Derivative Applications – Function Behavior and Graph Sketching

We preciously learned that the derivative can be used to find the local minimums and maximums of a function, which are important characteristics of any function. In this section we learn that the derivative can be used to find many more interesting characteristics of functions. The various characteristics are useful in themselves, but even more, they can be used to obtain a reasonably accurate sketch of the graph of an entire function. We begin with learning how to determine when a function is increasing or decreasing.

Increasing and Decreasing Behavior of Functions

Recall that the derivative fundamentally tells us how the output of the function changes as the input changes.

$$f'(x) = \lim_{\Delta x \to 0} \left\{ \frac{\Delta y}{\Delta x} \right\}$$

Therefore, when the derivative is positive the output *increases* for increasing input, and when the derivative is negative the output *decreases* for increasing input. This can be stated formally with the following theorem.

Increasing and Decreasing Function Behavior
Consider a differentiable function, f, over an open interval, (a, b). • If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b). • If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

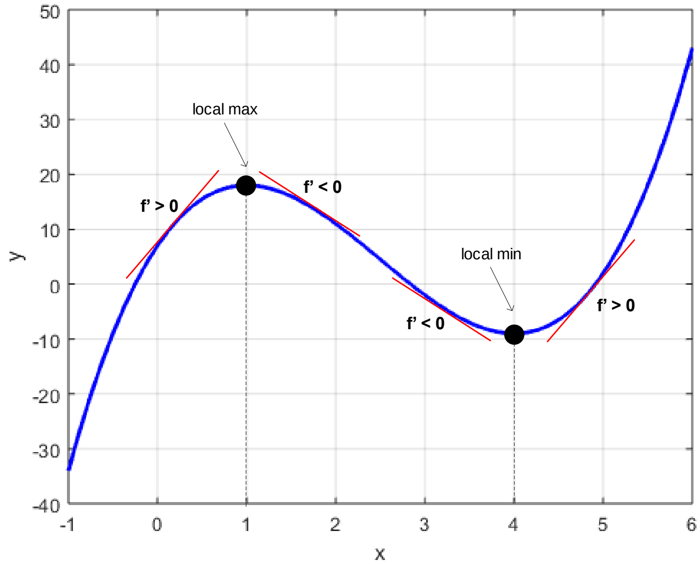
Let's combine this result with what we already know about extreme values and see what we can learn about a function from these two simple observations.

We'll use example 1 from the previous section. In that example we found two critical points for $f(x) = 2x^3 - 15x^2 + 24x + 7$ by taking the derivative and setting it to zero as follows.

$$f'(x) = 6x^2 - 30x + 24 = 0$$
$$(x - 4)(x - 1) = 0$$

The critical points are: $c_1 = 4$ and $c_2 = 1$.

Recall Fermat's theorem states that all local extreme values are critical points, however *not* all critical points are local extreme values. Therefore, how can we determine if *these* critical points are indeed local extreme values? Even further, if they are extreme values how can we determine if they are local minimums or local maximums? Let's see if we can discover a method for determining these aspects by examining the below plot of $f(x)$.



The first critical point is at $x = 1$, where the derivative if zero.

If we evaluate the derivative to the left of the point, e.g., $x = 0$, we find that it's positive.

$$f'(0) = 6 \cdot 0^2 - 30 \cdot 0 + 24 = 24$$

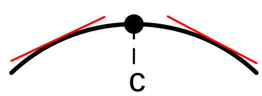
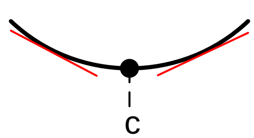
If we evaluate the derivative to the right of the point, e.g., $x = 2$, we find that it's negative.

$$f'(2) = 6 \cdot 2^2 - 30 \cdot 2 + 24 = -12$$

Therefore, the function is increasing before the critical point and decreasing after the critical point and we conclude that we have a local maximum at the critical point, $x = 1$.

