Derivative Applications – Applied Optimization

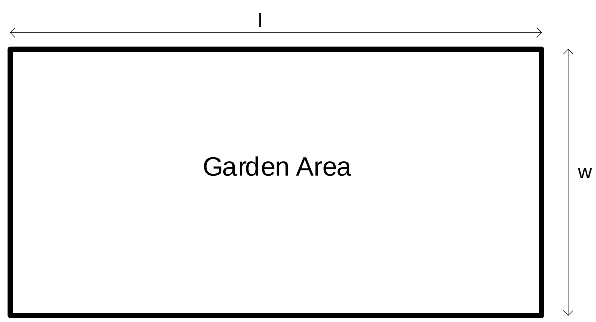
Locating the extreme values of a function, as we have done in the previous two lessons, has many practical applications. We are generally presented with a situation where we would like to find the maximum or minimum value of some parameter, e.g., profit, cost, area, etc., as it varies according to an independent variable, e.g., x. Our task is to find a mathematical relationship between the parameter and the independent variable. We represent this relationship with a function, $f(x)$, which is commonly referred to as an *objective function*. The process of finding the maximum or minimum value is called *optimization* and is carried out using the techniques we have previously learned. Since we already have the mathematical background to perform the optimization let's begin with examples.

Example 1:

We wish to build a rectangular shaped garden using P feet of fencing that we have available. What should the dimensions of the rectangular garden be to maximize the area?

Solution:

Let's start by drawing a picture.



Our objective function is the area function and is given by:

$$A(l,w) = lw$$

In this lesson we are restricted to optimizing functions of only one variable, therefore we need to somehow relate the two independent variables, l and w. We can do this using the fact that the perimeter of the garden is restricted by the amount of fence we have. This is sometimes called a constraint equation, and in this case is given as:

$$P = 2l + 2w$$

Solving for w in terms of l we have:

$$w = \frac{1}{2}P - l$$

Now we substitute this relationship, (the constraint function), giving us our desired single variable objective function.

$$A(l) = l\left(\frac{1}{2}P - l\right)$$
$$A(l) = \frac{lP}{2} - l^2$$

Optimizing, in this case finding the maximum area, is performed as we did in previous lessons, i.e., compute the derivative and set it to zero.

$$A'(l) = -2l + \frac{P}{2} = 0$$
$$l = \frac{P}{4}$$

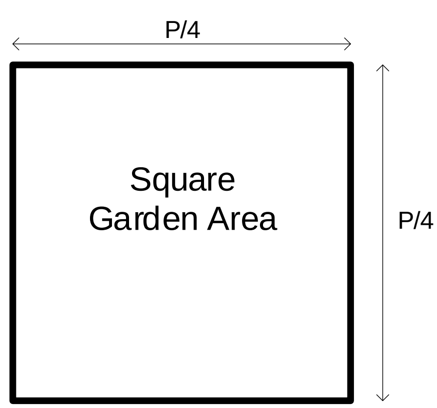
We can verify this critical point is a local maximum via the second derivative test.

$$A''(l) = -2$$

Since the second derivative is negative, we have a local maximum. We may also notice that the function we are maximizing is a concave down parabola, and therefore the local maximum is also an absolute maximum. Lastly, to find the width, w, we substitute the length, l, from above back into our constraint equation.

$$w = \frac{1}{2}P - \frac{P}{4}$$
$$= \frac{P}{4}$$

Therefore, a square shape, with $l = w = \frac{P}{4}$, will result in a garden of maximum area.



Example 2:

For this example, we want to build a rectangular garden that covers a certain area, A, such that the amount of fencing we need to purchase, i.e., the cost, is minimized.

Solution:

Contrary to example 1, in this case the objective function is the perimeter function and the constraint function is the area function.

Objective function: $P(l, w) = 2l + 2w$

Constraint function: $A = lw$

Similar to example 1 we can solve the constraint function for w and substitute into the objective function.

$$P(l) = 2l + \frac{2A}{l}$$

We can now optimize this function, i.e., differentiate and set equal to zero.

$$P'(l) = 2 - \frac{2A}{l^2} = 0$$
$$l^2 = A$$
$$l = \pm\sqrt{A}$$

Since a negative length is not physically possible, we have a single critical point at $l = \sqrt{A}$. Let's again use the second derivative test to see if we have our desired local minimum.

$$P''(l)|_{l=\sqrt{A}} = \frac{4A}{\left(\sqrt{A}\right)^3} = 4\sqrt[3]{A}$$

Which is positive and therefore our critical point is a local minimum as desired. To find the width we substitute $l = \sqrt{A}$ into the constraint equation.

