Derivative Applications – L'Hôpital's Rule

L'Hôpital's rule uses derivatives to help us evaluate limits, which otherwise may be difficult to evaluate. The rule applies to limits of quotients, i.e. $f(x)/g(x)$. Generally speaking, if we attempt to evaluate the limit of this quotient at $x = a$ and an indeterminate form of type $0/0$ or $\pm\infty/\pm\infty$ results, then we can replace $f(x)/g(x)$ with $f'(x)/g'(x)$ and attempt to evaluate again. The rule is formally stated below.

L'Hôpital's Rule

If $\lim_{x\to a}\{f(x)\} = \lim_{x\to a}\{g(x)\} = 0$ or $\pm\infty$, and the following hold:

1. $f(x)$ and $g(x)$ are both differentiable functions in an open interval containing a, except possibly at $x = a$.
2. $g'(x) \neq 0$, except possibly at $x = a$.

Then,

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to a}\left\{\frac{f'(x)}{g'(x)}\right\}$$

So long as the limit exists or is $\pm\infty$.

Note 1: The goal in differentiating the numerator and denominator is to make the quotient simpler to evaluate.
Note 2: This procedure can be applied repeatedly until the limit can be evaluated, assuming it exists.
Note 3: The rule is also valid for one-sided limits.
Note 4: The rule also applies for limits as $x \to \infty$ or $x \to -\infty$.

Let's start with a few basic examples to get a working understanding of the rule.

Example 1: Indeterminate form: $0/0$

Evaluate: $\lim_{x\to 3}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to 3}\left\{\frac{x^2+2x-15}{x^2-9x+18}\right\}$

We start by trying to evaluate with substitution:

$$\frac{3^2 + 2\cdot 3 - 15}{3^2 - 9\cdot 3 + 18} = \frac{0}{0}$$

Then since $f(x)$ and $g(x)$ are both differentiable, and $g'(x) = 2x - 9 \neq 0$ near $x = 3$, we can apply L'Hôpital's rule and attempt to evaluate.

$$\lim_{x\to 3}\left\{\frac{x^2+2x-15}{x^2-9x+18}\right\} = \lim_{x\to 3}\left\{\frac{\frac{d}{dx}(x^2+2x-15)}{\frac{d}{dx}(x^2-9x+18)}\right\}$$

$$= \lim_{x\to 3}\left\{\frac{2x+2}{2x-9}\right\} = \frac{2\cdot 3+2}{2\cdot 3-9} = -\frac{8}{3}$$

Example 2: Indeterminate form: ∞/∞

Evaluate: $\lim\limits_{x\to\infty}\left\{\frac{f(x)}{g(x)}\right\} = \lim\limits_{x\to\infty}\left\{\frac{6x+4}{2x-8}\right\}$

We start by noting that $\lim\limits_{x\to\infty}\{f(x)\} = \lim\limits_{x\to\infty}\{g(x)\} = \infty$. Next, since $f(x)$ and $g(x)$ are both differentiable, and $g'(x) = 2 \neq 0$, we can apply L'Hôpital's rule and attempt to evaluate.

$$\lim_{x\to\infty}\left\{\frac{6x+4}{2x-8}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}(6x+4)}{\frac{d}{dx}(2x-8)}\right\}$$
$$= \lim_{x\to\infty}\left\{\frac{6}{2}\right\} = 3$$

Example 3: Indeterminate form: $-\infty/-\infty$

Evaluate: $\lim\limits_{x\to\infty}\left\{\frac{f(x)}{g(x)}\right\} = \lim\limits_{x\to\infty}\left\{\frac{-4x+3}{-8x-2}\right\}$

We start by noting that $\lim\limits_{x\to\infty}\{f(x)\} = \lim\limits_{x\to\infty}\{g(x)\} = -\infty$. Next since $f(x)$ and $g(x)$ are both differentiable, and $g'(x) = -8 \neq 0$, we can apply L'Hôpital's rule and attempt to evaluate.

