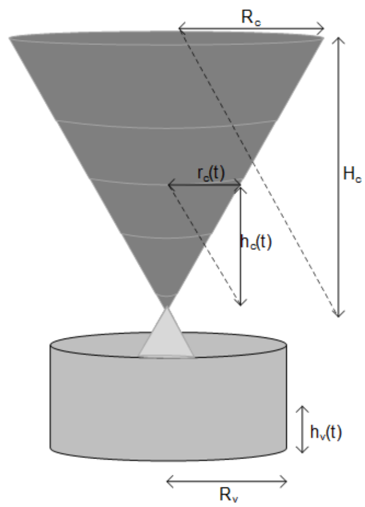
$$\frac{dh}{dt} = \left(\frac{37.5}{\pi}\right)\frac{1}{h^2}$$

Finally, we evaluate at $h = 5\ m$

$$\left.\frac{dh}{dx}\right|_{h=5} = \left(\frac{37.5}{\pi}\right)\frac{1}{25} = \frac{3}{2\pi}\ m/min$$

Example 5:

A solution is filtering through a conical filter with a height, H_c, of 18 inches and a radius, R_c, of 6 inches, into a cylindrical vessel whose radius, R_v, is 5 inches. When the depth of the solution in the filter is 12 inches, its level is falling at a rate of 1 inch per minute. At what rate is the level in the cylinder rising?



Starting with the cylindrical vessel, the two variables are the volume pouring into the vessel and the height of the solution in vessel, which we relate as

$$V_v = \pi R_v^{\,2} h_v$$

Differentiating and solving for the time rate of change of the vessel height we have

$$\frac{dV_v}{dt} = \left(\pi R_v^{\,2}\right)\frac{dh_v}{dt}$$
$$\frac{dh_v}{dt} = \left(\frac{1}{\pi R_v^{\,2}}\right)\frac{dV_v}{dt}$$

We are not able to solve this since we don't yet know dV_v/dt. However, since the rate of the solution falling into the cylinder will be equal to the rate of the solution falling out of the filter, let's analyze the cone. The volume of the solution in the filter can be related to the height in the same way we did for example 4.

$$V_c = \left(\frac{\pi R_c^{\,2}}{3H_c^{\,2}}\right) h_c^{\,3}$$

Differentiating, we find an expression for the rate of the solution falling out of the cone.

$$\frac{dV_c}{dt} = \left(\frac{\pi R_c^{\,2}}{\cancel{3}H_c^{\,2}}\right)\cancel{3}h_c^{\,2}\frac{dh_c}{dt}$$
$$\frac{dV_c}{dt} = \left(\frac{\pi R_c^{\,2}}{H_c^{\,2}}\right)h_c^{\,2}\frac{dh_c}{dt}$$

We can now substitute this expression in the rate equation we earlier derived for the vessel. Note the negative sign since the solution is flowing out of the filter and into the vessel.

$$\frac{dh_v}{dt} = \left(\frac{1}{\pi {R_v}^2}\right)\frac{dV_v}{dt}$$

$$\frac{dh_v}{dt} = \left(\frac{1}{\cancel{\pi} {R_v}^2}\right)\left(-\left(\frac{\cancel{\pi} {R_c}^2}{{H_c}^2}\right){h_c}^2\frac{dh_c}{dt}\right)$$

$$\frac{dh_v}{dt} = -\left[\left(\frac{R_c}{H_c R_v}\right)^2\frac{dh_c}{dt}\right]{h_c}^2$$