$$w = \frac{A}{\sqrt{A}} = \sqrt{A}$$

Therefore, the dimensions of the garden with an area of A that will minimize the cost of fencing is also a square, with:

$$l = w = \sqrt{A}$$

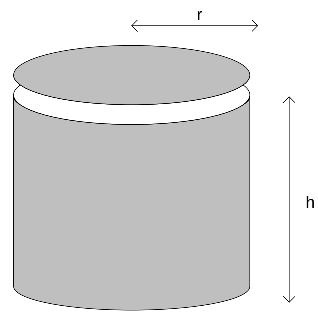
Which results in the amount of fencing, P, as shown.

$$P = 2\sqrt{A} + 2\sqrt{A}$$
$$P = 4\sqrt{A}$$

Example 3:

A cylindrical can, with a top lid, must contain $V\ cm^3$ of liquid. (A typical can of soda, for example, has $\mathrm{V} = 355\ cm^3$.) What dimensions, height and radius, will minimize the cost of the metal needed to construct the can?

Solution: We again start by drawing a picture.



We want to minimize the amount of material and therefore the objective function is represented by the surface area, which includes the top and bottom surfaces as well as the cylinder itself.

$$S(r,h) \ = \ \pi r^2 + \pi r^2 + 2\pi r h$$

Since the volume of liquid that the container must hold is fixed the constraint function is

$$V \ = \ \pi r^2 h$$

Solving for h and substituting into the objective function we have:

$$S(r) \ = \ 2\pi r^2 + 2\cancel{\pi r}\left(\frac{V}{\cancel{\pi} r^{\cancel{2}}}\right)$$

$$S(r) \ = \ 2\pi r^2 + \frac{2V}{r}$$

Next, we find the critical points by differentiating and setting to zero.

$$S'(r) \ = \ 4\pi r - \frac{2V}{r^2} = 0$$

$$r^3 = \frac{2V}{4\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

Using the second derivative test verifies that the critical point is a local minimum as desired.

$$S''(r)|_{r=\sqrt[3]{\frac{V}{2\pi}}} \ = \ 4\pi + \frac{4V}{\left(\sqrt[3]{\frac{V}{2\pi}}\right)^3} = \ 4\pi\left(1 + \frac{\cancel{V}2}{\cancel{V}}\right) = \ 12\pi > 0$$

Therefore, $r = \sqrt[3]{\frac{V}{2\pi}}$, is the radius of the container that minimizes the cost. To find the height we substitute this into the constraint equation. Note that we take a few extra algebra steps to show the insight that the height is exactly twice the ideal radius.

$$h = \frac{V}{\pi r^2}$$

$$h = \frac{V}{\pi \left(\frac{V}{2\pi}\right)^{2/3}}$$

$$h = \frac{V(2\pi)^{2/3}}{\pi V^{2/3}}$$

$$h = \frac{V^{1/3}(2)}{(2\pi)^{1/3}}$$

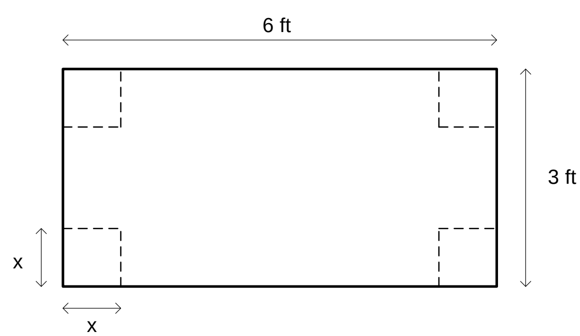
$$h = 2\left(\sqrt[3]{\frac{V}{2\pi}}\right) = 2r$$

Using the typical volume in a soda can, $\mathrm{V} = 355\ cm^3$, we find numerical values as follows:

$$r = \sqrt[3]{\frac{355}{2\pi}} \cong 3.84\ cm \qquad\qquad h = 2r \cong 7.67\ cm$$

Example 4:

A cardboard rectangle measuring $3x6$ ft shown below is to be cut equally at the corners and folded into an open top box. Find the side length of the corners so that the box attains maximum volume.



