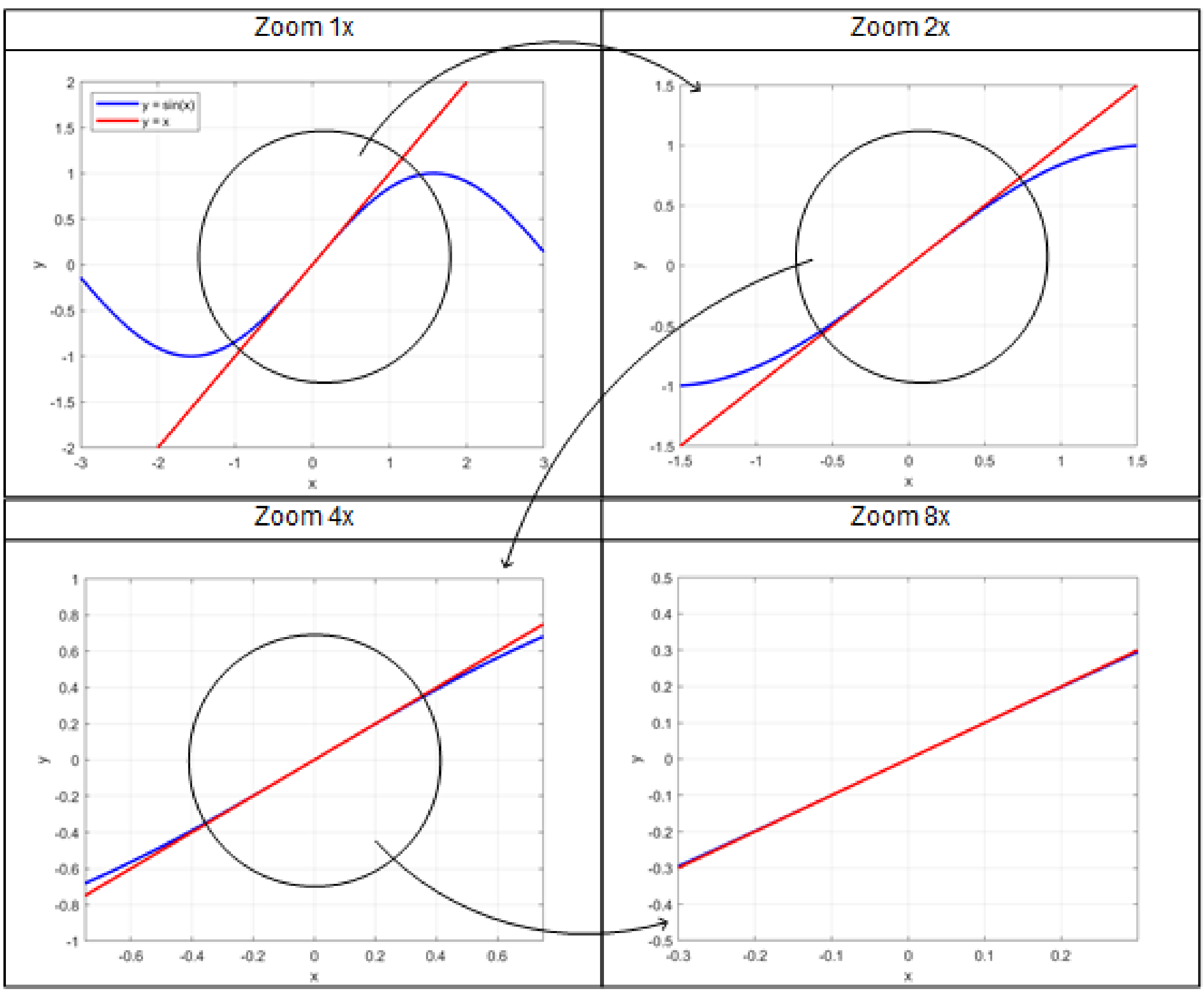
Derivative Applications – Linear Approximation