$$\lim_{x\to\infty}\left\{\frac{-4x+3}{-8x-2}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}(-4x+3)}{\frac{d}{dx}(-8x-2)}\right\}$$
$$= \lim_{x\to\infty}\left\{\frac{-4}{-8}\right\} = \frac{1}{2}$$

Example 4: Repeated use of L'Hôpital's Rule

Evaluate: $\lim\limits_{x\to 0}\left\{\frac{f(x)}{g(x)}\right\} = \lim\limits_{x\to 0}\left\{\frac{2\,sin(x)-sin(2x)}{x-sin(x)}\right\}$

We first confirm an indeterminate form:

$$\frac{2\,sin(0) - sin(0)}{0 - sin(0)} = \frac{0}{0}$$

However, when we apply L'Hôpital's rule and attempt to evaluate we still end up with an indeterminate form.

$$\lim_{x\to 0}\left\{\frac{\frac{d}{dx}(2\,sin(x) - sin(2x))}{\frac{d}{dx}(x - sin(x))}\right\} = \lim_{x\to 0}\left\{\frac{2\,cos(x) - 2\,cos(2x)}{1 - cos(x)}\right\}$$

$$\frac{2\,cos(0) - 2\,cos(0)}{1 - cos(0)} = \frac{2-2}{1-1} = \frac{0}{0}$$

Let's apply L'Hôpital's rule a second time and attempt to evaluate again. Unfortunately, we still end up with an indeterminate form.

$$\lim_{x \to 0} \left\{ \frac{\frac{d}{dx}(2\,cos(x) - 2\,cos(2x))}{\frac{d}{dx}(1 - cos(x))} \right\} = \lim_{x \to 0} \left\{ \frac{-2\,sin(x) + 4\,sin(2x)}{sin(x)} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{-2\,sin(0) + 4\,sin(0)}{sin(0)} \right\} = \frac{0}{0}$$

Therefore, we apply L'Hôpital's rule a third time and discover that we can now evaluate by direct substitution.

$$\lim_{x \to 0} \left\{ \frac{\frac{d}{dx}(-2\,sin(x) + 4\,sin(2x))}{\frac{d}{dx}(sin(x))} \right\} = \lim_{x \to 0} \left\{ \frac{-2\,cos(x) + 8\,cos(2x)}{cos(x)} \right\}$$

$$= \lim_{x \to 0} \left\{ \frac{-2\,cos(0) + 8\,cos(0)}{cos(0)} \right\} = \frac{-2 + 8}{1} = 6$$

As we have seen L'Hôpital's rule can be applied for the indeterminate forms, $0/0$, and $\pm\infty/\pm\infty$. What about other indeterminate forms such as: $0 \cdot \infty, \infty - \infty, 1^{\infty}, 0^{0}$ or ∞^{0}? Can we somehow apply L'Hôpital's rule in these cases? The quick answer is no. However, there are cases where we can algebraically transform the limit into the required forms, $0/0$, or $\pm\infty/\pm\infty$. Let's take a look at some of these cases.

Example 5: Indeterminate form: $0 \cdot \infty$

Evaluate: $\lim_{x \to 0^+} \{x\,ln(x)\}$

Evaluation by substitution yields the indeterminate form $0 \cdot \infty$, however we can algebraically rearrange the limit as follows:

$$\lim_{x \to 0^+} \{x\,ln(x)\} = \lim_{x \to 0^+} \left\{ \frac{ln(x)}{1/x} \right\}$$

Evaluating now results in the indeterminate form ∞/∞, for which we can apply L'Hôpital's rule.

$$\lim_{x \to 0^+} \{x\,ln(x)\} = \lim_{x \to 0^+} \left\{ \frac{ln(x)}{1/x} \right\}$$

$$= \lim_{x \to 0^+} \left\{ \frac{\frac{d}{dx}(ln(x))}{\frac{d}{dx}(1/x)} \right\}$$

$$= \lim_{x \to 0^+} \left\{ \frac{(1/x)}{-(1/x^2)} \right\}$$

$$= \lim_{x \to 0^+} \{-x\} = 0$$

Example 6: Indeterminate form: $\infty - \infty$

Evaluate: $\lim_{x \to 0} \left\{ \frac{1}{sin(x)} - \frac{1}{x} \right\}$

Again, we start by algebraically rearranging the expression to a quotient.