The objective function is the volume of the box to be maximized. Once the corners are folded up, the length and width of the box become, $6 - 2x$ and $3 - 2x$, respectively, and the height of the box is x.

$$V(x) = (6 - 2x)(3 - 2x)(x)$$

Since the objective function is already a function of one variable, we can immediately find the critical points. Of course, to differentiate it will be easier to expand the equation first.

$$\begin{aligned} V(x) &= (6-2x)(3-2x)x \\ &= (6-2x)(3x-2x^2) \\ &= 18x - 12x^2 - 6x^2 + 4x^3 \\ &= 4x^3 - 18x^2 + 18x \end{aligned}$$

$$V'(x) = 12x^2 - 36x + 18 = 0$$

The zeros are found using the quadratic formula as

$$x_{1,2}\frac{3 \pm \sqrt{3}}{2} \cong (2.366, 0.634)$$

Using the second derivative test we have

$$V''(2.366) = 24 \cdot 2.366 - 36 > 0$$
Local minimum

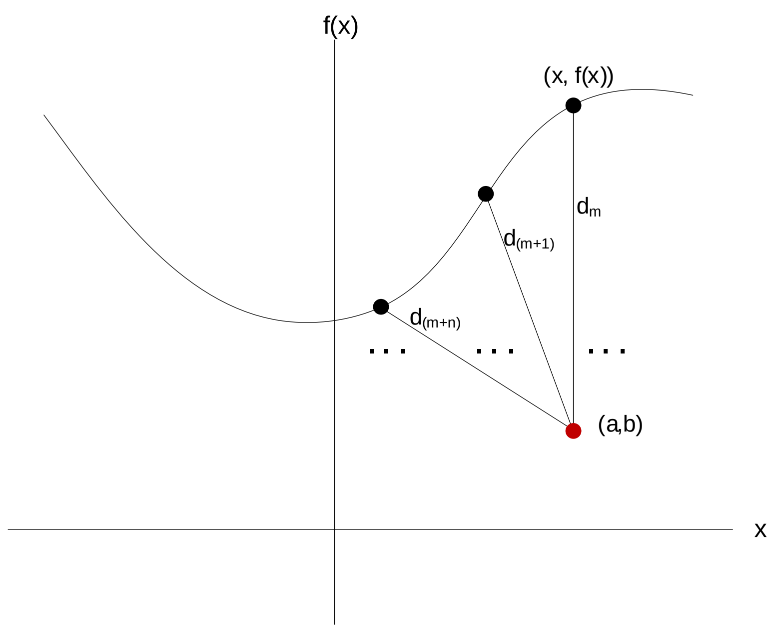
$$V''(\,0.634) = 24 \cdot 0.634 - 36 < 0$$
Local maximum

Therefore, to maximize the volume of the box we should cut and fold square corners with side length of 0.634 ft.

Example 5:

Find the point on the graph of $y = \sqrt{x^2+4}$ that is closest to the point $(2,0)$. What is the distance between these two points?

Solution: For illustrative purposes we draw a generic function and a generic point in a 2D plane below.



The question asks us to measure the distance from all points on the graph of the function, a generic point being represented as $(x, f(x))$, to some fixed point, (a, b). Then we are to locate the point on the graph that has the smallest distance. The distance between the two points is given by the Pythagorean Theorem.

$$D = \sqrt{(x-a)^2 + (f(x)-b)^2}$$

Note, the x value that results in minimizing D will also minimize $D^2 = \mathcal{D}$, therefore we can remove the square root and minimize the following function.

$$\mathcal{D}(x) = (x-a)^2 + (f(x)-b)^2$$

Using the function and the fixed point from above, the objective function in this example is

$$\begin{aligned}\mathcal{D}(x) &= (x-2)^2 + \left(\sqrt{x^2+4} - 0\right)^2 \\ &= x^2 - 4x + 4 + x^2 + 4 \\ &= 2x^2 - 4x + 8\end{aligned}$$

Minimizing we find the point on the graph as follows.

$$\begin{aligned}\mathcal{D}'(x) = 4x - 4 &= 0 \\ x &= 1\end{aligned} \qquad \therefore \qquad y = \sqrt{1^2+4} = \sqrt{5}$$

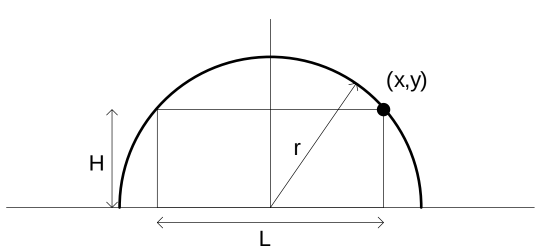
Therefore, the point on the graph that is closest to $(2,0)$ is $(1, \sqrt{5})$, and the distance between these two points is

$$\begin{aligned}D_{min} &= \sqrt{(2-1)^2 + \left(0-\sqrt{5}\right)^2} \\ &= \sqrt{6}\end{aligned}$$

Example 6:

Find the rectangle of maximum area that can be inscribed inside a semicircle of radius, r.