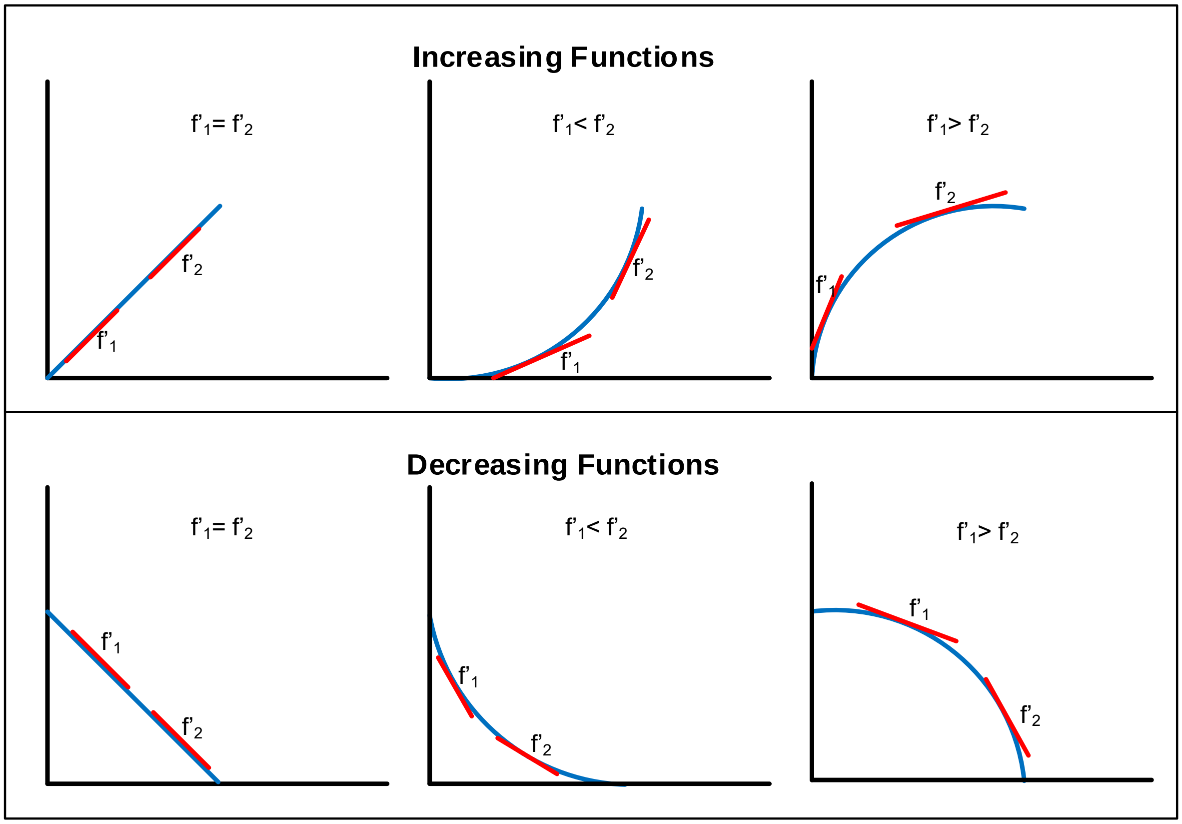
A similar argument can be made to conclude that we have a local minimum at $x = 4$.

This method for determining the nature of critical points is referred to this as the *first derivative test for critical points* and is stated below.

First Derivative Test for Critical Points
Let c be a critical point of the function, $f(x)$. Then: • If $f'(x)$ changes from + to – at c, $\Longrightarrow$ $f(c)$ is a local maximum.  • If $f'(x)$ changes from – to +at c, $\Longrightarrow$ $f(c)$ is a local minimum. 

Increasing and Decreasing Behavior of the *Derivative* of Functions – Concavity

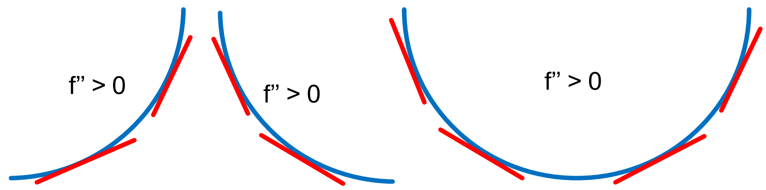
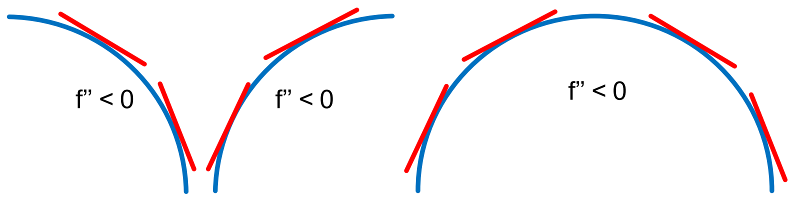
We have now learned how to determine whether a function is increasing or decreasing over certain intervals by looking at the sign of the derivative. However, simply knowing that a function is increasing or decreasing may not provide us with the level of detail we desire. As an example, look at the figures below. Each of the set of three functions are increasing or decreasing, but, as you can see, the shapes are very different.



The three columns can be distinguished as follows:

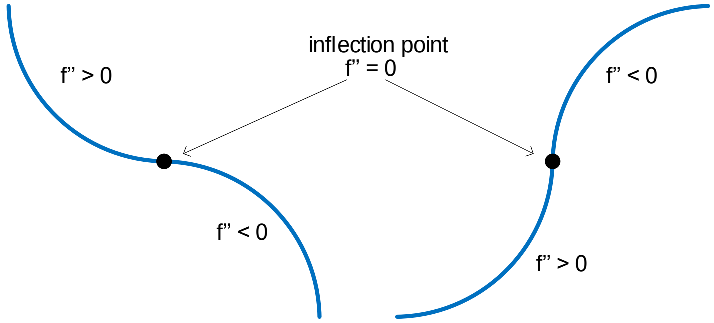
1. The derivative of the function is constant.
2. The derivative of the function is increasing.
3. The derivative of the function is decreasing.