$$\lim_{x \to 0} \left\{ \frac{1}{sin(x)} - \frac{1}{x} \right\} = \lim_{x \to 0} \left\{ \frac{x - sin(x)}{x\, sin(x)} \right\}$$

Evaluating the limit in this form results in the indeterminate form $0/0$, for which we can apply L'Hôpital's rule.

$$\begin{aligned} \lim_{x \to 0} \left\{ \frac{1}{sin(x)} - \frac{1}{x} \right\} &= \lim_{x \to 0} \left\{ \frac{x - sin(x)}{x\, sin(x)} \right\} \\ &= \lim_{x \to 0} \left\{ \frac{\frac{d}{dx}(x - sin(x))}{\frac{d}{dx}(x\, sin(x))} \right\} \\ &= \lim_{x \to 0} \left\{ \frac{(1 - cos(x))}{(x\, cos(x) + sin(x))} \right\} \end{aligned}$$

Substituting results again in an indeterminate form, therefore we apply L'Hôpital's rule again.

$$\begin{aligned} \lim_{x \to 0} \left\{ \frac{(1 - cos(x))}{(x\, cos(x) + sin(x))} \right\} &= \lim_{x \to 0} \left\{ \frac{\frac{d}{dx}(1 - cos(x))}{\frac{d}{dx}(x\, cos(x) + sin(x))} \right\} \\ &= \lim_{x \to 0} \left\{ \frac{sin(x)}{(-x\, sin(x) + cos(x) + cos(x))} \right\} \\ &= \lim_{x \to 0} \left\{ \frac{sin(x)}{-x\, sin(x) + 2\, cos(x)} \right\} \\ &= \frac{sin(0)}{-0 \cdot sin(0) + 2\, cos(0)} = \frac{0}{2} = 0 \end{aligned}$$

The next three indeterminate forms, $1^\infty, 0^0$ and ∞^0, arise from functions of the form $f(x)^{g(x)}$. We can sometimes evaluate these cases using the following technique.

1. Use the fact that $x = e^{ln(x)}$, to rewrite the original limit as follows:

$$\lim_{x\to a}\{f(x)^{g(x)}\} = \lim_{x\to a}\left\{e^{\ln(f(x)^{g(x)})}\right\} = \lim_{x\to a}\left\{e^{\left(\frac{\ln(f(x))}{1/g(x)}\right)}\right\}$$

2. Move the limit "inside", which is valid since e^x is a continuous function.

$$\lim_{x\to a}\left\{e^{\left(\frac{ln(f(x))}{1/g(x)}\right)}\right\} = e^{\lim_{x\to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}}$$

3. Apply L'Hôpital's rule as needed for the quotient in the exponent. Assuming it evaluates to the value, L, we can write.

$$\lim_{x\to a}\{f(x)^{g(x)}\} = e^{\lim_{x\to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}} = e^L$$

Let's do an example to illustrate this technique.

Example 7: Indeterminate form: 0^0

Evaluate: $\lim_{x\to 0^+}\{x^x\}$

According to the explanation above we start by rewriting the base as $x = e^{ln(x)}$. The function can then be expressed as $x^x = e^{x\,ln(x)} = e^{\frac{ln(x)}{1/x}}$. We can then move the limit to the exponent and attempt to use L'Hôpital's rule on the quotient, which we know from example 5 evaluates to 0. The steps are explicitly shown below.

$$\lim_{x\to 0^+}\{x^x\} = \lim_{x\to 0^+}\left\{e^{ln(x^x)}\right\} = \lim_{x\to 0^+}\left\{e^{\left(\frac{ln(x)}{1/x}\right)}\right\} = e^{\lim_{x\to 0^+}\left\{\frac{ln(x)}{1/x}\right\}} = e^0 = 1$$

Before moving on with some additional examples, let's look at how L'Hôpital's rule can be used to help us compare growth rates of functions.