Solution: Draw a picture:



The semicircle, (top half of a circle of radius r), can be represented as follows:

$$y = \sqrt{r^2 - x^2}$$

Maximizing the entire rectangle is the same as maximizing half of the rectangle, whose area is given by:

$$A(x, y) = xy$$

This is the objective function, which is a function of both x and y, therefore we must first substitute for y from the constraint equation given above.

$$A(x) = x\left(\sqrt{r^2 - x^2}\right)$$
$$= \sqrt{x^2r^2 - x^4}$$

We can now maximize this function as we have done before.

$$A'(x) = \frac{\cancel{2}r^2\cancel{x} - \cancel{4}2x^{\cancel{3}2}}{\cancel{2x}\sqrt{x^2r^2 - x^4}} = 0$$
$$\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}} = 0$$
$$x^2 = \frac{r^2}{2}$$
$$x = \pm\frac{r}{\sqrt{2}}$$

Note that these two values of x will result in the same rectangles, therefore we use the positive value and find the y coordinate using the constraint equation.

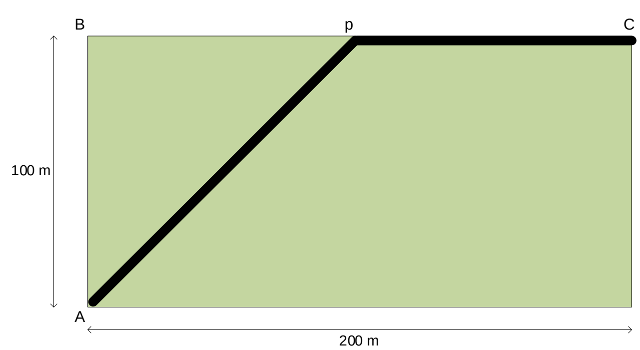
$$y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}} = \frac{r}{\sqrt{2}}$$

Finally, the dimensions of the maximum area rectangle are:

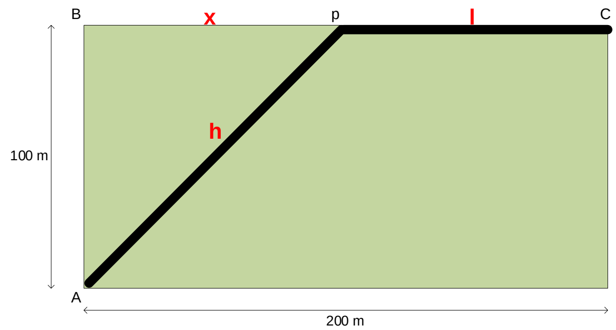
$$L = 2x = 2\frac{r}{\sqrt{2}} = \sqrt{2}r \qquad H = y = \frac{r}{\sqrt{2}} = \frac{\sqrt{2}r}{2} = \frac{L}{2}$$

Example 7:

A rectangular plot has a size of $100\ x\ 200\ m$. Pipe is to be laid from point A to a point P on side BC and from there to C. Going through the plot, i.e., from A to P, the pipe needs to go underground at a cost of $\$80/m$. However, the pipe can stay above ground from P to C at a cost of $\$45/m$. What is the most economical way to lay the pipe? What is the total cost?



Solution: Start by redrawing the figure with added notation so that we can write a general equation that describes the cost to lay the total length of pipe.



The length of pipe to be laid through the plot, h, and along the side, l, are given below

$$h = \sqrt{100^2 + x^2} \qquad\qquad l = 200 - x$$

The cost to lay the pipe, which is what we would like to minimize, is then given as:

$$C(x) = 80h + 45l = 80\left(\sqrt{100^2 + x^2}\right) + 45(200 - x)$$

The interval over which we minimize is $[0,200]$.

Differentiating the objective function and setting to zero we find the following.

$$C'(x) = \frac{80(\cancel{2}x)}{\cancel{2}\sqrt{100^2 + x^2}} + 45(-1) = 0$$

$$80x = 45\sqrt{100^2 + x^2}$$

$$80^2x^2 = 45^2(100^2 + x^2)$$

$$80^2x^2 - 45^2x^2 = 45^2 100^2$$

$$x = \pm\sqrt{\frac{45^2 100^2}{(80^2 - 45^2)}}$$

And since x cannot be negative we have a single critical point at $x \cong 68$. To be sure this is the minimum value we evaluate the cost at the endpoints of the interval and compare to $x = 68$.