Linear approximation is a technique used to approximate a nonlinear function *near* a point, e.g. $x = a$, using a linear equation. Finding the proper linear function requires use of the derivative, however, let's begin with an informal look at the basic idea. To do this we choose $y = sin(x)$ as our nonlinear function and zoom in on the graph centered on the point, $x = 0$. The figure below shows four different views of the sine function along with the linear function, $y = x$. The graphs are progressively magnified at $1x$, $2x$, $4x$, and $8x$, and show that the two functions become progressively indistinguishable from each other as the magnification increases. Therefore, if we are working with the function, $y = sin(x)$, but are only interested in values near the point $x = 0$, we can instead approximate this function using a simple linear function, $y = x$. As a matter of fact, it turns out that under high magnification, *any differentiable function* looks exactly like a line! This phenomenon is extremely important to engineers and scientists because it can be used to greatly simplify numerical computations, as well as help with theoretical developments. Of course, proper understanding and use of this technique is critical to avoid larger numerical errors than desired or theoretical developments that do not align well enough with experiments. With that let's now move to a formal treatment of the topic of linear approximation.



Linear Approximation:

As we have discovered above, any differentiable function can be approximated near a point using a linear function. The question now becomes: "*how do we find this linear function?*". To answer this question, we first recall that the equation of a line that passes through a point, (x_p, y_p), and has a slope, m, is given by the point slope formula.

$$\begin{aligned} y - y_p &= m(x - x_p) \\ y &= m(x - x_p) + y_p \end{aligned}$$

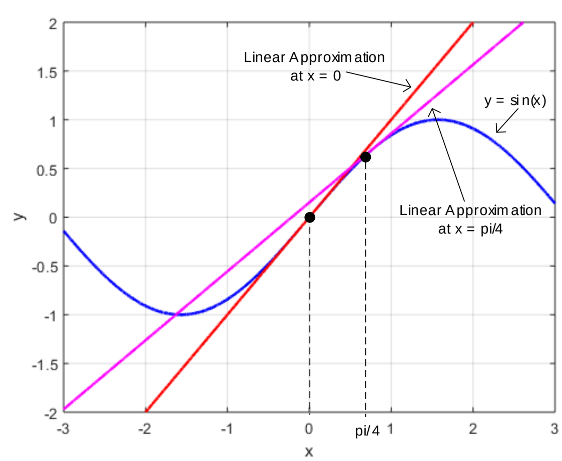
In the context of finding a linear equation to approximate a nonlinear function, $f(x)$, near the point $x = a$, our line must pass through the point, $(a, f(a))$, and should have a slope equal to the derivative of the function evaluated at the given point, $f'(a)$. Using this information and the point slope formula from above we can formally state the following.

Linear Approximation
The linear approximation, $L(x)$, to a differentiable function, $f(x)$, near the point $x = a$ is given by the tangent line to $f(x)$ at the given point. $$L(x) = f'(a)(x - a) + f(a)$$

As an example, let's return to the sine function and find linear approximations near the points $x = 0$ and $x = \pi/4$. We derive both below and show the graphs for illustration.

$$L(x) = sin'(0)\,(x - 0) + sin(0)$$
$$L(x) = cos(0)\,(x) + 0$$
$$L(x) = x$$

$$L(x) = sin'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)$$
$$L(x) = cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}0$$
$$L(x) = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}\pi}{8} + \frac{4\sqrt{2}}{8}$$
$$L(x) = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}(4 - \pi)}{8}$$



Linear Approximation of Δf:

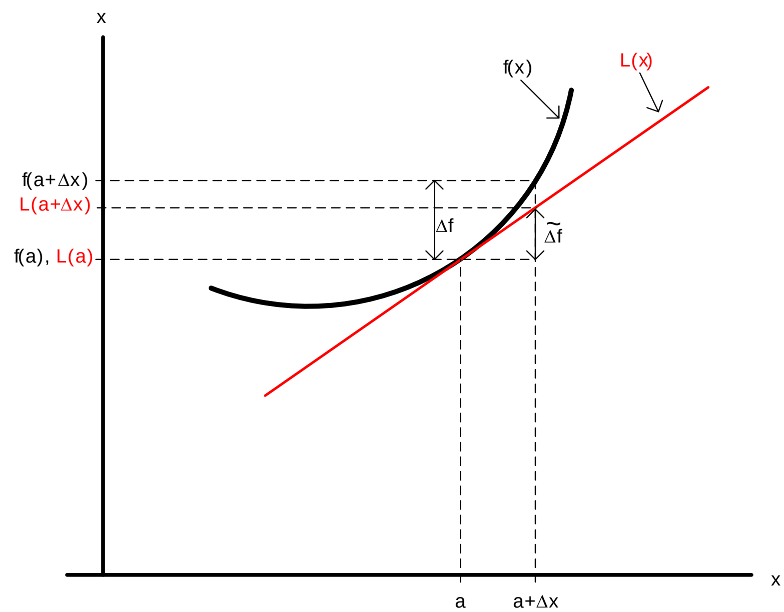
There are times where we would like to determine how a small change in the input changes the output of a function. For example, we may want to know how a small change in the unit price of an item will change the total revenue. Of course, to find the exact change in the output of a function when the input changes from $x = a$ to $x = a + \Delta x$ we compute the following:

$$\Delta f \;=\; f(a + \Delta x) - f(a)$$

However, if Δx is small it suffices to use the linear approximation of $f(x)$, i.e., $L(x)$, to estimate Δf, which we can call $\widetilde{\Delta f}$. We do this below.

$$\begin{aligned}
\Delta f \cong \widetilde{\Delta f} &= L(a + \Delta x) - L(a) \\
\widetilde{\Delta f} &= [f'(a)(\cancel{a} + \Delta x \cancel{- a}) + f(a)] - [f'(a)(a - a) + f(a)] \\
\widetilde{\Delta f} &= f'(a)(\Delta x) + \cancel{f(a)} - f'(a)(0) \cancel{- f(a)} \\
\widetilde{\Delta f} &= f'(a)\Delta x
\end{aligned}$$

Therefore, the quantity $f'(a)\Delta x$ can be used to make a linear approximation of Δf. The figure below illustrates this concept graphically.



We can also arrive at this formula by starting with the following definition of the derivative.

$$f'(a) = \lim_{\Delta x \to 0} \left\{ \frac{f(a + \Delta x) - f(a)}{\Delta x} \right\} = \lim_{\Delta x \to 0} \left\{ \frac{\Delta f}{\Delta x} \right\}$$

If Δx is sufficiently small, we may remove the limit and say

$$f'(a) \cong \frac{\Delta f}{\Delta x}$$

Solving for Δf we have the same result we derive above.

$$\Delta f \;\cong\; f'(a)\Delta x$$

Linear Approximation of Δf
If $f(x)$ is differentiable at $x = a$ and Δx is small, then $$\Delta f \cong \widetilde{\Delta f} = f'(a)\Delta x$$ Where, $$\Delta f = f(a + \Delta x) - f(a)$$ and $$\widetilde{\Delta f} = L(a + \Delta x) - L(a)$$

Let's do some examples to practice these concepts.

Example 1:

Find the linear approximation of the following functions at $x = a$.

a. $f(x) = \frac{1}{\sqrt{1+x}}$, $a = 0$ *b.* $f(x) = x^4$, $a = 1$ *c.* $f(x) = sin^2(x)$, $a = \frac{\pi}{4}$

a.

$$L(x) = f'(0)(x - 0) + f(0)$$
$$L(x) = \left[\left(-\frac{1}{2}(1 + x)^{-3/2}\right)\Big|_{x=0}\right](x) + 1$$
$$L(x) = -\frac{1}{2}x + 1$$

b.