Comparing Growth of Functions

Certain computer algorithms are sometimes compared on the basis of how long the algorithm takes to complete a task. A classic example is the task of sorting a list with n entries. Let's say the time to complete the task for the first algorithm has an order of magnitude of n^2, while a second algorithm scales as $n\,ln(n)$. We would like to know which algorithm is faster when n is very large. Although n is a whole number we can analyze this problem by mapping n to a real number x, and compare the growth rates of $f(x) = x^2$ and $g(x) = x\,ln(x)$ as $x \to \infty$. In this case we say that $f(x)$ grows faster that $g(x)$ if

$$\lim_{x\to\infty}\left\{\frac{f(x)}{g(x)}\right\} = \infty \qquad \text{Or, equivalently} \qquad \lim_{x\to\infty}\left\{\frac{g(x)}{f(x)}\right\} = 0$$

As an example, let's compare the growth rates of these two functions to see which algorithm is faster.

Example 8: Growth Rate Comparison 1

Which function grows faster, $f(x) = x^2$ or $g(x) = x\,ln(x)$?

Using the quotient and applying L'Hôpital's rule we have

$$\lim_{x\to\infty}\left\{\frac{x^{\not{2}}}{\not{x}\,ln(x)}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(ln(x))}\right\} = \lim_{x\to\infty}\left\{\frac{1}{\frac{1}{x}}\right\} = \lim_{x\to\infty}\{x\} = \infty$$

Which says that $f(x)$ grows faster than $g(x)$, and therefore the second algorithm is faster, (will take less time to complete when n is large)

Let's look at another interesting example.

Example 9: Growth Rate Comparison 2

Compare the growth rates of an exponential function, $f(x) = b^x$, where $b > 1$, and a power function, $g(x) = x^a$, where a is any whole number greater than one.

We again form a quotient and attempt to evaluate using L'Hôpital's rule.

$$\lim_{x\to\infty}\left\{\frac{b^x}{x^a}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}(b^x)}{\frac{d}{dx}(x^a)}\right\} = \lim_{x\to\infty}\left\{\frac{ln(b)\,(b^x)}{a(x^{a-1})}\right\} = \left(\frac{ln(b)}{a}\right)\lim_{x\to\infty}\left\{\frac{b^x}{x^{(a-1)}}\right\}$$

For which we can apply L'Hôpital's rule again.

$$\left(\frac{ln(b)}{a}\right)\lim_{x\to\infty}\left\{\frac{\frac{d}{dx}(b^x)}{\frac{d}{dx}(x^{(a-1)})}\right\}=\left(\frac{ln(b)}{a}\right)\lim_{x\to\infty}\left\{\frac{ln(b)\,(b^x)}{(a-1)(x^{a-2})}\right\}=\left(\frac{ln^2(b)}{a(a-1)}\right)\lim_{x\to\infty}\left\{\frac{b^x}{x^{(a-2)}}\right\}$$

Note that since the derivate of an exponential function is itself an exponential the numerator will continue to evaluate to ∞. The denominator, on the other hand, is reduced by one order each time we take the derivative. After applying L'Hôpital's rule a times the result is as follows.

$$\lim_{x\to\infty}\left\{\frac{b^x}{x^a}\right\}=\left(\frac{ln^a(b)}{a(a-1)(a-2)\cdots(1)}\right)\lim_{x\to\infty}\{b^x\}=\infty$$

Therefore, we have the interesting results the growth rate of an exponential functions is *always* greater than a power function, regardless of the order (i.e. size of a).

We finish this section with a few more examples of applying L'Hôpital's rule.

Example 10:

Evaluate: $\lim_{x\to 0}\left\{\frac{sin(4x)}{x^2+3x+1}\right\}$

If we start by applying L'Hôpital's rule, we get the following

$$\lim_{x\to 0}\left\{\frac{sin(4x)}{x^2+3x+1}\right\}=\lim_{x\to 0}\left\{\frac{\frac{d}{dx}(sin(4x))}{\frac{d}{dx}(x^2+3x+1)}\right\}=\lim_{x\to 0}\left\{\frac{4\,cos(4x)}{2x+3}\right\}=\frac{4\,cos(0)}{0+3}=\frac{4}{3}$$

However, recall that in order to apply L'Hôpital's rule we must have an indeterminate form. Let's instead start by checking for an indeterminate form.

$$\lim_{x\to 0}\left\{\frac{sin(4x)}{x^2+3x+1}\right\}=\frac{sin(0)}{0^2+3\cdot 0+1}=\frac{0}{1}=0$$

Note that $\frac{0}{1}$ is not an indeterminate form and therefore L'Hôpital's rule cannot be used. The correct answer is 0.