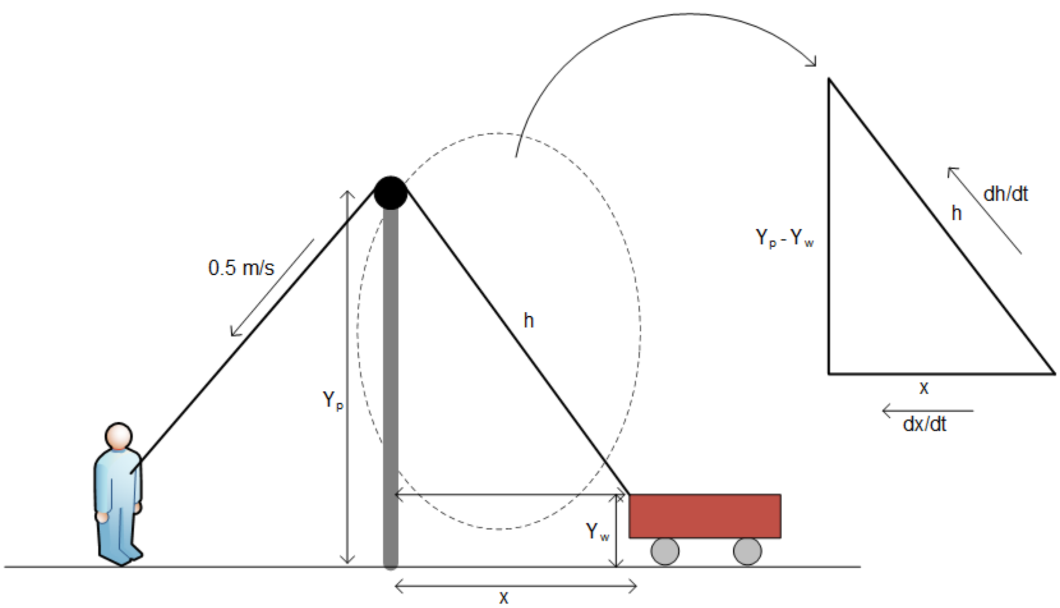
Finally, we can evaluate this expression when the height of the cone, h_c, is 12 inches, and the rate of change of the cone height is $-1\ in/min$.

$$\left.\frac{dh_v}{dt}\right|_{h_c=12} = -\left[\left(\frac{6}{18 \cdot 5}\right)^2(-1)\right]12^2$$

$$\left.\frac{dh_v}{dt}\right|_{h_c=12} = \frac{16}{25}\ in/min$$

Example 6:

A man pulls a rope attached to a wagon though a pulley at a rate of $0.5\ m/s$. The height of the pulley and the wagon are $Y_p = 3\ m$ and $Y_w = 0.6\ m$ respectively. Find the speed of the wagon when it is $0.7\ m$ from the pole.



The figure shows that we can extract a right triangle for which the height is constant, $Y_p - Y_w$, while the base, x, and the hypotenuse, h, are functions of time. These variables are related through the Pythagorean Theorem.

$$h^2 = x^2 + \left(Y_p - Y_w\right)^2$$

Differentiating and solving for dx/dt, we have

$$\not{2}h\frac{dh}{dt} = \not{2}x\frac{dx}{dt} + 0$$

$$\frac{dx}{dt} = \left(\frac{h}{x}\right)\frac{dh}{dt}$$

$$\frac{dx}{dt} = \left(\frac{\sqrt{x^2 + \left(Y_p - Y_w\right)^2}}{x}\right)\frac{dh}{dt}$$

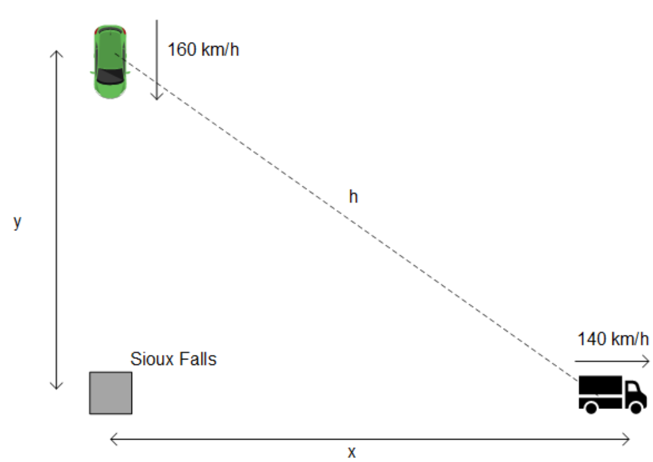
Since the pulley simply changes the direction of the motion, $dh/dt = -0.5\ m/s$. Substituting and evaluating when $x = 0.7\ m$, the velocity of the wagon is as shown.

$$\left.\frac{dx}{dt}\right|_{x=0.7} = \left(\frac{\sqrt{0.7^2 + (3 - 0.6)^2}}{0.7}\right)(-0.5)$$

$$\left.\frac{dx}{dt}\right|_{x=0.7} = -1.79\ m/s$$

Example 7:

A police car is traveling south towards Sioux Falls at $160\ km/h$, while a truck is traveling east away from Sioux Falls at $140\ km/h$. At $t = 0$, the police car is $20\ km$ north and the truck is $30\ km$ east of Sioux Falls. Find the rate at which the distance between the vehicles is changing at $t = 0$ and at $t = 5\ min$.