Recall that the derivative of a function is itself a function. Therefore, we can use the same logic we used to determine the increasing and decreasing behavior of functions, but this time we look at the increasing and decreasing behavior of the *derivative of the function*, i.e. the second derivative of the function. We refer to this characteristic of a function as its concavity.

Concavity of a Function
Consider a differentiable function, f, over an open interval, (a, b). • If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).  • If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b). 

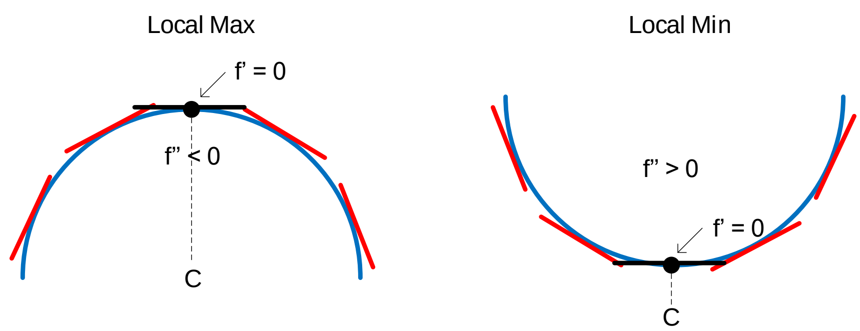
Inflection Points

As we have seen, of special interest with regard to the increasing and decreasing behavior of functions are the points on the graph where the behavior changes from increasing to decreasing or vice versa. They are of special interest because they are locations where local extreme values may occur. Similarly, points on the graph where the derivative of a function, i.e., the concavity of a function, changes sign are also of special interest. We refer to these points as *inflection points* and the theorem below specifies how they can be identified.

Test for Inflection Points
A number c in the domain of $f(x)$ is called an **inflection point** if either of the following are true: • $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$ • $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$ Examples: 

Second Derivative Test for Critical Points

Figuring out whether a critical point is either a local maximum or minimum is very useful when looking at the behavior of $f(x)$. Earlier we used the so-called first derivative test for this task. Based on what we have learned with regard to the concavity of a function we can introduce a simpler method for testing critical points, called the *second derivative test*. Using the figures shown in the definition of the concavity of a function for illustration, we can formally state the second derivative test as below.



Second Derivative Test for Critical Points

Let c be a critical point of the function, $f(x)$. Then:
 	
• If $f''(c) < 0 \implies f(c)$ is a local maximum.
• If $f''(c) > 0 \implies f(c)$ is a local minimum.
• If $f''(c) = 0 \implies$ The test is inconclusive.
  ○ In this case $f(c)$ may be a local maximum, a local minimum, an inflection point, or none of the above.
  ○ Use the first derivative test and/or the test for inflection points to investigate further.

Before moving to graph sketching let's do some examples using what we have learned so far.

Example 1:

Find the any local minimums and maximums, inflection points, and the intervals over which $f(x)$ is increasing, decreasing, concave up, and concave down.

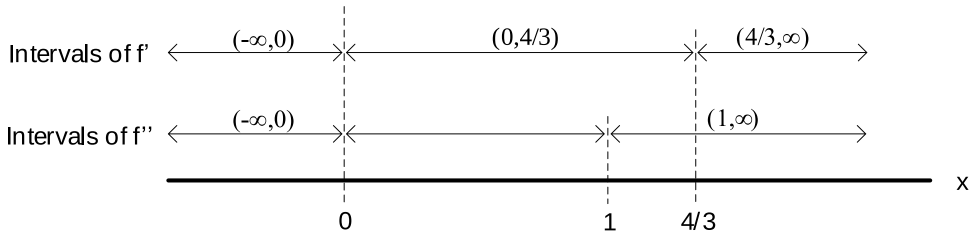
$$f(x) = 3x^5 - 5x^4 + 1$$

Solution:

We start by finding the zeros of the first and second derivatives. We then divide the x-axis into intervals based on these points and use "test points" within each interval to find the sign of these functions.

Zeros of First Derivative, $z1_n$	Zeros of Second Derivative, $z2_m$
$f'(x) = 15x^4 - 20x^3 = 0$ $5x^3(3x - 4) = 0$	$f''(x) = 60x^3 - 60x^2$ $= 60x^2(x - 1)$
$z1_1 = 0, \ \ z1_2 = 4/3$	$z2_1 = 0, \ \ z2_2 = 1$

The x-axis intervals for the first and second derivatives are shown below.