$$C(0) = 80\left(\sqrt{100^2 + 0}\right) + 45(200 - 0) = \$17{,}000$$

$$C(200) = 80\left(\sqrt{100^2 + 200^2}\right) + 45(200 - 200) = \$17{,}888.54$$

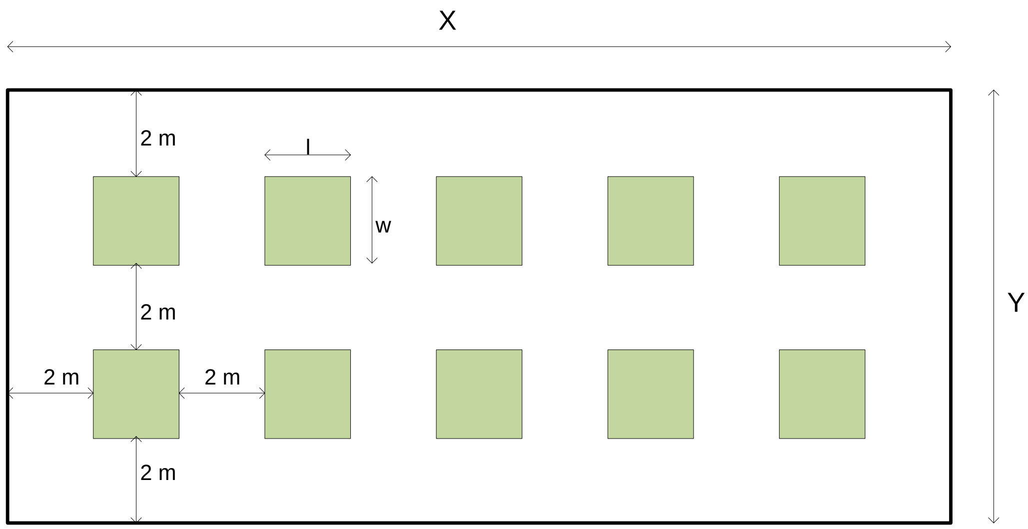
$$C(200) = \$17{,}888.54$$

$$C(68) = 80\left(\sqrt{100^2 + 68^2}\right) + 45(200 - 68) = \$\mathbf{15{,}614.38}$$

Therefore, the minimum cost to lay the pipe is \$15,614.38, which occurs when the point, P, is 68 meters from point B.

Example 8:

A garden of rectangular shape must be designed in the form consisting of 10 identical rectangles of planting areas arranged in two rows and five columns. The width of the sidewalk is 2 meters so that tourists can walk through. The total area of the garden is to be 2500 square meters. How should the length, X, and width, Y, of the garden be designed so that it has the largest planting area?



Solution:

We would like to maximize the planting area, which is given by:

$$A(l, w) = 10lw$$

The constraint function is the area of the entire garden area.

$$2500 = XY$$

Furthermore, X and Y are related to l and w as follows:

$$X = 12 + 5l \qquad\qquad Y = 6 + 2w$$

$$l = \frac{X - 12}{5} \qquad\qquad w = \frac{Y - 6}{2}$$

The objective function can then be rewritten as

$$A(X, Y) = 10\left(\frac{X - 12}{5}\right)\left(\frac{Y - 6}{2}\right)$$

$$A(X, Y) = (X - 12)(Y - 6)$$

Finally, we use the constraint function to substitute for Y.

$$\begin{aligned} A(X) &= (X-12)\left(\frac{2500}{X}-6\right) \\ &= 2500 - 6X - \frac{30000}{X} + 72 \\ &= 2572 - 6X - \frac{30000}{X} \end{aligned}$$

We can now optimize the single variable objective function

$$A'(X) = -6 + \frac{30000}{X^2} = 0$$

$$X^2 = \frac{30000}{6}$$

$$X = \sqrt{5000}$$

Where we ignored the negative solution.

The garden dimensions that will maximize the planting area is then given by:

$$X = \sqrt{5000}$$
$$X \cong 70.7\ m$$

$$Y = \frac{2500}{\sqrt{5000}}$$
$$Y \cong 35.4\ m$$

***Final Summary for Derivative Applications* – Applied Optimization**

Applied Optimization Problem Solving
We can identify 5 main steps to help with solving applied optimization problems. 1. Understand what the problem is asking - often by drawing a diagram. 2. Write an expression for the objective function, i.e. the function that requires optimization. 3. If the objective function is not of a single variable, look for additional function(s) that relates the variables, i.e. constraint function(s). 4. Rewrite the objective function as a function of a single variable if required. 5. Identify the interval of optimization if required and optimize the objective function.

By: ferrantetutoring