Example 11:

Evaluate: $\lim_{x\to 0}\left\{\frac{sin(4x)}{sin(7x)}\right\}$

Evaluating via substitution results in the indeterminate form, $0/0$, therefore we apply L'Hôpital's rule.

$$\lim_{x\to 0}\left\{\frac{sin(4x)}{sin(7x)}\right\} = \lim_{x\to 0}\left\{\frac{\frac{d}{dx}(sin(4x))}{\frac{d}{dx}(sin(7x))}\right\} = \lim_{x\to 0}\left\{\frac{4(cos(4x))}{7(cos(7x))}\right\} = \frac{4(cos(0))}{7(cos(0))} = \frac{4}{7}$$

Example 12:

Evaluate: $\lim_{x\to 0^+}\{sin(x)\,ln(x)\}$

Evaluating via substitution results in the indeterminate form, $0 \cdot -\infty$, therefore we rearrange and attempt to use L'Hôpital's rule.

$$\lim_{x\to 0^+}\left\{\frac{ln(x)}{1/sin(x)}\right\} = \lim_{x\to 0^+}\left\{\frac{\frac{d}{dx}(ln(x))}{\frac{d}{dx}(csc(x))}\right\} = \lim_{x\to 0^+}\left\{\frac{1/x}{(-\,csc(x)\,cot(x))}\right\} = \lim_{x\to 0^+}\left\{-\frac{sin^2(x)}{x\,cos(x)}\right\}$$

This still has an indeterminate form, and we can try to apply L'Hôpital's rule again. However, we may also evaluate using the product law of limits as shown below.

$$\lim_{x\to 0^+}\left\{-\frac{sin^2(x)}{x\,cos(x)}\right\} = -\lim_{x\to 0^+}\left\{\frac{sin(x)}{x}\right\}\cdot \lim_{x\to 0^+}\{sin(x)\}\cdot \lim_{x\to 0^+}\left\{\frac{1}{cos(x)}\right\}$$

$$= -(1)\cdot(0)\cdot(1) = 0$$

Example 13:

Evaluate: $\lim_{x\to 0}\left\{(\cos(x))^{3/x^2}\right\}$

This limit has the indeterminate form, 1^∞, therefore attempt to apply the logarithm technique from above, where we let $(\cos(x))^{3/x^2} = \left(e^{\ln(\cos(x))}\right)^{3/x^2}$. Therefore, we have

$$\lim_{x\to 0}\left\{(\cos(x))^{3/x^2}\right\} = \lim_{x\to 0}\left\{\left(e^{\frac{3\ln(\cos(x))}{x^2}}\right)\right\} = e^{\lim_{x\to 0}\left(\frac{3\ln(\cos(x))}{x^2}\right)}$$

Now, taking the limit of the exponent we have

$$3\lim_{x\to 0}\left\{\frac{\ln(\cos(x))}{x^2}\right\} = 3\lim_{x\to 0}\left\{\frac{\frac{d}{dx}(\ln(\cos(x)))}{\frac{d}{dx}(x^2)}\right\}$$

$$= 3\lim_{x\to 0}\left\{\frac{\frac{-\sin(x)}{\cos(x)}}{2x}\right\}$$

$$= -3\lim_{x\to 0}\left\{\frac{\tan(x)}{2x}\right\}$$

$$= -3\lim_{x\to 0}\left\{\frac{\frac{d}{dx}(\tan(x))}{\frac{d}{dx}(2x)}\right\}$$

$$= -3\lim_{x\to 0}\left\{\frac{\sec^2(x)}{2}\right\}$$

$$= -3\left(\frac{\sec^2(0)}{2}\right) = -\frac{3}{2}$$

Finally, we find

$$\lim_{x\to 0}\left\{(\cos(x))^{3/x^2}\right\} = e^{-3/2}$$

Example 14:

Evaluate: $\lim_{x \to \pi/2} \{sec(x) - tan(x)\}$

We rearrange before applying L'Hôpital's rule.

$$\lim_{x\to\pi/2}\{sec(x) - tan(x)\} = \lim_{x\to\pi/2}\left\{\frac{1 - sin(x)}{cos(x)}\right\}$$

$$= \lim_{x\to\pi/2}\left\{\frac{\frac{d}{dx}(1 - sin(x))}{\frac{d}{dx}(cos(x))}\right\}$$

$$= \lim_{x\to\pi/2}\left\{\frac{-cos(x)}{-sin(x)}\right\} = \frac{cos(\pi/2)}{sin(\pi/2)} = \frac{0}{1} = 0$$

Example 15:

Prove: $\lim_{x\to\infty}\left\{\left(1 + \frac{1}{x}\right)^x\right\} = e$

Attempting to evaluate this limit results in 1^∞. Therefore, we proceed as we did in example 13.

$$\lim_{x\to\infty}\left\{x\, ln\left(1 + \frac{1}{x}\right)\right\} = \lim_{x\to\infty}\left\{\frac{ln\left(1 + \frac{1}{x}\right)}{1/x}\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}\left(ln\left(1 + \frac{1}{x}\right)\right)}{\frac{d}{dx}(1/x)}\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{\left(\frac{1}{1 + \frac{1}{x}}\right)\cancel{\frac{d}{dx}(1/x)}}{\cancel{\frac{d}{dx}(1/x)}}\right\}$$

$$= \lim_{x\to\infty}\left\{\left(\frac{1}{1 + \frac{1}{x}}\right)\right\} = \left(\frac{1}{1 + \frac{1}{\infty}}\right) = \frac{1}{1 + 0} = 1$$

The proof is complete after exponentiating.

$$\lim_{x\to\infty}\left\{\left(1 + \frac{1}{x}\right)^x\right\} = e^1 = e$$

***Final Summary for Derivative Applications* – L'Hôpital's Rule**

L'Hôpital's Rule for Quotient Indeterminate Forms
If $\lim_{x\to a}\{f(x)\} = \lim_{x\to a}\{g(x)\} = 0$ or $\pm\infty$, and the following hold: 3. $f(x)$ and $g(x)$ are both differentiable functions in an open interval containing a, except possibly at $x = a$. 4. $g'(x) \neq 0$, except possibly at $x = a$. Then, $$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to a}\left\{\frac{f'(x)}{g'(x)}\right\}$$ So long as the limit exists or is $\pm\infty$. Note 1: The goal in differentiating the numerator and denominator is to make the quotient simpler to evaluate. Note 2: This procedure can be applied repeatedly until the limit can be evaluated, assuming it exists. Note 3: The rule is also valid for one-sided limits. Note 4: The rule also applies for limits as $x \to \infty$ or $x \to -\infty$.
L'Hôpital's Rule for Other Indeterminate Forms
• Indeterminate forms of the type, $0 \cdot \infty$ or $\infty - \infty$, may also be evaluate using L'Hôpital's Rule once that are algebraically rearranged to be in the quotient form. • Indeterminate forms of the type, $1^\infty, 0^0, or\ \infty^0$, arising from functions of the form $f(x)^{g(x)}$ may also be evaluated with L'Hôpital's Rule according to $$\lim_{x\to a}\{f(x)^{g(x)}\} = e^{\lim_{x\to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}} = e^L$$ Where, L, if exists, is equal to $\lim_{x\to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}$
L'Hôpital's Rule to Compare Growth of Functions
If $f(x)$ and $g(x)$ are both differentiable functions, we can say that $f(x)$ grows faster that $g(x)$ if $\lim_{x\to\infty}\left\{\frac{f(x)}{g(x)}\right\} = \infty$ Or, equivalently $\lim_{x\to\infty}\left\{\frac{g(x)}{f(x)}\right\} = 0$

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