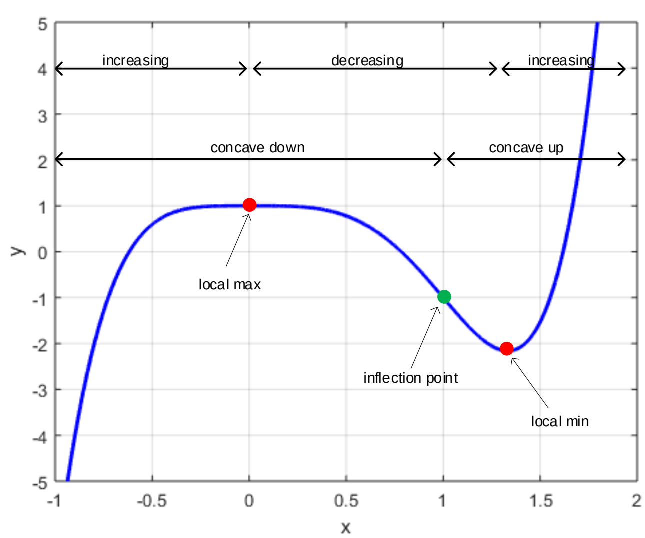


We now use test points within each of the above intervals to determine the function behavior.

Intervals based on the zeros of $f'(x)$	**Test value**	**Sign**	**Behavior**
$(-\infty, 0)$	$f'(-1) = 35$	$+$	Increasing
$(0, 4/3)$	$f'(0.5) = -1.5625$	$-$	Decreasing
$(4/3, \infty)$	$f'(2) = 80$	$+$	Increasing

Intervals based on the zeros of $f''(x)$	**Test value**	**Sign**	**Behavior**
$(-\infty, 0)$	$f''(-1) = -120$	$-$	Concave Down
$(0,1)$	$f''(0.5) = -7.5$	$-$	Concave Down
$(1, \infty)$	$f''(2) = 240$	$+$	Concave Up

Applying the first derivative test using the information from the first table we find a local maximum at $x = 0$, and a local minimum at $x = 4/3$. Furthermore, applying the test for inflection points using information from the second table we find an inflection point at $x = 1$. Finally, the graph of the function is shown below for illustration purposes.



Example 2:

Find the any local minimums and maximums, inflection points, and the intervals over which $f(x)$ is increasing, decreasing, concave up, and concave down.

$$f(x) = x^4 - 8x^2 + 1$$

We proceed as we did in example 1.

Zeros of First Derivative, $z1_n$	**Zeros of Second Derivative, $z2_m$**
$f'(x) = 4x^3 - 16x = 0$ $4x(x^2 - 4) = 0$ $4x(x-2)(x+2) = 0$	$f''(x) = 12x^2 - 16 = 0$ $x = \pm\frac{2\sqrt{3}}{3}$
$z1_1 = 0,\ \ z1_2 = 2,\ \ z1_3 = -2$	$z2_1 = \frac{2\sqrt{3}}{3},\ \ z2_2 = -\frac{2\sqrt{3}}{3}$

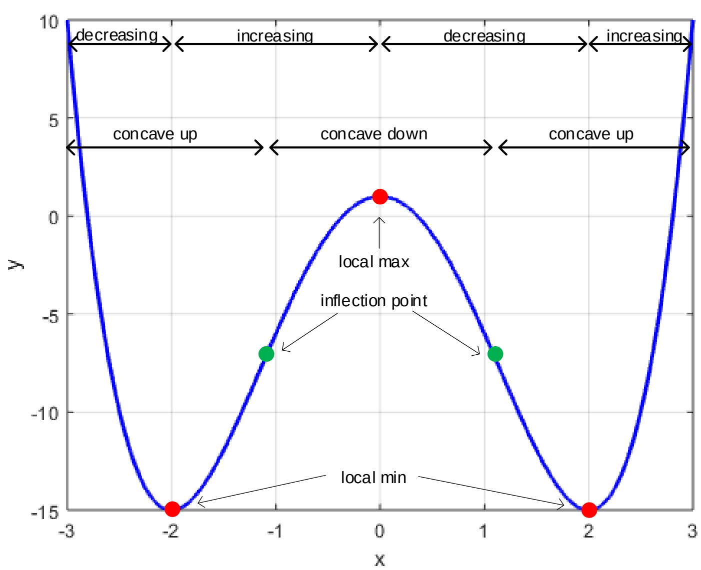
In this case, we'll use the second derivative test to determine if the critical points are indeed local extrema. We do this by evaluating the second derivative at the critical points.

$f''(0) = 12(0)^2 - 16 = -16\ < 0$	$\Longrightarrow$	$x = 0$ is a local maximum.
$f''(2) = 12(2)^2 - 16 = 32\ > 0$	$\Longrightarrow$	$x = 2$ is a local minimum.
$f''(-2) = 12(-2)^2 - 16 = 32\ > 0$	$\Longrightarrow$	$x = -2$ is a local minimum.

For inflection point identification we need to verify that the concavity changes at $z2_1$ and $z2_2$. To do this we use "test points" within intervals determined by $z2_1$ and $z2_2$.