$$L(x) = f'(1)(x - 1) + f(1)$$
$$L(x) = [(4x^3)|_{x=1}](x - 1) + 1$$
$$L(x) = 4(x - 1) + 1$$
$$L(x) = 4x - 3$$

c.

$$L(x) = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)$$
$$L(x) = \left[(2\,sin(x)\,cos(x))|_{x=\frac{\pi}{4}}\right]\left(x - \frac{\pi}{4}\right) + \frac{1}{2}$$
$$L(x) = 1\left(x - \frac{\pi}{4}\right) + \frac{1}{2}$$
$$L(x) = x + \left(\frac{2 - \pi}{4}\right)$$

Example 2:

Estimate Δf when the input changes from 5 to 4.6, for the function $f(x) = 2x^2 - x$. Then find the true value of Δf and compute the error.

Using the formula from above, $\widetilde{\Delta f} = f'(a)\Delta x$, we let $a = 5$ and $\Delta x = 4.6 - 5 = -0.4$

$$\begin{aligned}\widetilde{\Delta f} &= f'(a)\Delta x \\ \widetilde{\Delta f} &= (4x-1)|_{x=5}(-0.4) \\ \widetilde{\Delta f} &= 19(-0.4) \\ \widetilde{\Delta f} &= -7.6\end{aligned}$$

The true value is computed as follows:

$$\begin{aligned}\Delta f &= f(4.6) - f(5) \\ \Delta f &= (2(4.6)^2 - 4.6) - (2(5)^2 - 5) \\ \Delta f &= (37.72) - (45) \\ \Delta f &= -7.28\end{aligned}$$

The error in the estimate is then:

$$\begin{aligned}E &= \Delta f - \widetilde{\Delta f} \\ E &= (-7.28) - (-7.6) \\ E &= 0.32\end{aligned}$$

Example 3:

Approximate using linearization the following values:

a. $\frac{1}{\sqrt{17}}$ *b.* $ln(1.07)$

A general procedure for approximating these values is stated below.

1. Define a function that matches the value to be evaluated, e.g. $\frac{1}{\sqrt{x}}$ to evaluate $\frac{1}{\sqrt{17}}$
2. Choose the point for linearization close to the value to be evaluated, but that can be easily computed without a calculator, e.g. use $a = 16$ for the function $\frac{1}{\sqrt{x}}$ above.
3. Write the linear approximation for the function defined in step 1.
$$L(x) = f'(a)(x-a) + f(a)$$
4. Evaluate the desired x value, i.e., $x = a + \Delta x$

$$L(a+\Delta x) = f'(a)\big((\cancel{a} + \Delta x) - \cancel{a}\big) + f(a)$$

$$L(a+\Delta x) = f'(a)\Delta x + f(a)$$

a. With the explanation from above we can estimate $\frac{1}{\sqrt{17}}$ as $L(17)$, using $f(x) = \frac{1}{\sqrt{x}}$, $a = 16$, and hence $\Delta x = 1$

$$L(a + \Delta x) = f'(a)\Delta x + f(a)$$
$$L(17) = f'(16)(1) + f(16)$$
$$L(17) = \left(-\frac{1}{2(\sqrt{x})^3}\right)\Bigg|_{x=16} (1) + \frac{1}{\sqrt{16}}$$
$$L(17) = \left(-\frac{1}{128}\right) + \frac{32}{128}$$
$$L(17) = \frac{31}{128} \cong \frac{1}{\sqrt{17}}$$

b. In this case we use $f(x) = ln(x)$, $a = 1$, and hence $\Delta x = 0.07$

$$L(a + \Delta x) = f'(a)\Delta x + f(a)$$
$$L(1.07) = f'(1)(0.07) + f(1)$$
$$L(1.07) = \left(\frac{1}{x}\right)\Bigg|_{x=1} (0.07) + ln(1)$$
$$L(1.07) = 1(0.07) + 0$$
$$L(1.07) = 0.07 \cong ln(1.07)$$

Example 4:

Approximate the value of $\sqrt{26} - \sqrt{25}$.