In this case we have three variable that are changing, which can all be related through the Pythagorean Theorem as shown.

$$h^2 = x^2 + y^2$$

Differentiating, we find a relation between the rate of change for the distance between the two vehicles, dh/dt, and the rate of change for the distance of each vehicle to Sioux falls, dx/dt and dy/dt.

$$\cancel{2}h\frac{dh}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\frac{dh}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{h}$$

$$\frac{dh}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}$$

Both the speed of the police car, dy/dt, and the speed of the truck, dx/dt, are known. We are asked to find dh/dt at times $t = 0$ and $t = 5\ min$, for which we need to determine the distances. We are given the distances at $t = 0$, which, along with the speeds we can find the distances at $t = (5/60\ )hr$ as shown below.

$$x(t) = x(0) + v_x(t)$$

$$x(5/60) = 30 + 140\left(\frac{5}{60}\right)$$

$$x(5/60) = 41\frac{2}{3}km$$

$$y(t) = y(0) + v_y(t)$$

$$y(5/60) = 20 + -160\left(\frac{5}{60}\right)$$

$$y(5/60) = 6\frac{2}{3}km$$

Substituting we can find the desired rates.

$$\left.\frac{dh}{dt}\right|_{t=0} = \frac{30(140) + 20(-160)}{\sqrt{30^2 + 20^2}}$$

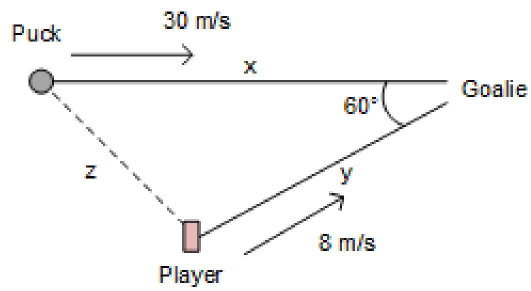
$$\left.\frac{dh}{dt}\right|_{t=0} = 27.74\ km/h$$

$$\left.\frac{dh}{dt}\right|_{t=5\ min} = \frac{41\frac{2}{3}(140) + 6\frac{2}{3}(-160)}{\sqrt{\left(41\frac{2}{3}\right)^2 + \left(6\frac{2}{3}\right)^2}}$$

$$\left.\frac{dh}{dt}\right|_{t=5\ min} = 112.96\ km/h$$

Example 8:

A hockey puck slides along the ice with a velocity of $30\, m/s$ towards the goalie located $15\ m$ to its right. A second player skates at $8\, m/s$ along a line that makes a 60° angle with the path of the puck. Determine the rate of change of the distance between the puck and the second player if the player is $4\ m$ from the goalie at the instant shown.



This scenario does not form a right triangle; however, we can relate the side distances using the Law of Cosines.

$$z^2 = x^2 + y^2 - 2xy\cos(\theta)$$

As usual we differentiate with respect to time, using the product rule for the last term.

$$\cancel{2}z\frac{dz}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} - \cancel{2}\cos(\theta)\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} - \cos(\theta)\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right)}{z}$$

$$\frac{dz}{dt} = \frac{\frac{dx}{dt}(x - y\cos(\theta)) + \frac{dy}{dt}(y - x\cos(\theta))}{\sqrt{x^2 + y^2 - 2xy\cos(\theta)}}$$

Finally, we evaluate at $x = 15$ and $y = 4$

$$\left.\frac{dz}{dt}\right|_{x=15,y=4} = \frac{(-30)(15 - 4\cos(60)) + (-8)(4 - 15\cos(60))}{\sqrt{15^2 + 4^2 - 2\cdot 15\cdot 4\cos(60)}}$$

$$\left.\frac{dz}{dt}\right|_{x=15,y=4} = \frac{(-390) + (28)}{\sqrt{181}}$$

$$\left.\frac{dz}{dt}\right|_{x=15,y=4} = -26.91\, m/s$$

<u>Final Summary for Derivative Applications – Related Rates</u>

Related Rate Problem Summary
We are generally presented with two variables, e.g., x, and y, that are both functions of time and that can be related either explicitly or implicitly, e.g., explicit: $y = f(x)$. We are then given the time rate of change of one of the variables, e.g. dx/dt, and we are asked to find the time rate of change of the other variable, e.g. dy/dt. Solving this problem generally involves the following steps: 1. Determine the relationship, if not given, between the two variables and write as either an explicit or implicit equation. 2. Take the time derivative of the entire equation from step 1. 3. Solve the equation from step 2 for the unknown rate. 4. Evaluate for the unknown rate using the values given in the problem.

By: ferrantetutoring