Intervals based on the zeros of $f''(x)$	**Test value**	**Sign**	**Behavior**
$\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$	$f''(-1.5) = 11$	$+$	Concave Up
$\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$	$f''(0) = -16$	$-$	Concave Down
$\left(\frac{2\sqrt{3}}{3}, \infty\right)$	$f''(1.5) = 11$	$+$	Concave Up

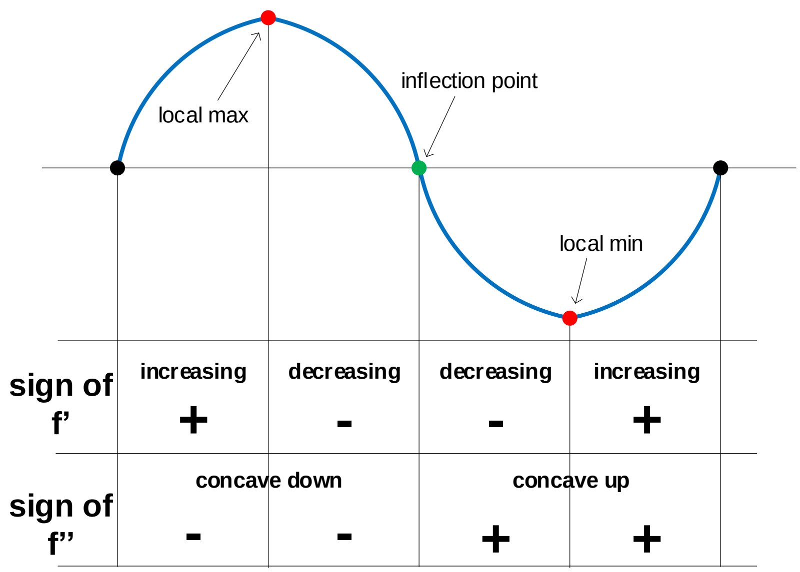
Since the sign of the second derivative changes at both zeros, these points are both inflection points. The graph of the function is again shown below for illustration purposes.



Sketching Graphs

As mentioned in the introduction we can use the characteristics from above to obtain a reasonably accurate sketch of the graph of a function. However, to obtain a good sketch it is also important to take note of the asymptotic behavior of $f(x)$ – i.e. the behavior as x approaches $\pm\infty$ and any vertical asymptotes.

Most graphs are made up of smaller *arcs* that have one of four basic shapes. The four shapes can be derived from all the possible sign combinations of the first and second derivatives of the function. These basic shapes are separated by so-called *transition points* - points where the sign of either f' or f'' change. Let's start by introducing the four basic shapes. The four shapes are most plainly seen by looking at one complete cycle of a sine wave, as shown below.



Using this table along with the transition points and any asymptotic behavior we can obtain a sketch of the graph of many different functions. Let's do some examples and see if we can develop a general procedure.

Example 3:

Sketch the graph of $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$.

As polynomials are continuous everywhere with a domain $(-\infty, \infty)$, we can start by finding any transition points, (i.e. extrema and inflection points), using the first and second derivatives.

$$f'(x) = 12x^3 - 24x^2 + 12x = 0$$
$$12x(x^2 - 2x + 1) = 0$$
$$12x(x-1)^2 = 0$$

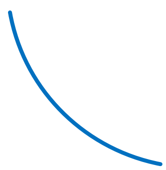
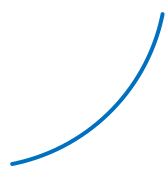
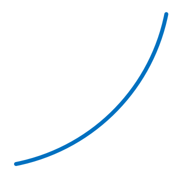
Therefore, two of the possible transition points are at $x = 0$, and $x = 1$

We do the same with the second derivative.

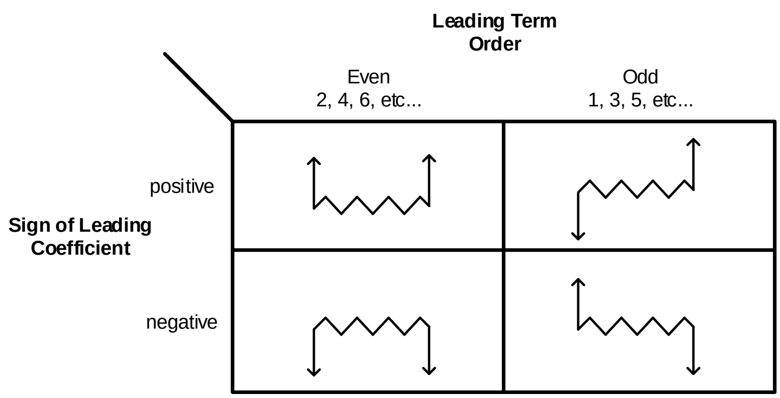
$$f''(x) = 36x^2 - 48x + 12 = 0$$
$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$

Additional possible transition points are at $x = \frac{1}{3}$, and $x = 1$

Let's make a table using intervals defined by the transition points, where we will evaluate the first and second derivatives to determine the shape of the graphs in those intervals.