Letting $f(x) = \sqrt{x}$, this question is equivalent to approximating Δf when $\Delta x = 1$, and $a = 25$.

$$\widetilde{\Delta f} = f'(a)\Delta x$$
$$\widetilde{\Delta f} = \left(\frac{1}{2\sqrt{x}}\right)\Bigg|_{x=25} (1)$$
$$\widetilde{\Delta f} = \frac{1}{10} = 0.1$$

The true value using a calculator evaluate to 0.09902

Example 5:

A stone is tossed vertically in the air reaches a maximum height, h, given by

$$h(v) = \frac{1}{1960}v^2\ cm$$

Where, v is the initial velocity in cm/s.

a. Approximate Δh if the initial velocity, v, is $700\ cm/s$ and $\Delta v = 1\, cm/s$
b. Approximate Δh if the initial velocity, v, is $1000\ cm/s$ and $\Delta v = 1\, cm/s$

$\mathbf{700\ cm/s}$ **initial velocity**	$\mathbf{1000\ cm/s}$ **initial velocity**
$\Delta h \cong h'(700)\Delta v$ $\Delta h \cong \left(\frac{2}{1960}700\right)(1)$ $\Delta h \cong 0.714\ cm$	$\Delta h \cong h'(1000)\Delta v$ $\Delta h \cong \left(\frac{2}{1960}1000\right)(1)$ $\Delta h \cong 1.02\ cm$

Note the effect is bigger at higher initial velocities.

Example 6:

The side of a square carpet is measured to be $s = 6\ m$. What is the error in the area if the error in the measurement of the side is $2\ cm$?

The area of the carpet is given as

$$A(s) = s^2$$

The measurement error of $2\ cm$ results in an approximate error in the area as follows:

$$\Delta A \cong A'(6)(\Delta s)$$
$$\Delta A \cong (2 \cdot 6)(0.02)$$
$$\Delta A \cong 0.24\ m^2$$

Final Summary for Derivative Applications – Linear Approximation

Linear Approximation
The linear approximation, $L(x)$, to a differentiable function, $f(x)$, near the point $x = a$ is given by the tangent line to $f(x)$ at the given point $$L(x) = f'(a)(x - a) + f(a)$$

Linear Approximation of Δf
If $f(x)$ is differentiable at $x = a$ and Δx is small, then $$\Delta f \cong \widetilde{\Delta f} = f'(a)\Delta x$$ Where, $$\Delta f = f(a + \Delta x) - f(a)$$ and $$\widetilde{\Delta f} = L(a + \Delta x) - L(a)$$

Approximating Values
Using the value $\frac{1}{\sqrt{17}}$ as an example, the procedure is described as follows. 1. Define a function that matches the value to be evaluated, e.g., $\frac{1}{\sqrt{x}}$ for this example. 2. Choose the point for linearization close to the value to be evaluated, but that can be easily computed without a calculator, e.g., use $a = 16$ for the function $\frac{1}{\sqrt{x}}$ above. 3. Write the linear approximation for the function defined in step 1. $$L(x) = f'(a)(x - a) + f(a)$$ *4.* Evaluate the desired x value, i.e., $x = a + \Delta x$ $$L(a + \Delta x) = f'(a)\big((\cancel{a} + \Delta x) - \cancel{a}\big) + f(a)$$ $$L(a + \Delta x) = f'(a)\Delta x + f(a)$$ Using the above example: $$L(17) = f'(16)(1) + f(16)$$ $$L(17) = \left(-\frac{1}{2\left(\sqrt{x}\right)^3}\right)\Bigg\rvert_{x=16}(1) + \frac{1}{\sqrt{16}}$$ $$L(17) = \left(-\frac{1}{128}\right) + \frac{32}{128}$$ $$L(17) = \frac{31}{128} \cong \frac{1}{\sqrt{17}}$$

By: ferrantetutoring