Intervals	$(-\infty, \mathbf{0})$	$(\mathbf{0}, \mathbf{1/3})$	$(\mathbf{1/3}, \mathbf{1})$	$(\mathbf{1}, \infty)$
$\boldsymbol{f'(testPoint)}$	$f'(-1) = -48$	$f'(.5) = 1.5$		$f'(2) = 24$
$\boldsymbol{f''(testPoint)}$	$f''(-1) = 96$		$f''(2/3) = -4$	$f''(2) = 60$
Sign of $\boldsymbol{f'(testPoint)}$	$-$	$+$	$+$	$+$
Sign of $\boldsymbol{f''(testPoint)}$	$+$	$+$	$-$	$+$
Curve Shape				

Interpreting this table in terms of the first derivative test as well as the inflection point test we see that $x = 0$ is a local minimum, while $x = 1/3$ and $x = 1$ are inflection points. Next, we determine the asymptotic behavior. As you may recall the asymptotic behavior of polynomials is determined by the leading term, as indicated by the table below.

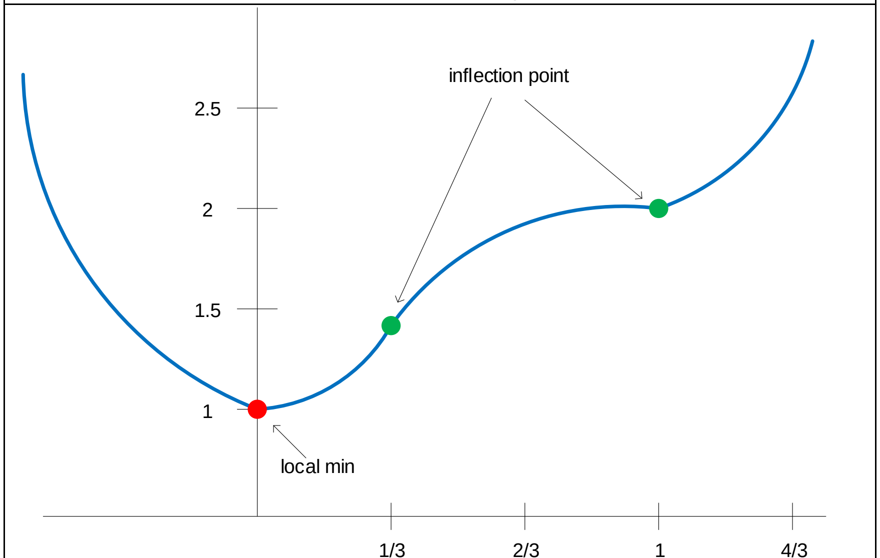


In this case we have a leading term with an even power and a positive coefficient, therefore $f(x)$ tends to ∞ as $x \to -\infty$ and as $x \to \infty$. Before we attempt to sketch the graph, we evaluate $f(x)$ at the transition points.

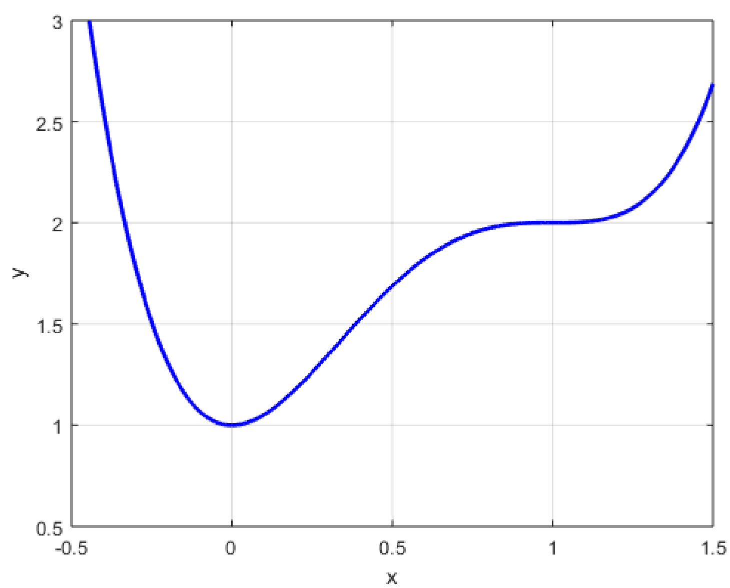
$f(0) = 1$ $\quad\quad$ $f(1/3) = 1.4$ $\quad\quad$ $f(1) = 2$

Finally, the graph can now be sketched by connecting the transition points with the proper shaped arc, while also considering the asymptotic behavior. For comparison the graph of the function produced by a computer is also shown.

Sketched Graph



Computer Generated Graph



Example 4:

Sketch the graph of $f(x) = \frac{1}{x^2-1}$.

We again start by finding any transition points, (i.e. extrema, inflection points, or where the derivatives are undefined), using the first and second derivatives.

$$f'(x) = -\frac{2x}{(x^2-1)^2}$$

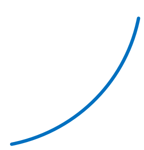
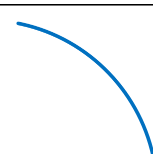
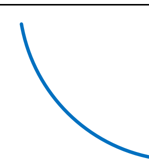
Which has three critical points, one at $x = 0$, where the function evaluates to zero, and two at $x = \pm 1$, where the derivate is undefined.

Similarly, with the second derivative we have:

$$\begin{aligned} f''(x) &= -\frac{2(x^2-1)^2 - 2x(2(x^2-1)2x)}{(x^2-1)^4} \\ &= -\frac{\cancel{(x^2-1)}(2(x^2-1) - 8x^2)}{(x^2-1)^{43}} \\ &= \frac{6x^2+2}{(x^2-1)^3} \end{aligned}$$

In this case the only possible transitions are at $x = \pm 1$, where the derivate is again undefined.

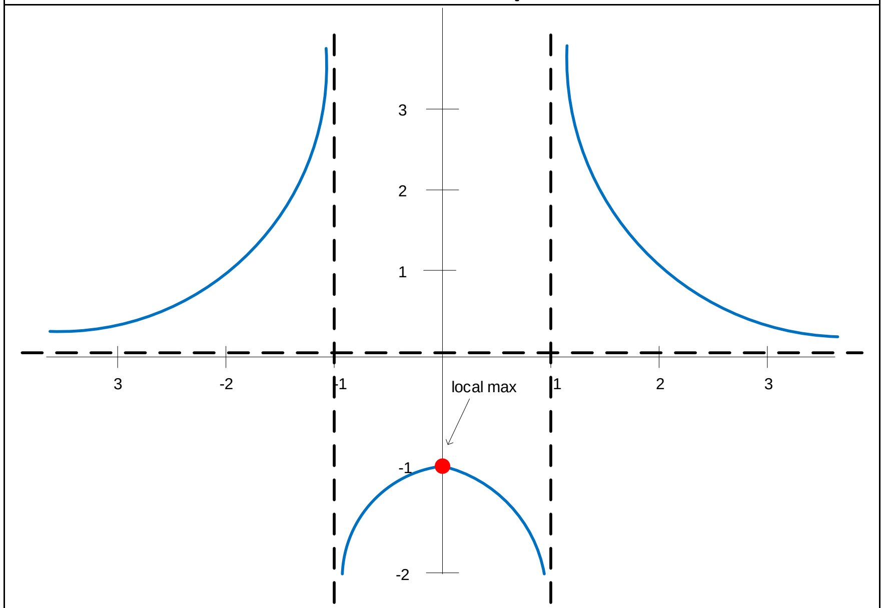
Next, we construct the table below as we did in the previous example.

Intervals	$(-\infty, -\mathbf{1})$	$(-\mathbf{1}, \mathbf{0})$	$(\mathbf{0}, \mathbf{1})$	$(\mathbf{1}, \infty)$
$\boldsymbol{f'(testPoint)}$	$f'(-2) = .44$	$f'(-0.5) = 1.78$	$f'(0.5) = -1.78$	$f'(2) = -0.44$
$\boldsymbol{f''(testPoint)}$	$f''(-2) = 0.96$	$f''(0) = -2$		$f''(2) = 0/96$
Sign of $\boldsymbol{f'(testPoint)}$	$+$	$+$	$-$	$-$
Sign of $\boldsymbol{f''(testPoint)}$	$+$	$-$	$-$	$+$
Curve Shape				

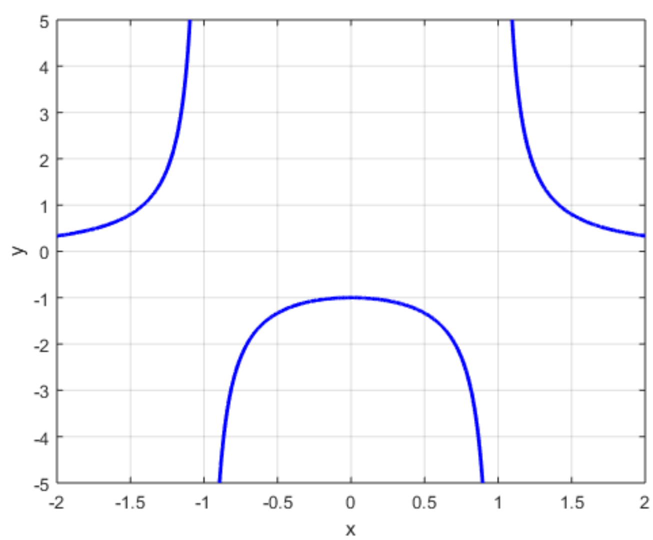
Interpreting this table in terms of the first derivative test we see that $x = 0$ is a local maximum. Next, we determine the asymptotic behavior. As this is a rational function, we identify the vertical asymptotes by setting the denominator to zero. Therefore, we have vertical asymptotes at $x = \pm 1$. Additionally, the function has a horizontal asymptote at $y = 0$ since the leading term in the denominator has a power that is greater than the equivalent numerator term. Lastly, before we attempt to sketch, we evaluate the function at the local maximum and find $f(0) = -1$

Finally, the graph can now be sketched by connecting the transition points with the proper shaped arc, while also considering the asymptotic behavior. For comparison the graph of the function produced by a computer is again shown.

Sketched Graph



Computer Generated Graph



A general procedure to sketch functions as we have done in the last two examples is given in the summary table below.

Final Summary for Derivative Applications – Function Behavior and Graph Sketching

Increasing and Decreasing Function Behavior

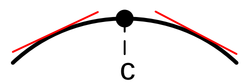
Consider a differentiable function, f, over an open interval, (a, b).

- If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).
- If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

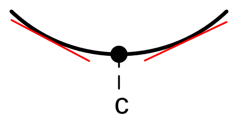
First Derivative Test for Critical Points

Let c be a critical point of the function, $f(x)$. Then:

- If $f'(x)$ changes from $+$ to – at c, $\implies$ $f(c)$ is a local maximum.



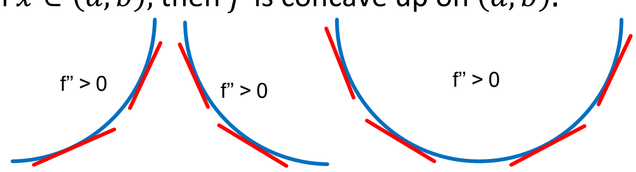
- If $f'(x)$ changes from – to $+$at c, $\implies$ $f(c)$ is a local minimum.



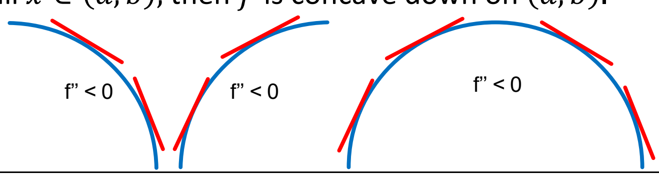
Concavity of a Function

Consider a differentiable function, f, over an open interval, (a, b).

- If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).



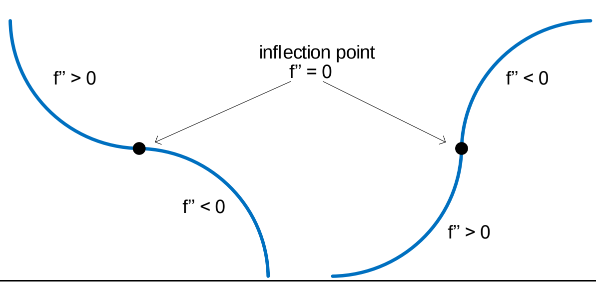
- If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).



Test for Inflection Points

A number c in the domain of $f(x)$ is called an **inflection point** if either of the following are true:

- $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$
- $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$



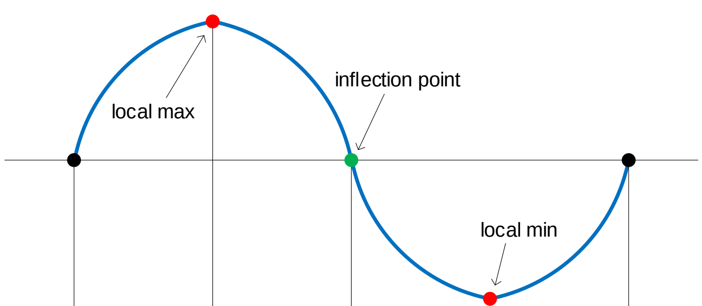
Second Derivative Test for Critical Points
Let c be a critical point of the function, $f(x)$. Then:

- If $f''(c) < 0 \implies f(c)$ is a local maximum.
- If $f''(c) > 0 \implies f(c)$ is a local minimum.
- If $f''(c) = 0 \implies$ The test is inconclusive.
  - In this case $f(c)$ may be a local maximum, a local minimum, an inflection point, or none of the above.
  - Use the first derivative test and/or the test for inflection points to investigate further.

Graph Sketching Procedure
The following steps can be used to obtain a sketch of a function, $f(x)$.

1. Determine the Domain of $f(x)$.
2. Compute $f'(x)$ and $f''(x)$ and identify all potential transition points, i.e., points where either of these functions is zero or undefined.
3. Divide the x-axis into intervals based on the transition points in step 2.
4. Choose a "test point" in each interval and evaluate $f'(x)$ and $f''(x)$ to determine the sign.
5. Determine any local minimums, local maximums, inflection points, and the asymptotic behavior of $f(x)$.
6. Compute $f(x)$ at all local minimums, local maximums, and inflection points.
7. Draw arcs of the appropriate shape based on the sign combination found in step 4 in each interval and "connect" them at the computed values from step 6, noting the asymptotic behavior.

The basic function shapes, determined by the sign of $f'(x)$ and $f''(x)$, are shown below.



sign of f'	increasing +	decreasing -	decreasing -	increasing +
sign of f''	concave down -	-	concave up +	+

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